Dynamic Programming

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20 December, 2021

Memoizing recursive implementations

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value
  return(value)
```

In general

```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic

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Evaluating `fib(5)`

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`fib(4)`

`fib(3)`

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`fib(1)`

`fib(0)`

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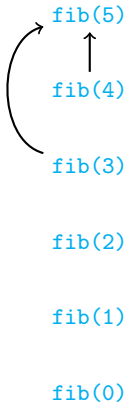

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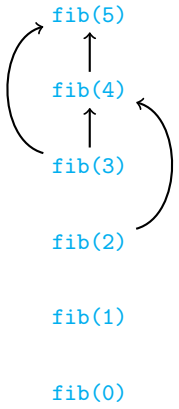

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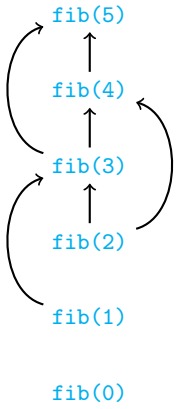

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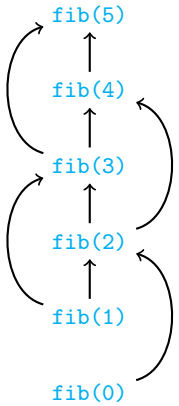

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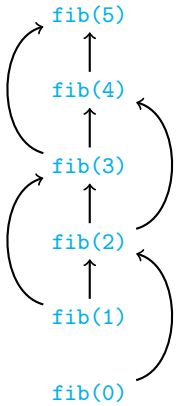

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  - Start with base cases — no dependencies

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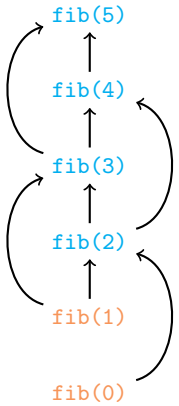

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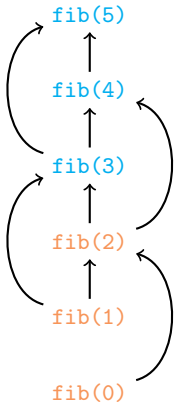

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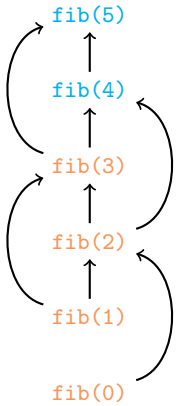

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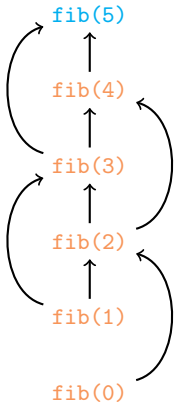

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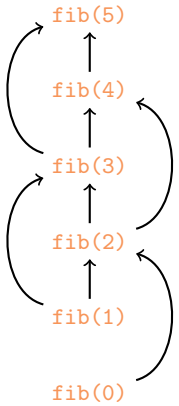

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  - Fill table iteratively
  - Never need to make a recursive call

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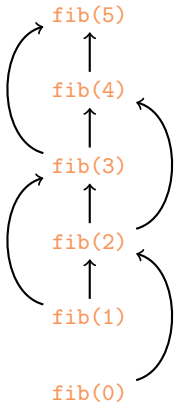
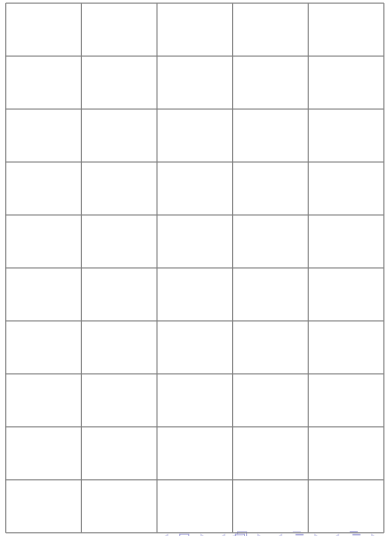


Grid paths

- Rectangular grid of one-way roads



Grid paths

- Rectangular grid of one-way roads
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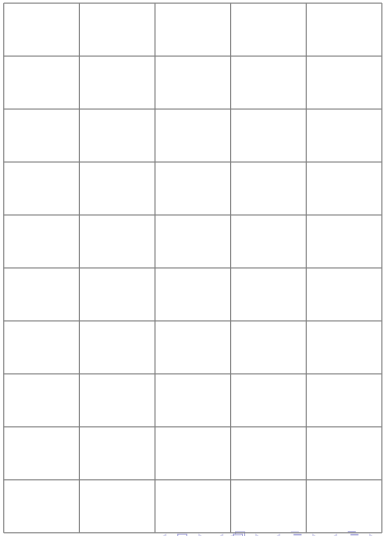

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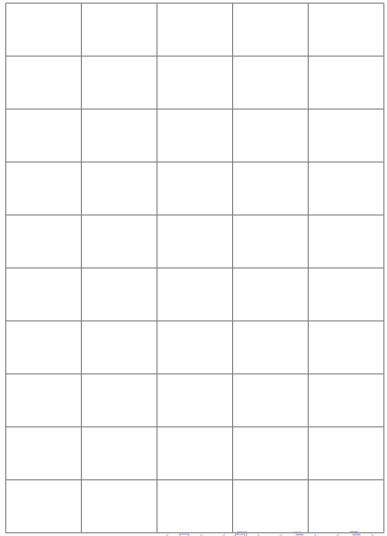

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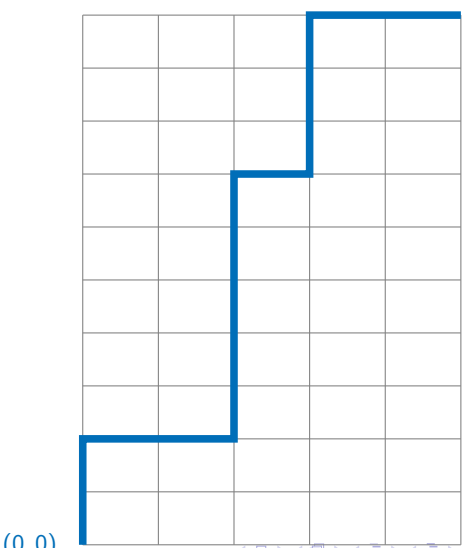

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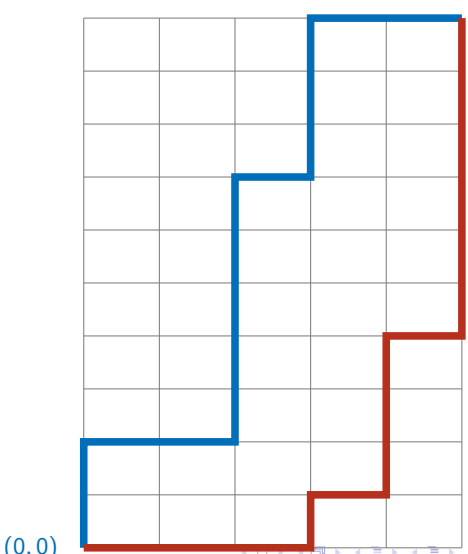

