Dynamic Programming

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23 December, 2021

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- "The lecture taught the students to appreciate how the concept of optimal substructures can be used in designing algorithms"

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- In our example, 24 characters inserted, 18 ~~deleted~~, 2 substituted

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Edit distance

- Minimum number of edit operations needed
- In our example, 24 characters inserted, 18 ~~deleted~~, 2 ~~substituted~~
- Edit distance is at most 44

Edit distance

- Minimum number of editing operations needed to transform one document to the other
  - Insert a character
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- LCS equivalent to edit distance without substitution

Inductive structure for edit distance

- $u = a_0 a_1 \dots a_{m-1}$
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- If $a_i = b_j$,
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  - $ED(m,n) = 0$
  - $ED(i,n) = m - i$ for all $0 \leq i \leq m$
    Delete $a_i a_{i+1} \ldots a_{m-1}$ from u

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  - $ED(m,j) = n - j$ for all $0 \leq j \leq n$
    Insert $b_j b_{j+1} \ldots b_{n-1}$ into u

Subproblem dependency

- Subproblems are $ED(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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- Table of $(m + 1) \cdot (n + 1)$ values

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
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4	c							
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6	•							

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**6**
1	i							**5**
2	s							**4**
3	e							**3**
4	c							**2**
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2	s		3	3	2	2	3	4
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Reading off the solution

- Transform `bisect` to `secret`

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6	•	6	5	4	3	2	1	0

Subproblem dependency

- Subproblems are $ED(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$
- Table of $(m+1)\cdot(n+1)$ values
- Like LCS, $ED(i,j)$ depends on $ED(i{+}1,j{+}1)$, $ED(i,j{+}1)$, $ED(i{+}1,j)$
- No dependency for $ED(m,n)$ — start at bottom right and fill by row, column or diagonal

Reading off the solution

- Transform `bisect` to `secret`
- Delete `b`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
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		s	e	c	r	e	t	•
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Reading off the solution