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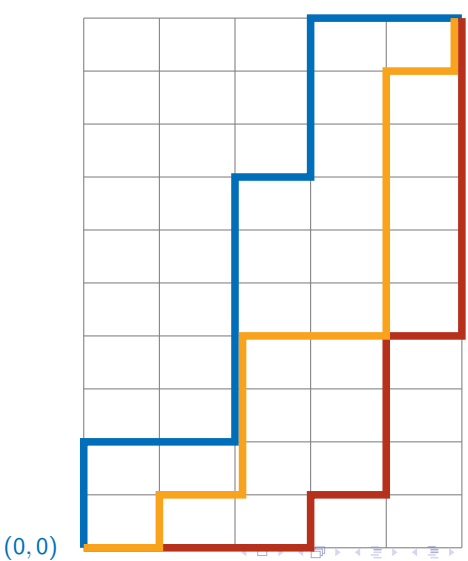
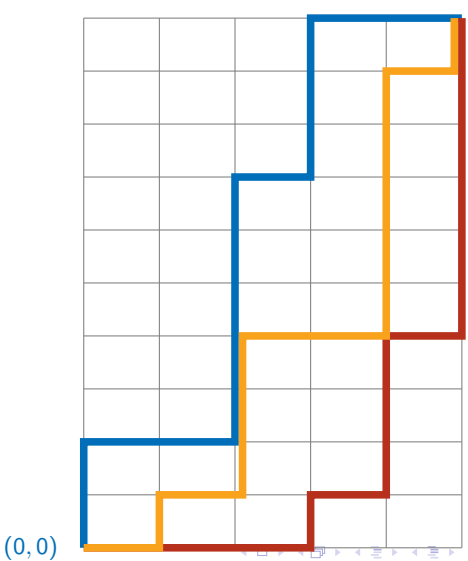


Combinatorial solution

- Every path from $(0, 0)$ to $(5, 10)$ has 15 segments
  - In general $m+n$ segments from $(0, 0)$ to (m, n)



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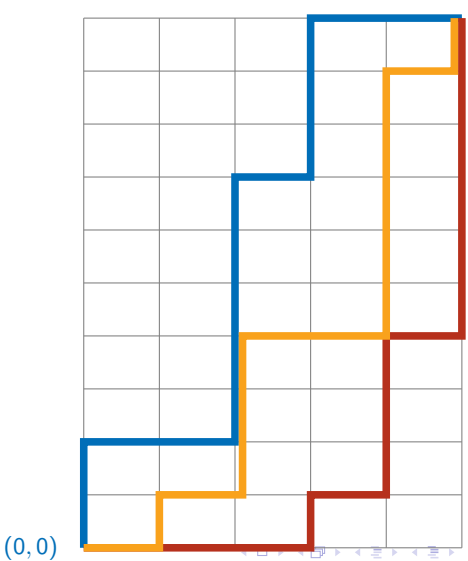

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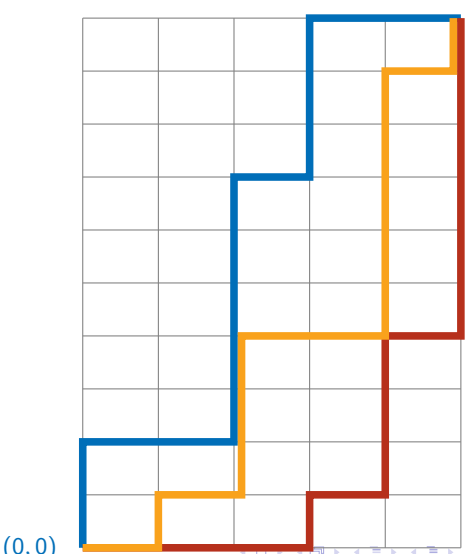
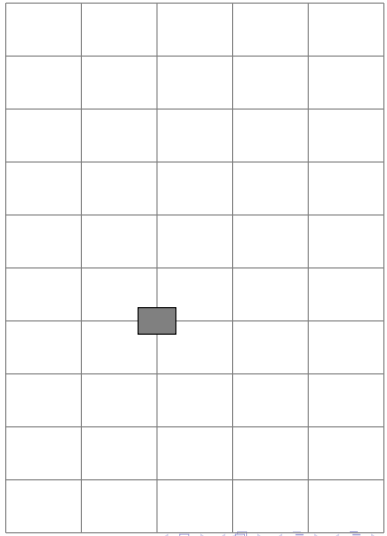
  - In general $m+n$ segments from $(0, 0)$ to (m, n)
- Out of 15, exactly 5 are right moves, 10 are up moves
- Fix the positions of the 5 right moves among the 15 positions overall
  - $\binom{15}{5} = \frac{15!}{10! \cdot 5!} = 3003$
  - Same as $\binom{15}{10}$ — fix the 10 up moves



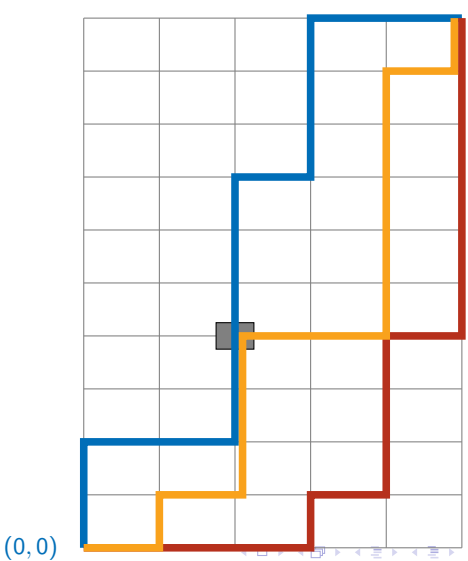
Holes

- What if an intersection is blocked?
  - For instance, $(2, 4)$



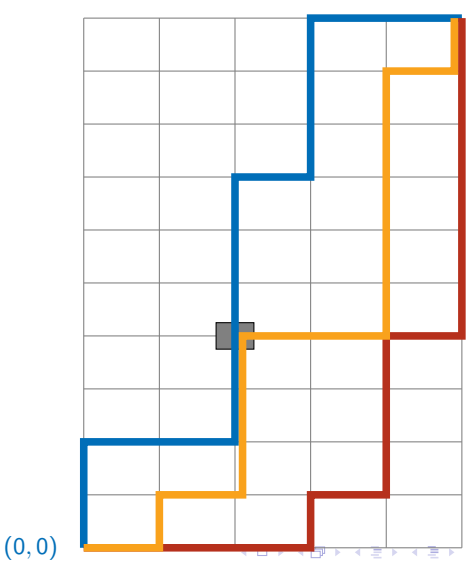
Holes

- What if an intersection is blocked?
  - For instance, $(2, 4)$
- Need to discard paths passing through $(2, 4)$
  - Two of our earlier examples are invalid paths



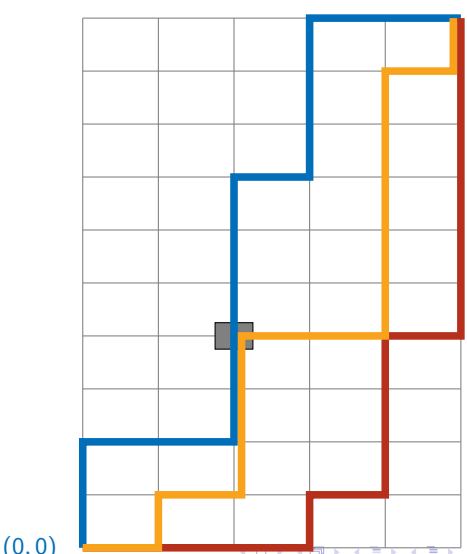
Combinatorial solution for holes

- Discard paths passing through $(2, 4)$



Combinatorial solution for holes

- Discard paths passing through $(2, 4)$
- Every path via $(2, 4)$ combines a path from $(0, 0)$ to $(2, 4)$ with a path from $(2, 4)$ to $(5, 10)$
  - Count these separately
  - $\binom{2+4}{2} = 15$ paths $(0, 0)$ to $(2, 4)$
  - $\binom{3+6}{3} = 84$ paths $(2, 4)$ to $(5, 10)$



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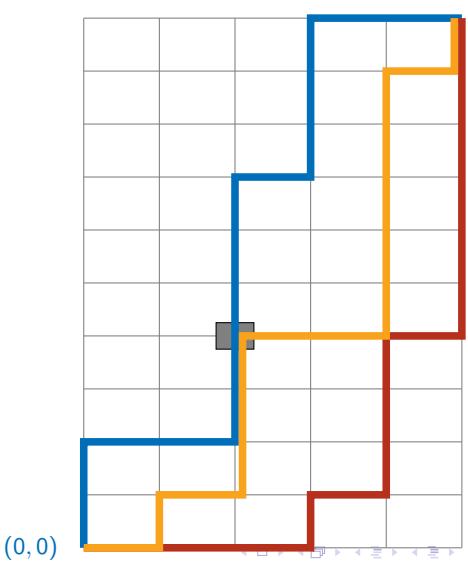
- $15 \times 84 = 1260$ paths via $(2, 4)$



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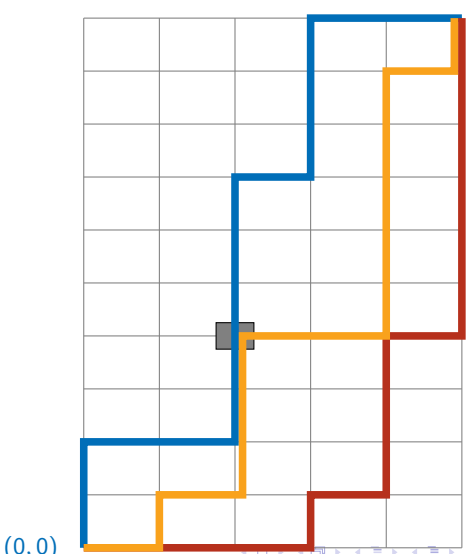
- $3003 - 1260 = 1743$ valid paths avoiding $(2, 4)$



More holes

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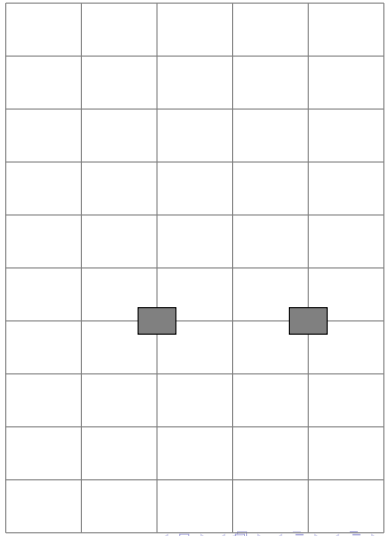
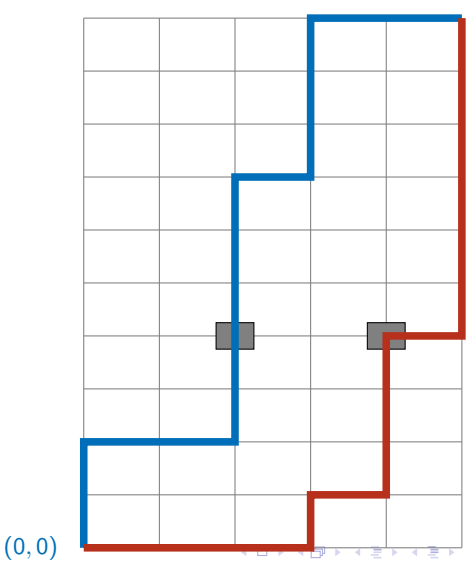


More holes

- What if two intersections are blocked?
- Discard paths via $(2, 4)$, $(4, 4)$



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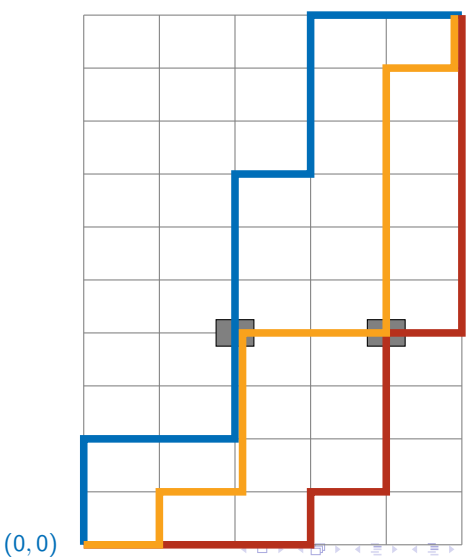

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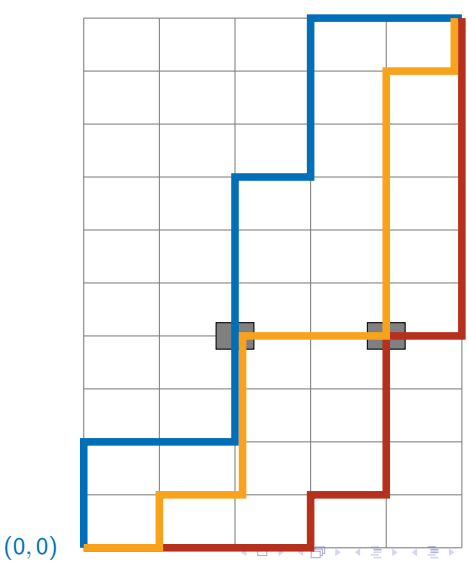
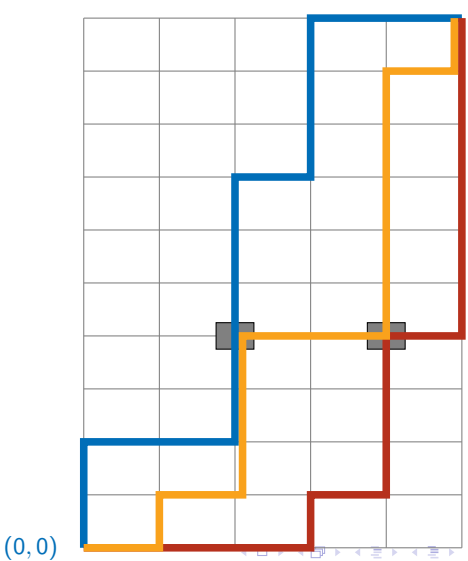
- Add back the paths that pass through both holes



More holes

- What if two intersections are blocked?
- Discard paths via $(2, 4)$, $(4, 4)$
  - Some paths are counted twice
- Add back the paths that pass through both holes
- Inclusion-exclusion — counting is messy



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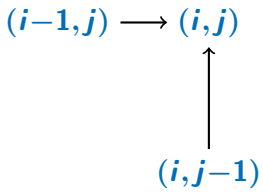

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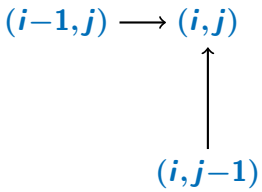