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		s	e	c	r	e	t	•
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		s	e	c	r	e	t	•
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Implementation

```
def ED(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  ed = np.zeros((m+1,n+1))

  for i in range(m-1,-1,-1):
    ed[i,n] = m-i
  for j in range(n-1,-1,-1):
    ed[m,j] = n-j

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        ed[i,j] = ed[i+1,j+1]
      else:
        ed[i,j] = 1 + min(ed[i+1,j+1],
                          ed[i,j+1],
                          ed[i+1,j])
  return(ed[0,0])
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Complexity

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Complexity

- Again $O(mn)$, using dynamic programming or memoization

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```

Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$

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  - $BC : 100 \times 100$, takes $100 \cdot 1 \cdot 100 = 10000$ steps to compute

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- Given n matrices $M_0 : r_0 \times c_0$, $M_1 : r_1 \times c_1$, . . . , $M_{n-1} : r_{n-1} \times c_{n-1}$

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  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed

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  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed
- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
  for some $0 < k < n$

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- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$

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- $C(0, n-1) = C(0, k-1) + C(k, n-1) + r_0 r_k c_{n-1}$

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- $C(0, n-1) = C(0, k-1) + C(k, n-1) + r_0 r_k c_{n-1}$

- Which k should we choose?

Inductive structure

- Final step combines two subproducts $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$ for some $0 < k < n$
- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$
- Let $C(0, n{-}1)$ denote the overall cost
- Final multiplication is $O(r_0 r_k c_{n-1})$
- Inductively, costs of factors are $C(0, k{-}1)$ and $C(k, n{-}1)$
- $C(0, n{-}1) =$
  $$C(0, k{-}1) + C(k, n{-}1) + r_0 r_k c_{n-1}$$
- Which k should we choose?
  - Try all and choose the minimum!

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- Subproblems?

Inductive structure

- Final step combines two subproducts $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$ for some $0 < k < n$
- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$