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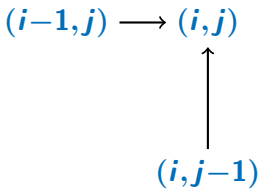
  - $P(i, j) = P(i-1, j) + P(i, j-1)$



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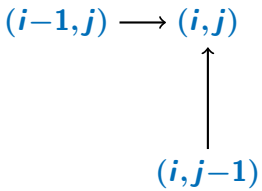

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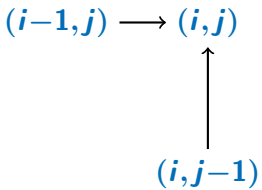

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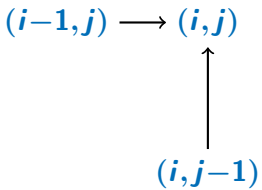

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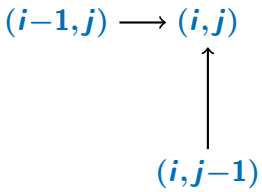
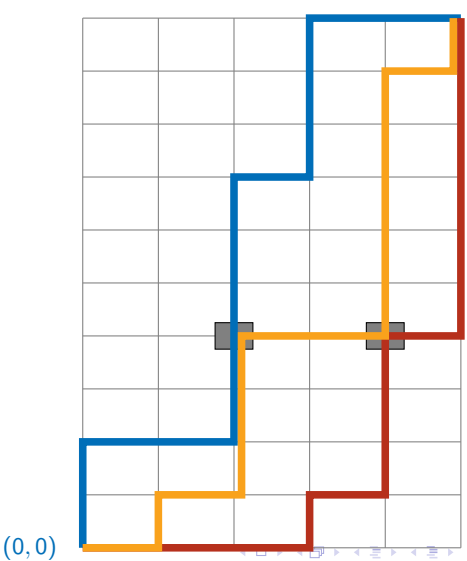
- $P(i, j) = 0$ if there is a hole at (i, j)



Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly



Computing $P(i,j)$

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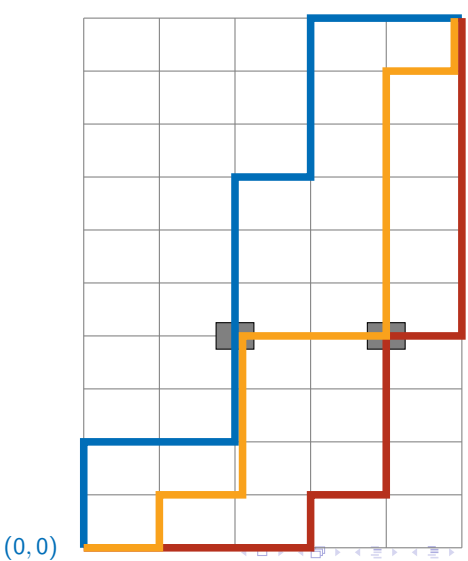
  - $P(5,10)$ requires $P(4,10)$, $P(5,9)$



Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly
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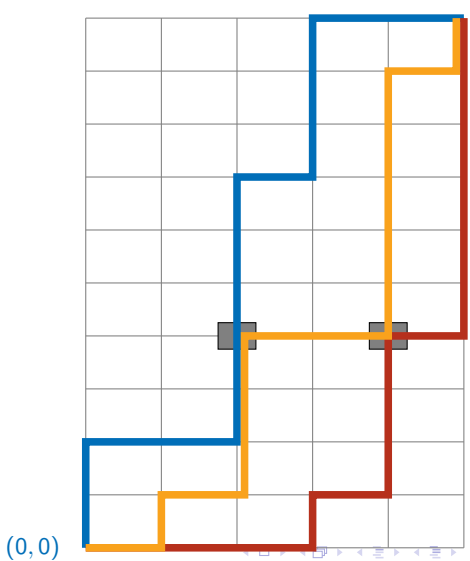
  - Both $P(4,10)$, $P(5,9)$ require $P(4,9)$



Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5,10)$ requires $P(4,10)$, $P(5,9)$
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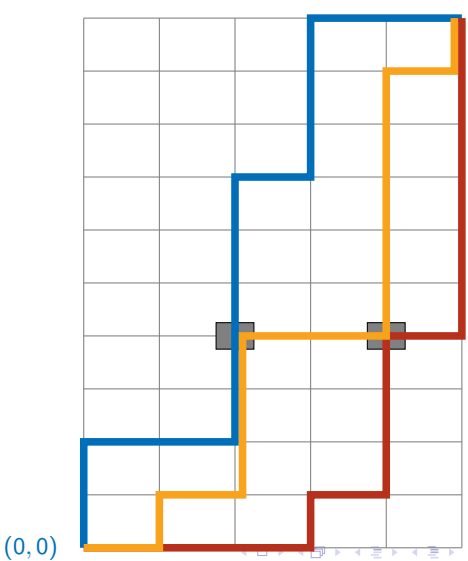
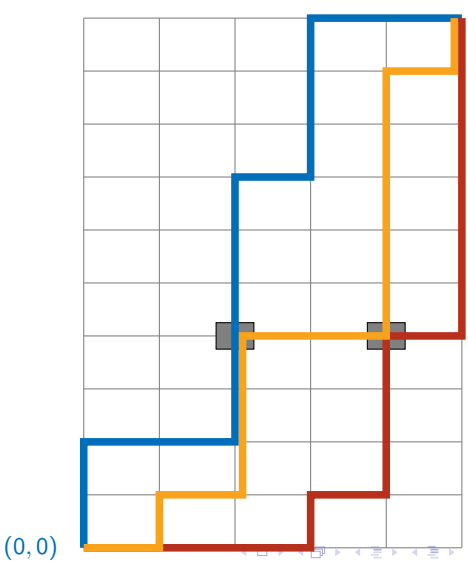
- Use memoization ...



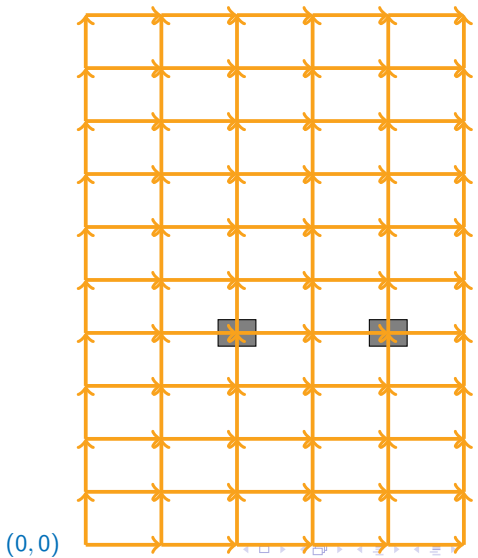
Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5,10)$ requires $P(4,10)$, $P(5,9)$
  - Both $P(4,10)$, $P(5,9)$ require $P(4,9)$
- Use memoization ...
- ... or find a suitable order to compute the subproblems



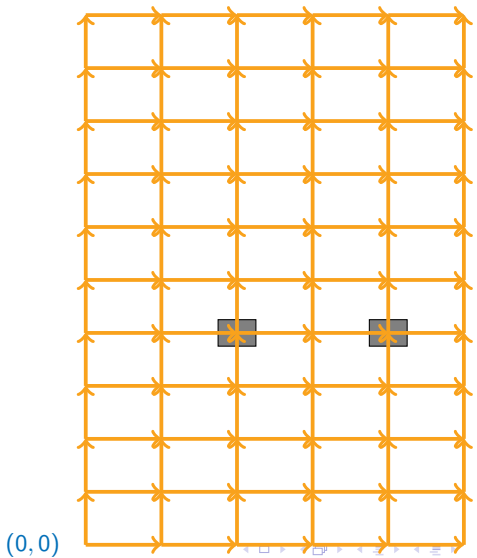
Dynamic programming

- Identify subproblem structure



Dynamic programming

- Identify subproblem structure
- $P(0, 0)$ has no dependencies



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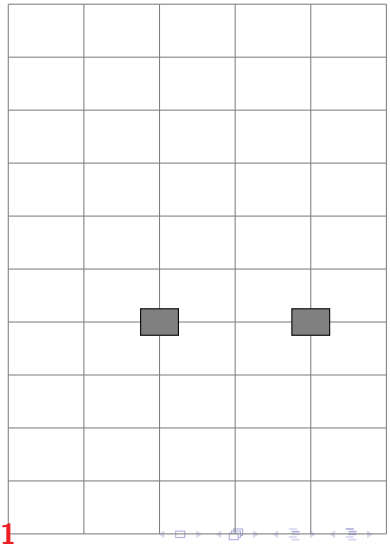

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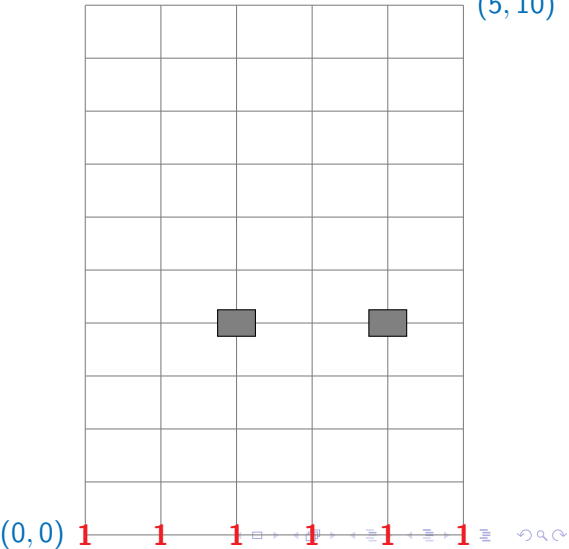

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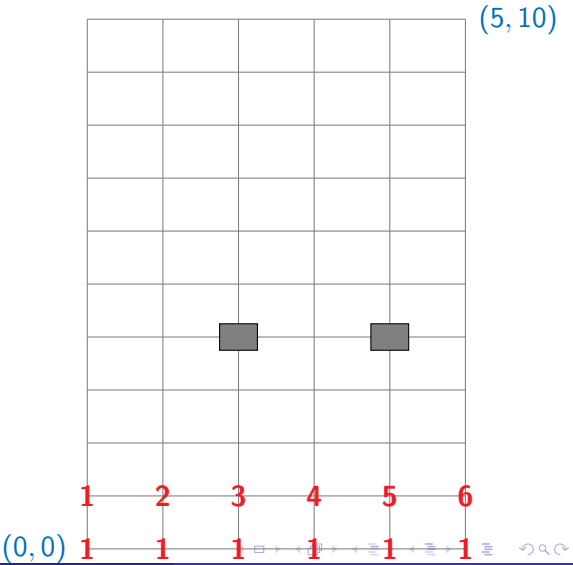

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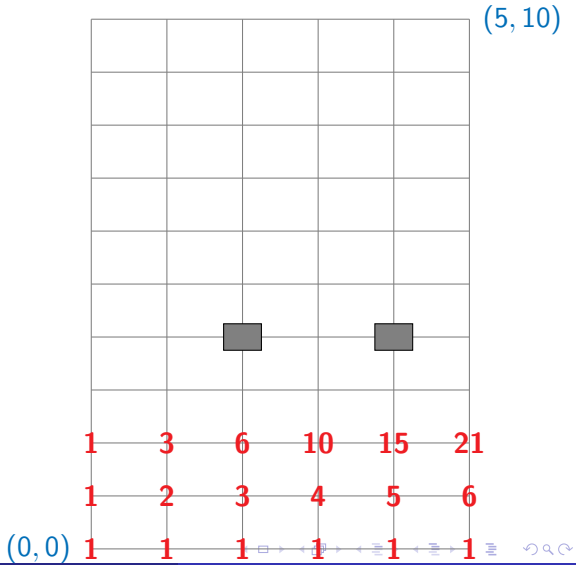

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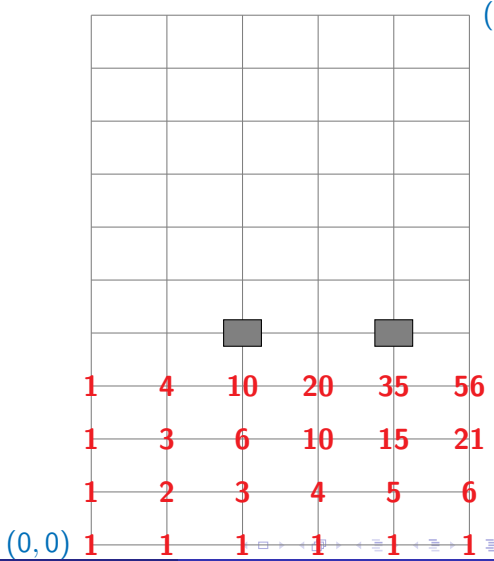

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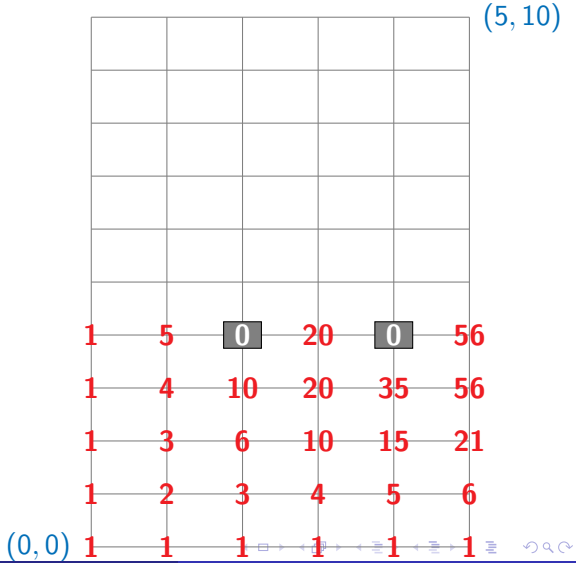

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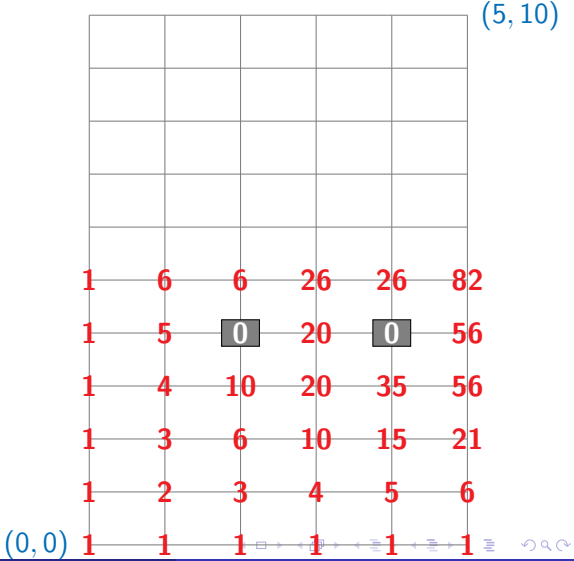