- Let $C(0, n-1)$ denote the overall cost
- Final multiplication is $O(r_0 r_k c_{n-1})$
- Inductively, costs of factors are $C(0, k-1)$ and $C(k, n-1)$
- $C(0, n-1) = C(0, k-1) + C(k, n-1) + r_0 r_k c_{n-1}$

- Which k should we choose?
  - Try all and choose the minimum!
- Subproblems?
  - $M_0 \cdot M_1 \cdots M_{k-1}$ would decompose as $(M_0 \cdots M_{j-1}) \cdot (M_j \cdots M_{k-1})$
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- Base case: $C(j, j) = 0$ for $0 \leq j < n$

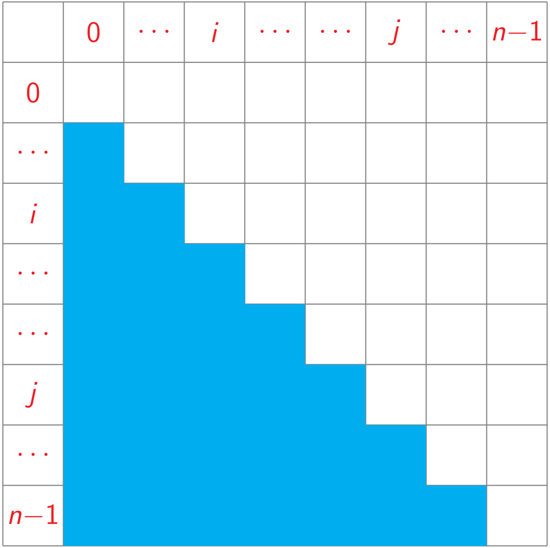
Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n-1$
0								
$\cdots$								
i								
$\cdots$								
$\cdots$								
j								
$\cdots$								
$n-1$								

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
  - Only for $i \leq j$
  - Entries above main diagonal



Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
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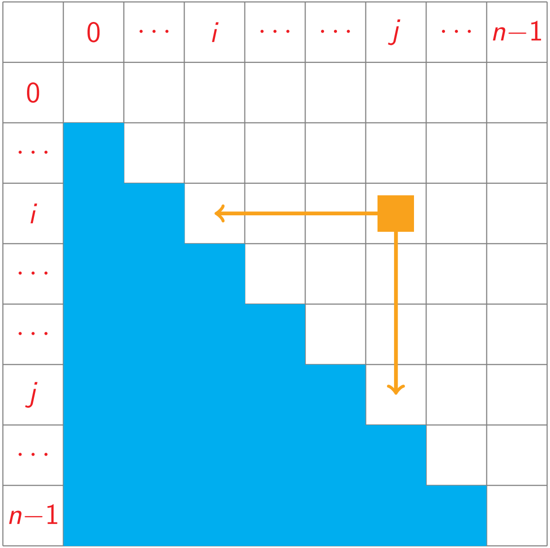
- $C(i,j)$ depends on $C(i,k-1)$, $C(k,j)$ for every $i < k \leq j$



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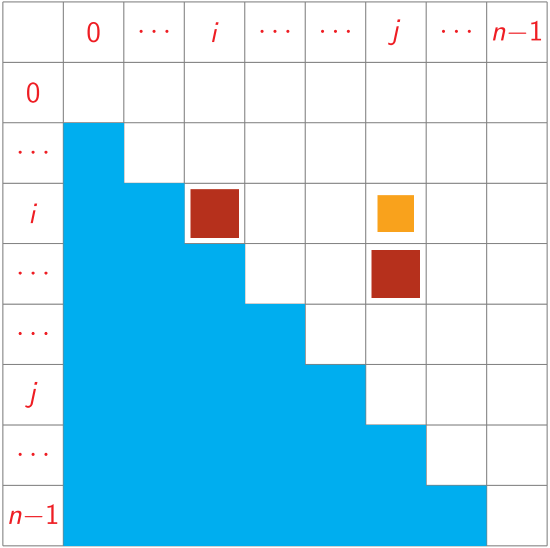

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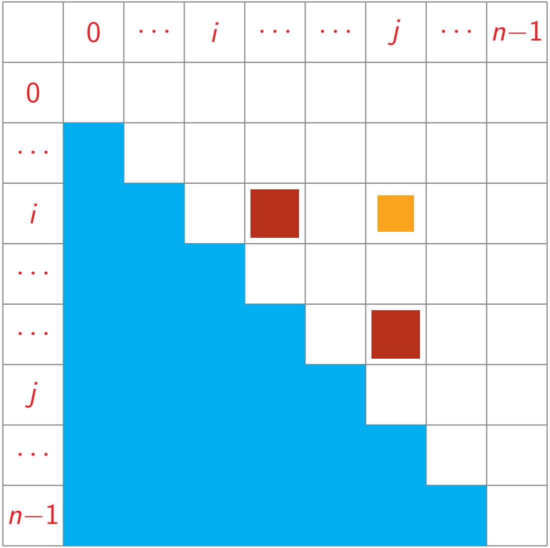

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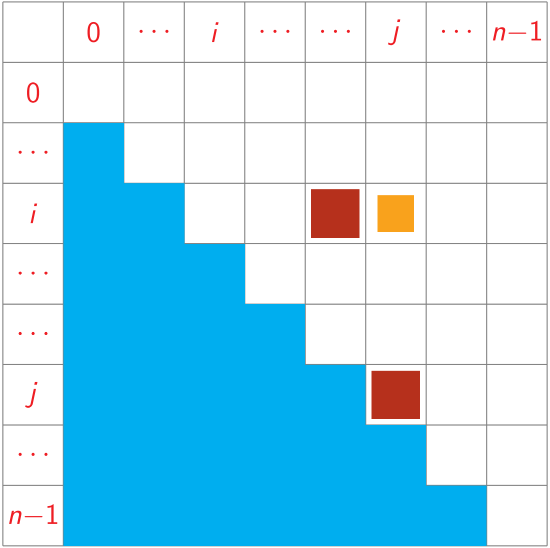