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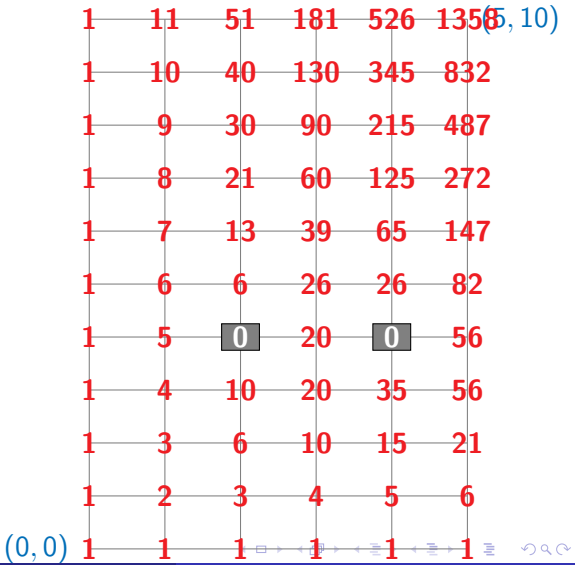

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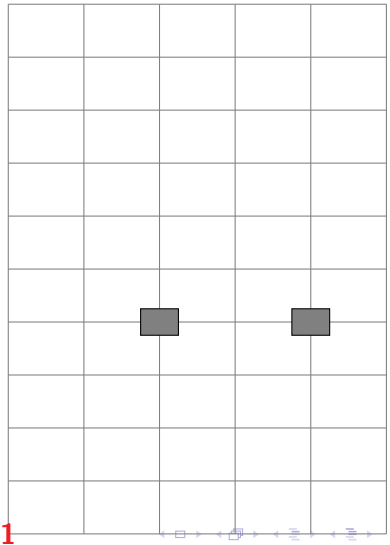

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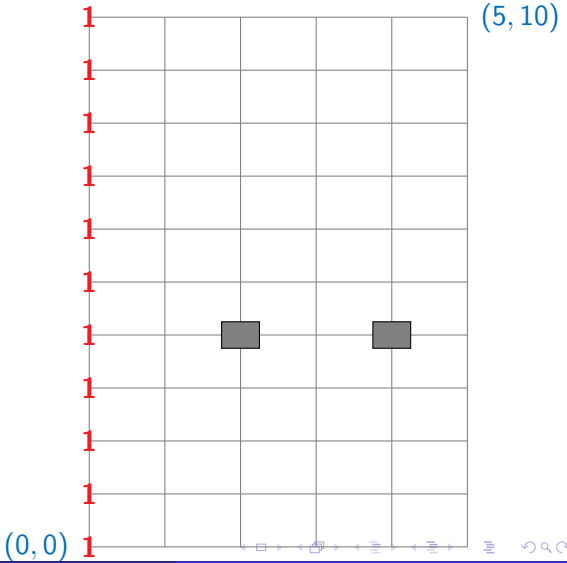

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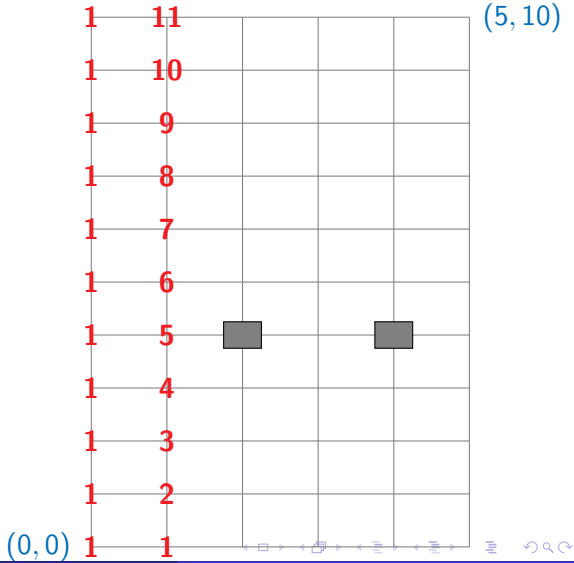

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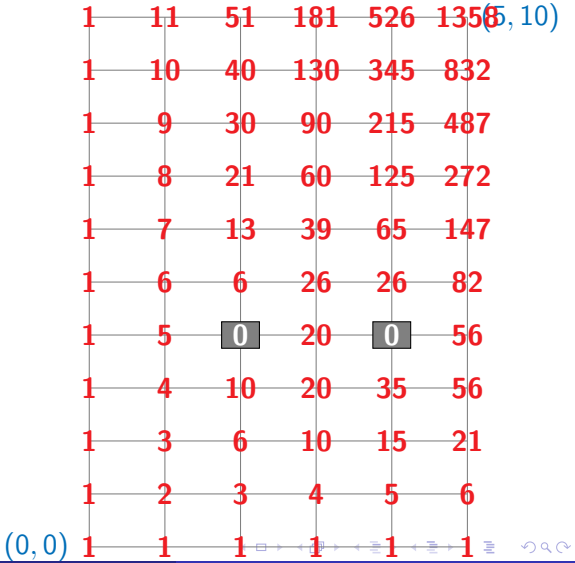

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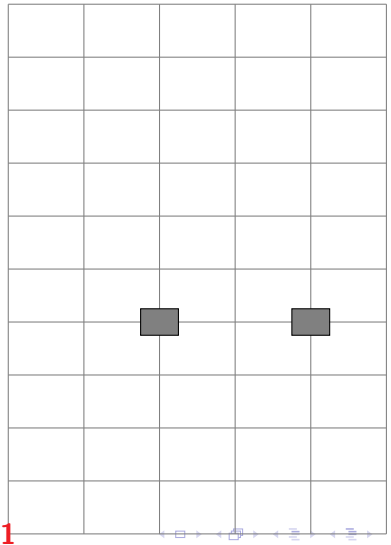

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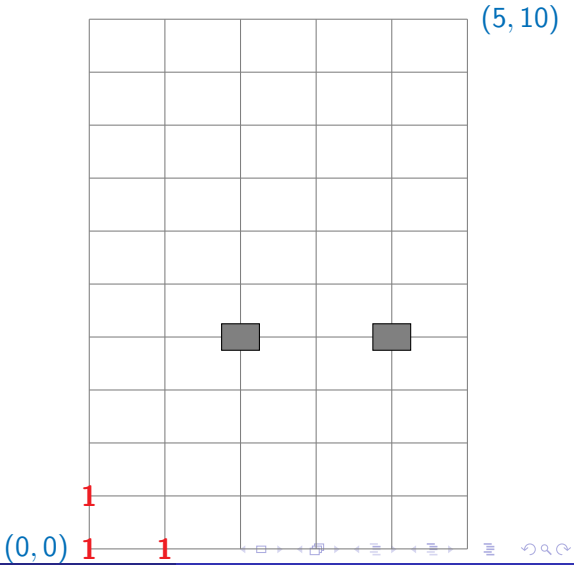

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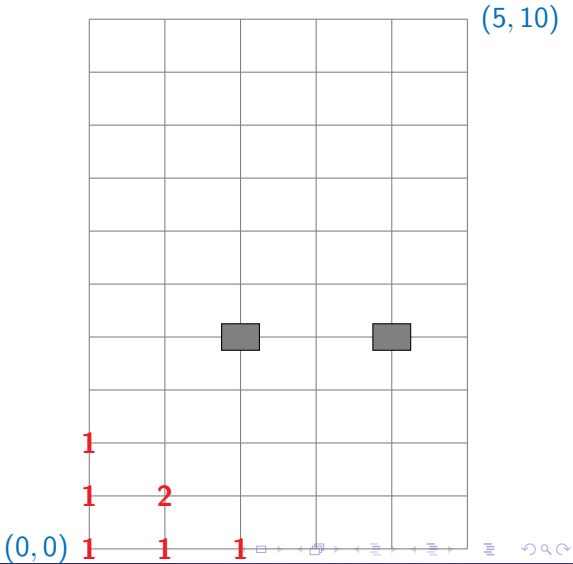

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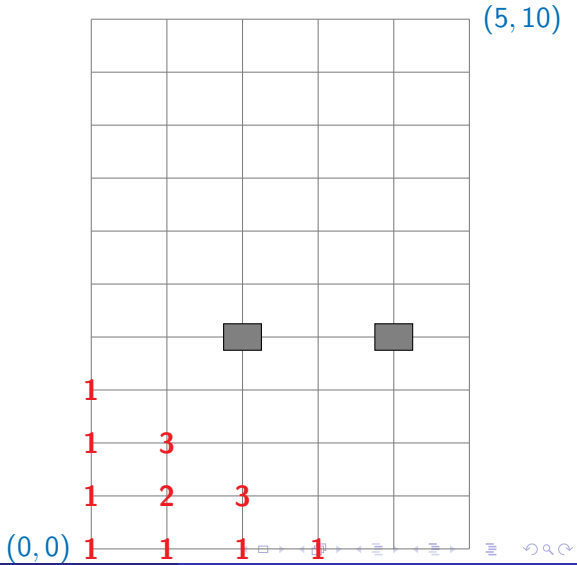
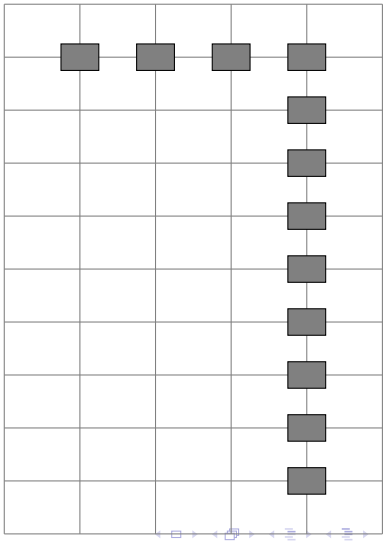


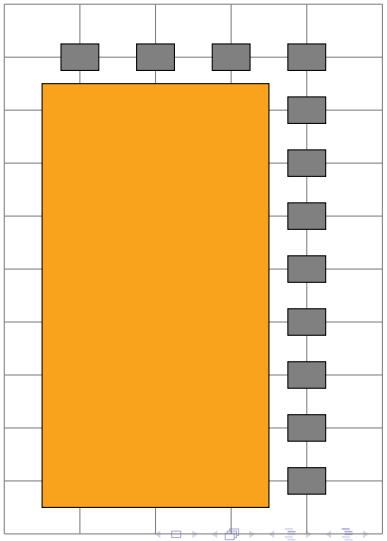
Memoization vs dynamic programming

- Barrier of holes just inside the border



Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region



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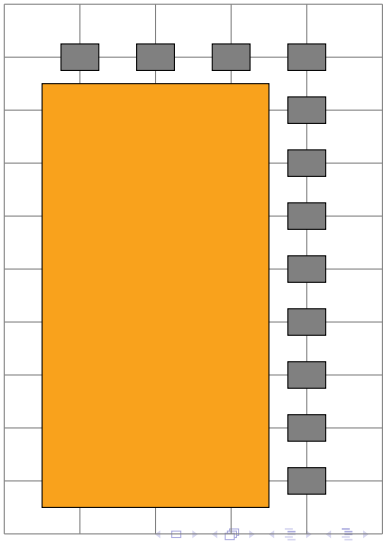

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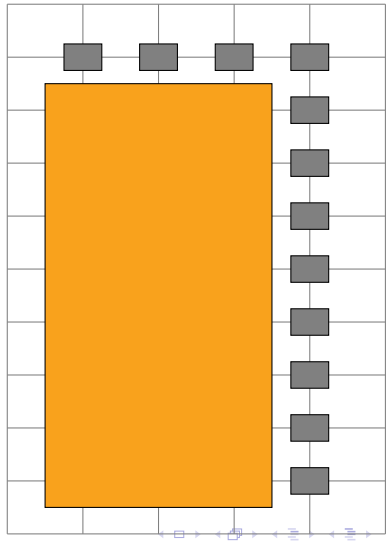

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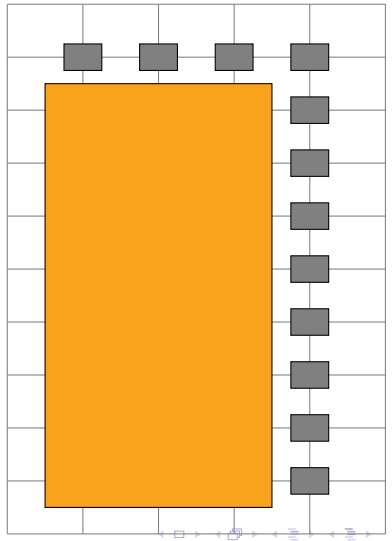
- Dynamic programming blindly fills all mn cells of the table
- Tradeoff between recursion and iteration
  - "Wasteful" dynamic programming still better in general



Longest common subword

- Given two strings, find the (length of the) longest common subword
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sec"`, length 3
  - `"director"`, `"secretary"` — `"ec"`, `"re"`, length 2

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- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$
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  - Find the largest such k — length of the longest common subword

Brute force

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \ldots$ as far as possible
  - Keep track of longest match

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- Assuming $m > n$, this is $O(mn^2)$
  - mn pairs of starting positions
  - From each starting position, scan could be $O(n)$

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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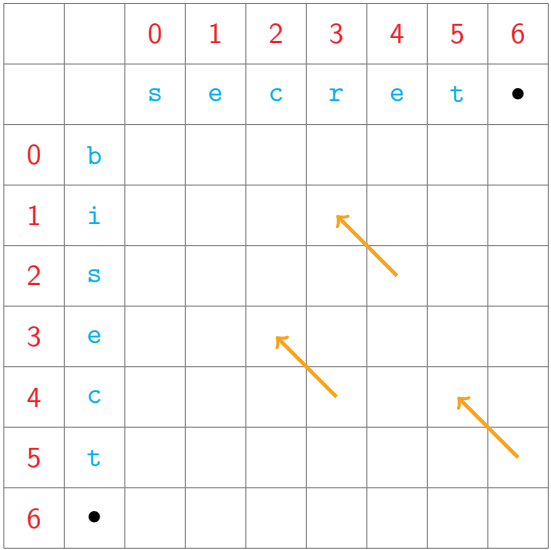

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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					**0**	**0**	**0**
1	i					**0**	**0**	**0**
2	s					**0**	**0**	**0**
3	e					**1**	**0**	**0**
4	c					**0**	**0**	**0**
5	t					**0**	**1**	**0**
6	•					**0**	**0**	**0**