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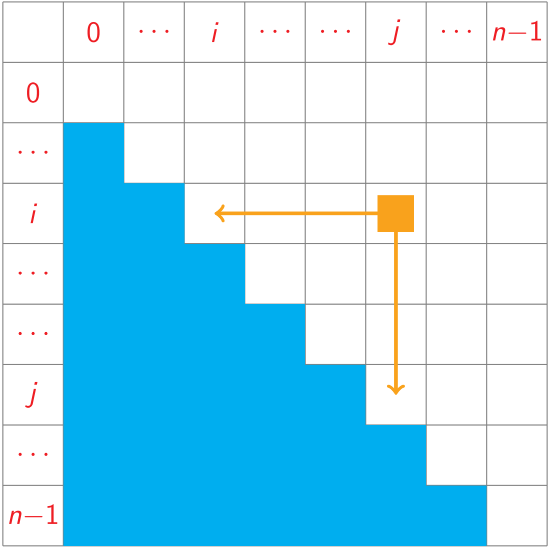

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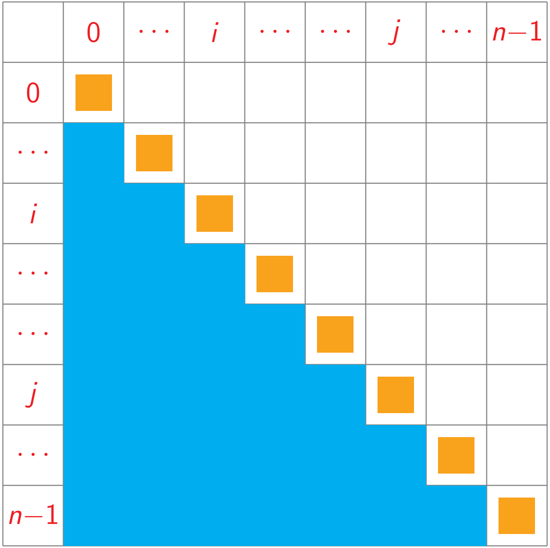

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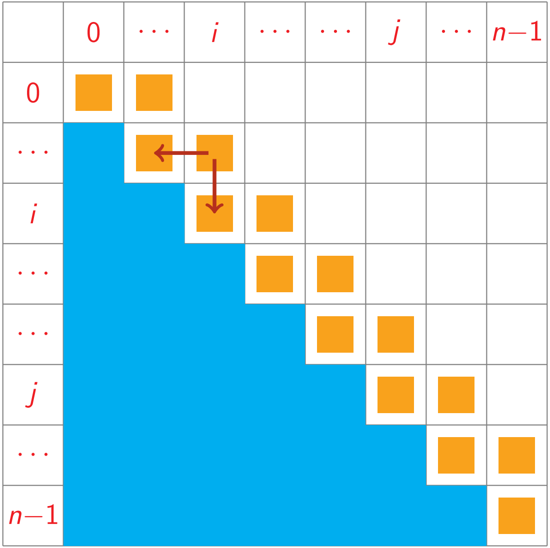
- Fill matrix by diagonal, from main diagonal



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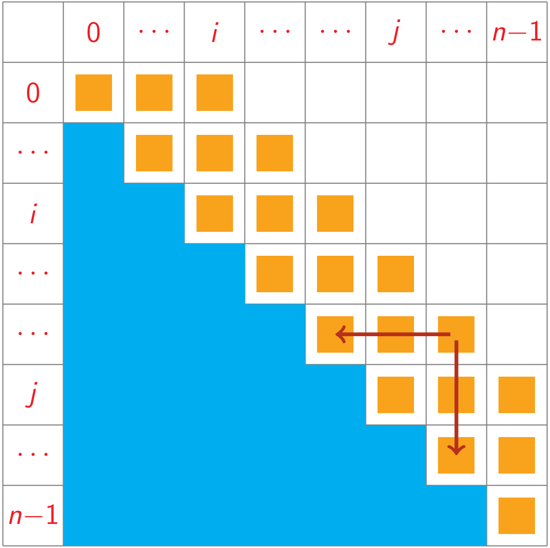

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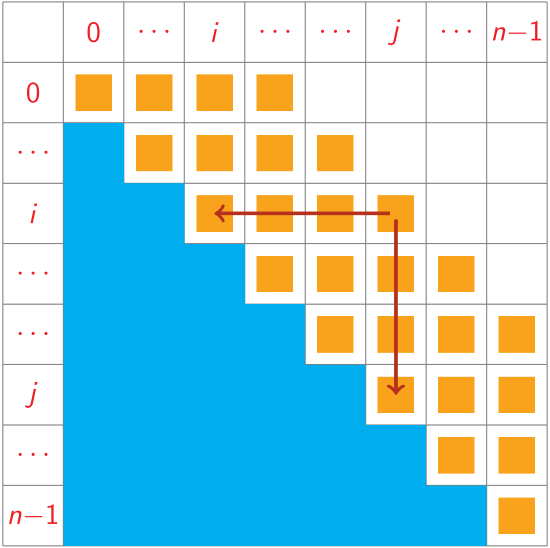

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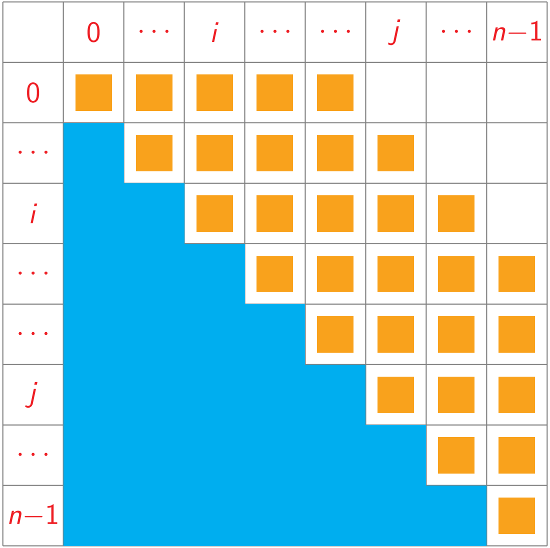

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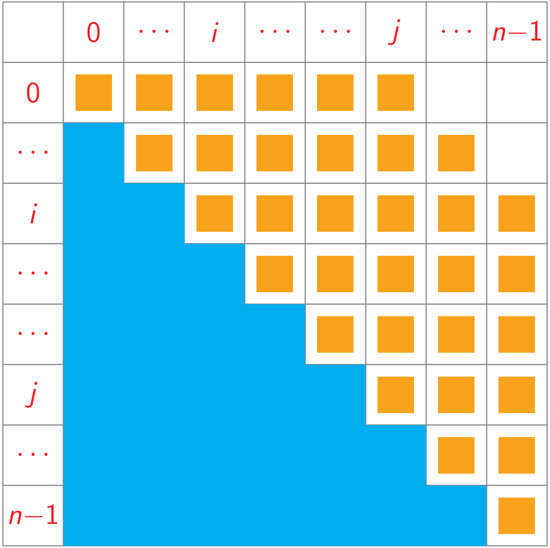

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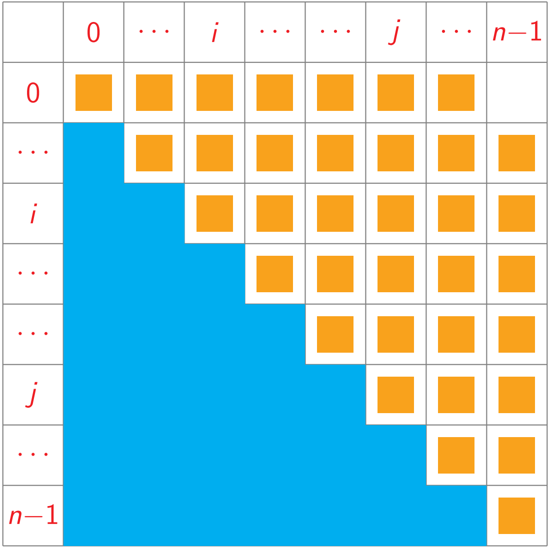

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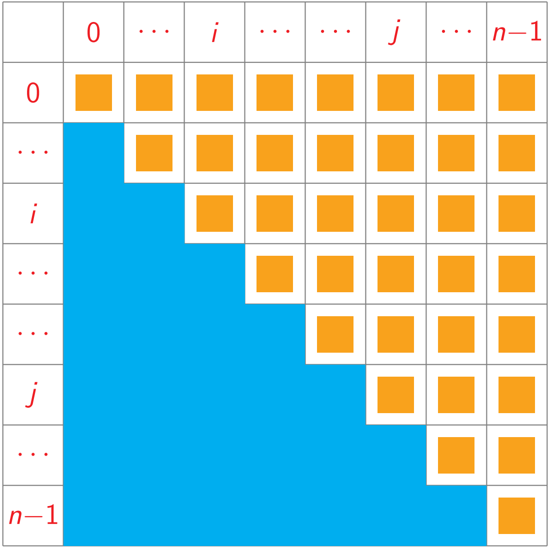

Implementation

```
def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
               dim[i][0]*dim[i+1][0]*dim[j][1]
      for k in range(i+1,j+1):
        C[i,j] = min(C[i,j],
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Complexity

- We have to fill a table of size $O(n^2)$

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Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$