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2	s				**0**	**0**	**0**	**0**
3	e				**0**	**1**	**0**	**0**
4	c				**0**	**0**	**0**	**0**
5	t				**0**	**0**	**1**	**0**
6	•				**0**	**0**	**0**	**0**

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3	e			**0**	**0**	**1**	**0**	**0**
4	c			**1**	**0**	**0**	**0**	**0**
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3	e		**2**	**0**	**0**	**1**	**0**	**0**
4	c		**0**	**1**	**0**	**0**	**0**	**0**
5	t		**0**	**0**	**0**	**0**	**1**	**0**
6	•		**0**	**0**	**0**	**0**	**0**	**0**

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1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

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- Find entry (i, j) with largest LCW value

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		s	e	c	r	e	t	•
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1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

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- Find entry (i, j) with largest LCW value
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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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4	c	0	0	1	0	0	0	0
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6	•	0	0	0	0	0	0	0

Implementation

```
def LCW(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcw = np.zeros((m+1,n+1))

  maxlcw = 0

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
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Complexity

- Recall that brute force was $O(mn^2)$

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- Recall that brute force was $O(mn^2)$
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Complexity

- Recall that brute force was $O(mn^2)$
- Inductive solution is $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Longest common subsequence

- Subsequence — can drop some letters in between
- Given two strings, find the (length of the) longest common subsequence
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sect"`, length 4
  - `"director"`, `"secretary"` — `"ectr"`, `"retr"`, length 4

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  - "bisect", "secret" — "sect", length 4
  - "director", "secretary" — "ectr", "retr", length 4
- LCS is the longest path connecting non-zero LCW entries, moving right/down

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

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1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
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4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Applications

- Analyzing genes
  - DNA is a long string over A, T, G, C
  - Two species are similar if their DNA has long common subsequences

		0	1	2	3	4	5	6
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0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Applications

- Analyzing genes
  - DNA is a long string over `A`, `T`, `G`, `C`
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- `diff` command in Unix/Linux
  - Compares text files
  - Find the longest matching subsequence of lines
  - Each line of text is a "character"

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	1	0	0	0	0	0	0
3	e	0	1	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
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Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$

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  - Which one should we drop?
  - Solve $LCS(i, j+1)$ and $LCS(i+1, j)$ and take the maximum
- Base cases as with LCW
  - $LCS(i, n) = 0$ for all $0 \leq i \leq m$
  - $LCS(m, j) = 0$ for all $0 \leq j \leq n$

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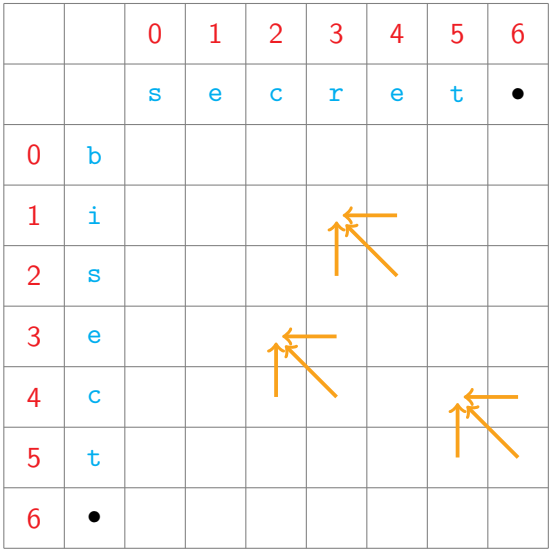

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		s	e	c	r	e	t	•
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1	i							**0**
2	s							**0**
3	e							**0**
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6	•							**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		**3**	**2**	**1**	**1**	**0**	**0**
1	i		**3**	**2**	**1**	**1**	**0**	**0**
2	s		**3**	**2**	**1**	**1**	**0**	**0**
3	e		**3**	**2**	**1**	**1**	**0**	**0**
4	c		**2**	**2**	**1**	**1**	**0**	**0**
5	t		**1**	**1**	**1**	**1**	**1**	**0**
6	•		**0**	**0**	**0**	**0**	**0**	**0**

Subproblem dependency

- Subproblems are $LCS(i, j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$
- Table of $(m + 1) \cdot (n + 1)$ values
- $LCS(i, j)$ depends on $LCS(i+1, j+1)$, $LCS(i, j+1)$,$LCS(i+1, j)$,
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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**4**	**3**	**2**	**1**	**1**	**0**	**0**
1	i	**4**	**3**	**2**	**1**	**1**	**0**	**0**
2	s	**4**	**3**	**2**	**1**	**1**	**0**	**0**
3	e	**3**	**3**	**2**	**1**	**1**	**0**	**0**
4	c	**2**	**2**	**2**	**1**	**1**	**0**	**0**
5	t	**1**	**1**	**1**	**1**	**1**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

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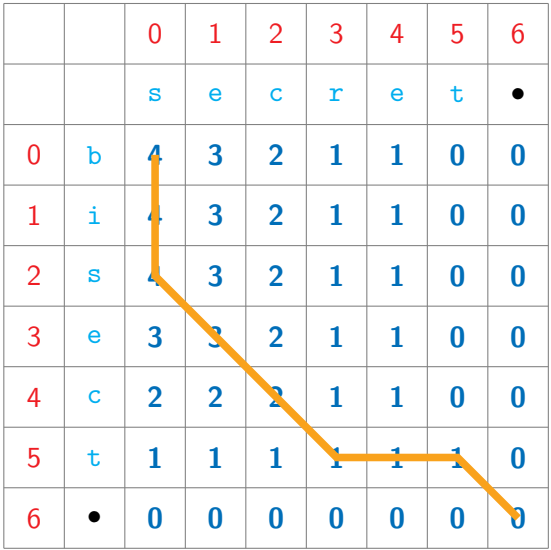
Reading off the solution

- Trace back the path by which each entry was filled



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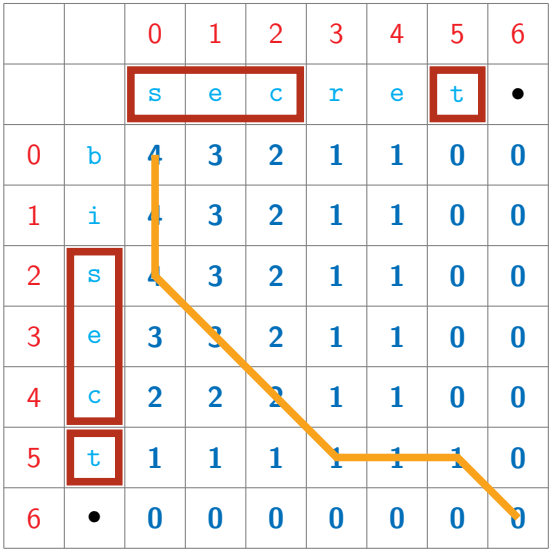
Reading off the solution

- Trace back the path by which each entry was filled
- Each diagonal step is an element of LCS



Implementation

```
def LCS(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcs = np.zeros((m+1,n+1))

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        lcs[i,j] = 1 + lcs[i+1,j+1]
      else:
        lcs[i,j] = max(lcs[i+1,j],
                       lcs[i,j+1])
  return(lcs[0,0])
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Complexity

- Again $O(mn)$, using dynamic programming or memoization

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Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute