Stacks, Queues, Priority Queues, Heaps

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Stacks

- Stack is a last-in, first-out list
  - `push(s,x)` — add `x` to stack `s`
  - `pop(s)` — return most recently added element
- Maintain stack as list, push and pop from the right
  - `push(s,x)` is `s.append(x)`
  - `s.pop()` — Python built-in, returns last element

Stacks

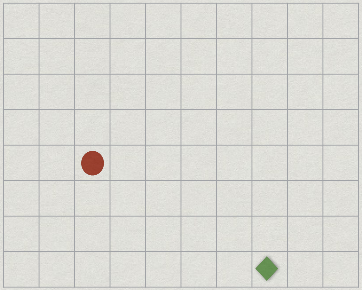
- Stacks are natural to keep track of recursive function calls
- In 8 queens, use a stack to keep track of queens added
  - Push the latest queen onto the stack
  - To backtrack, pop the last queen added

Queues

- First-in, first-out sequences
  - `addq(q,x)` — adds `x` to rear of queue `q`
  - `removeq(q)` — removes element at head of `q`
- Using Python lists, left is rear, right is front
  - `addq(q,x)` is `q.insert(0,x)`
    - `l.insert(j,x)`, insert `x` before position `j`
  - `removeq(q)` is `q.pop()`

Systematic exploration

- Rectangular m x n grid
- Chess knight starts at (sx,sy)
  - Usual knight moves
- Can it reach a target square (tx,ty)?



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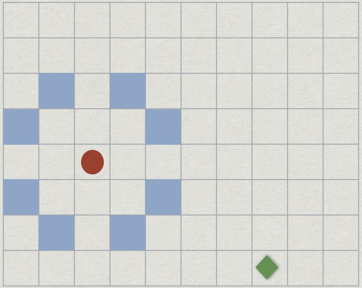

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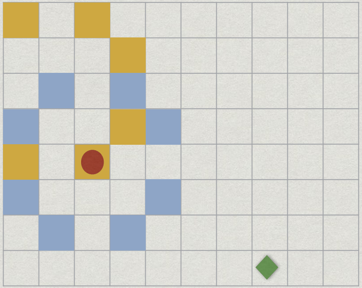

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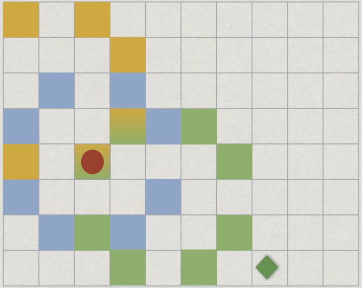

Systematic exploration

- X1 — all squares reachable in one move from (sx,sy)
- X2 — all squares reachable from X1 in one move
- . . .
- Don’t explore an already marked square
- When do we stop?
  - If we reach target square
  - What if target is not reachable?

Systematic exploration

- Maintain a queue Q of cells to be explored
- Initially Q contains only start node (sx,sy)
  - Remove (ax,ay) from head of queue
  - Mark all squares reachable in one step from (ax,ay)
  - Add all newly marked squares to the queue
- When the queue is empty, we have finished

Dealing with priorities

Job scheduler

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Priority queue

- Need to maintain a collection of items with priorities to optimise the following operations
- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the collection

Implementing priority queues with one dimensional structures

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Implementing priority queues with one dimensional structures

- Unsorted list
  - `insert()` is $O(1)$
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Implementing priority queues with one dimensional structures

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  - `insert()` is $O(1)$
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- Sorted list
  - `delete_max()` is $O(1)$
  - `insert()` is $O(n)$
- Processing n items requires $O(n^2)$

- `delete_max()`
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  - Need not be unique
- `insert()`
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Moving to two dimensions

First attempt

- Assume N processes enter/leave the queue

$N = 25$

3	19	23	35	58
12	17	25	43	67
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- Assume N processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array

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Moving to two dimensions

First attempt

- Assume N processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array
- Each row is in sorted order

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insert()

- Keep track of the size of each row

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insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine

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- Insert 15

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insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine
- Insert 15
- Takes time $O(\sqrt{N})$
  - Scan size column to locate row to insert, $O(\sqrt{N})$
  - Insert into the first row with free space, $O(\sqrt{N})$

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- Maximum in each row is the last element

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delete_max()

- Maximum in each row is the last element
- Position is available through size column

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- Maximum in each row is the last element
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- Identify the maximum amongst these

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- Delete it

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delete_max()

- Maximum in each row is the last element
- Position is available through size column
- Identify the maximum amongst these
- Delete it
- Again $O(\sqrt{N})$
  - Find the maximum among last entries, $O(\sqrt{N})$
  - Delete it, $O(1)$

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Summary

- 2D $\sqrt{N} \times \sqrt{N}$ array with sorted rows
  - `insert()` is $O(\sqrt{N})$
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- Flexible — need not fix N in advance

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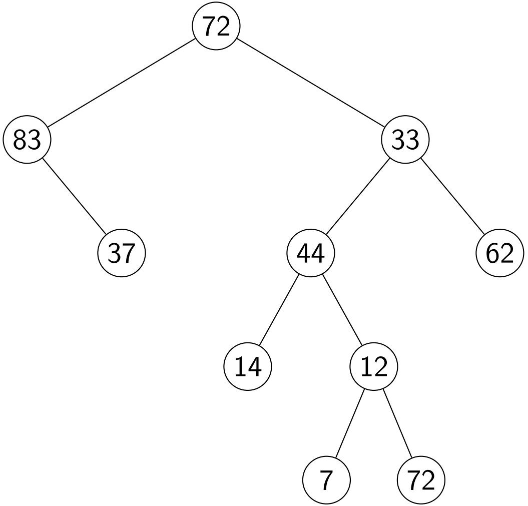
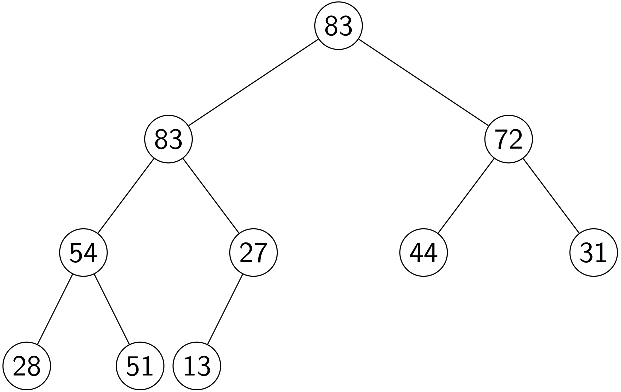
Binary trees

- Values are stored as nodes in a rooted tree
- Each node has up to two children
  - Left child and right child
  - Order is important
- Other than the root, each node has a unique parent
- Leaf node — no children
- Size — number of nodes
- Height — number of levels



Heap

- Binary tree filled level by level, left to right
- The value at each node is at least as big the values of its children
  - max-heap



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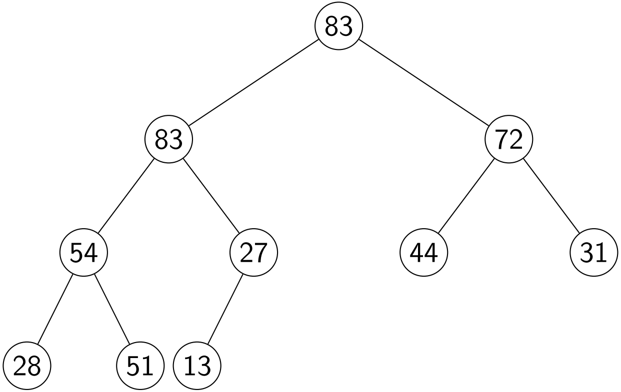
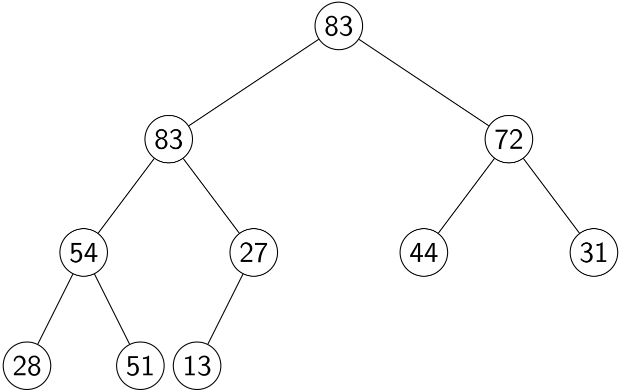


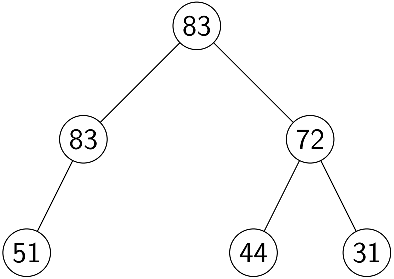
Heap

- Binary tree filled level by level, left to right
- The value at each node is at least as big the values of its children
  - max-heap
- Binary tree on the right is an example of a heap
- Root always has the largest value
  - By induction, because of the max-heap property



Non-examples

No “holes” allowed



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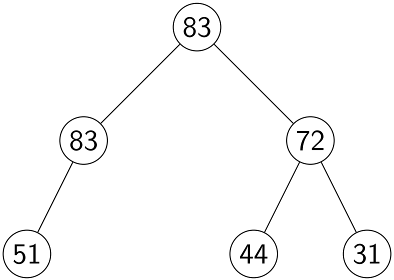
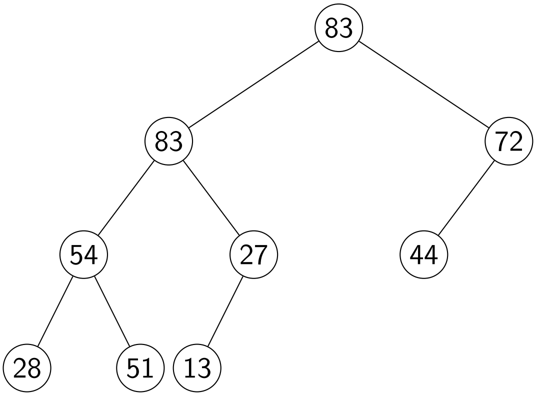


Cannot leave a level incomplete



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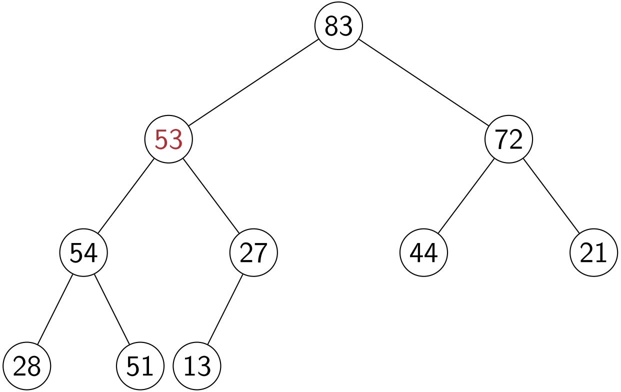
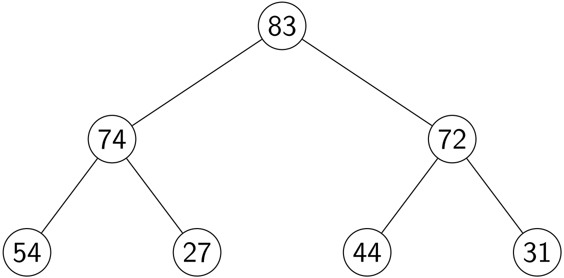
Heap property is violated



insert()

- `insert(77)`



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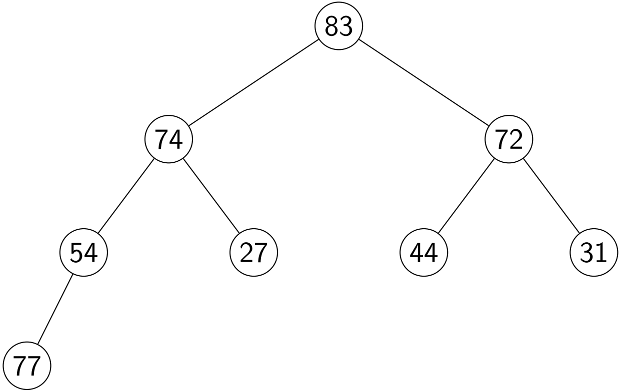
- insert(77)
- Add a new node at dictated by heap structure



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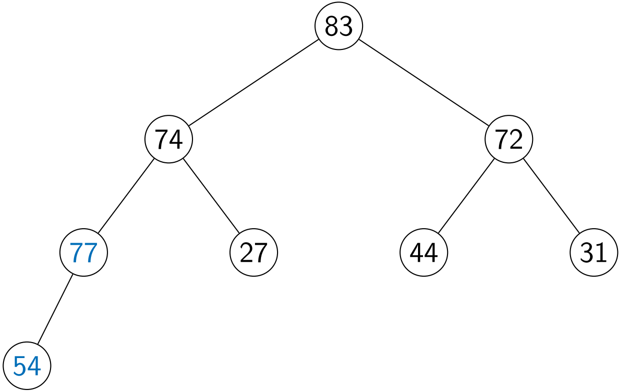
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root



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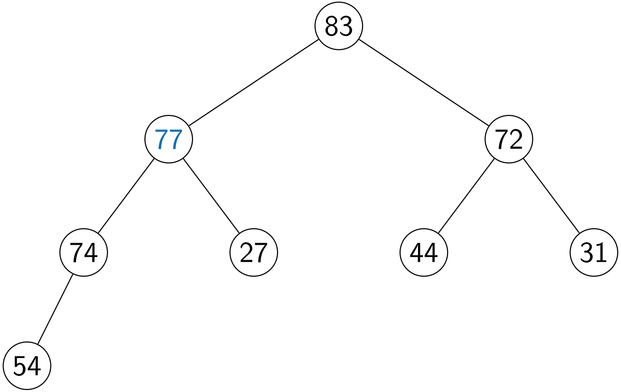

insert()

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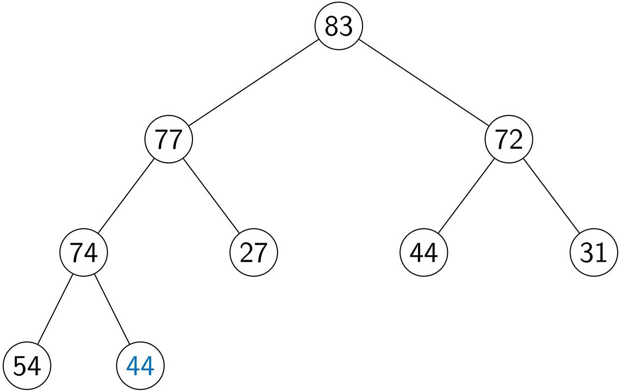

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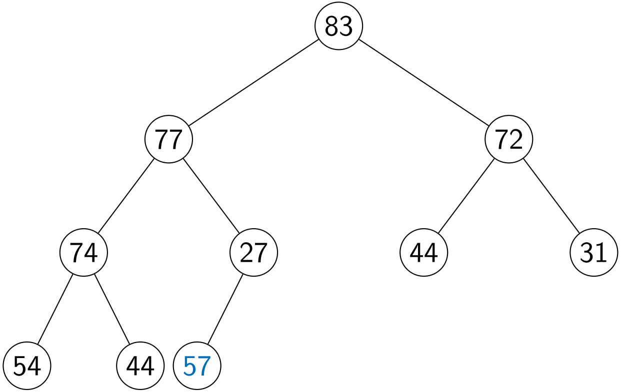

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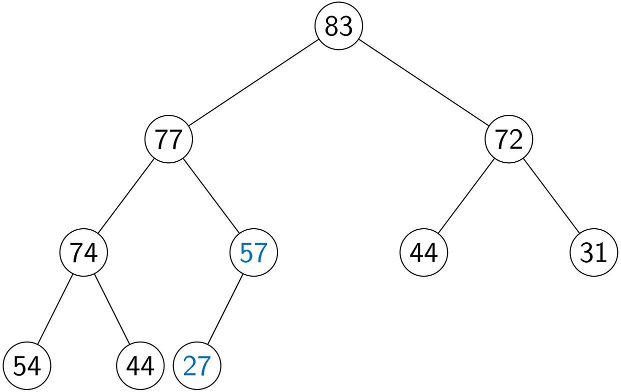
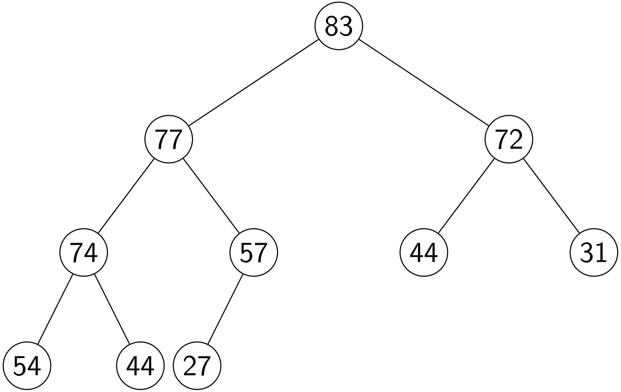


Complexity of `insert()`

- Need to walk up from the leaf to the root
  - Height of the tree



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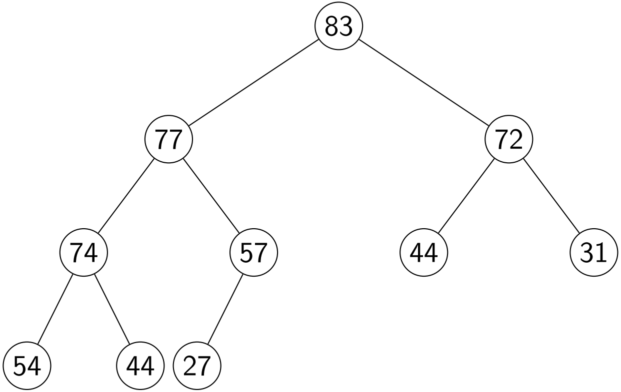
- Number of nodes at level 0 is $2^0 = 1$



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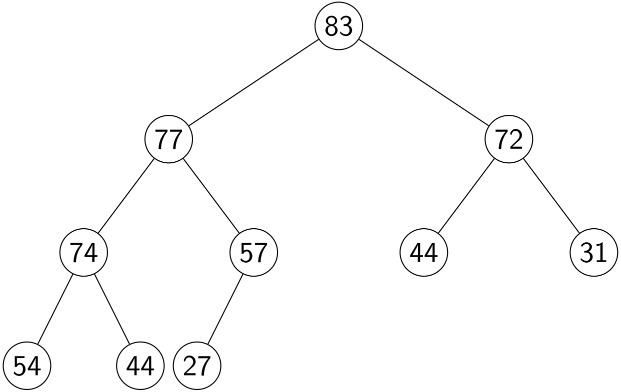

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- Number of nodes at level 0 is $2^0 = 1$
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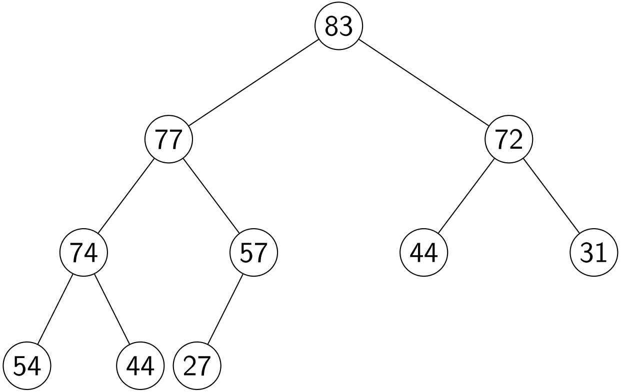
- If we fill k levels, $2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$ nodes



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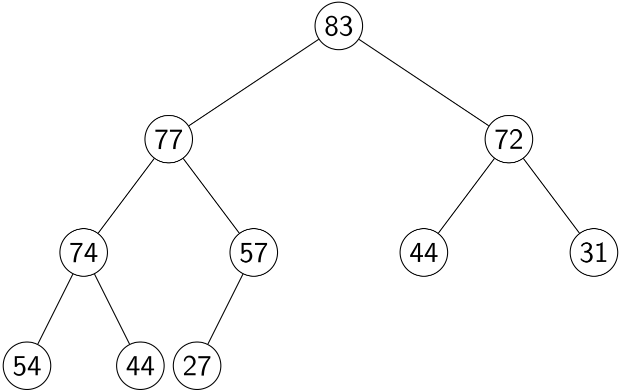
- If we have N nodes, at most $1 + \log N$ levels



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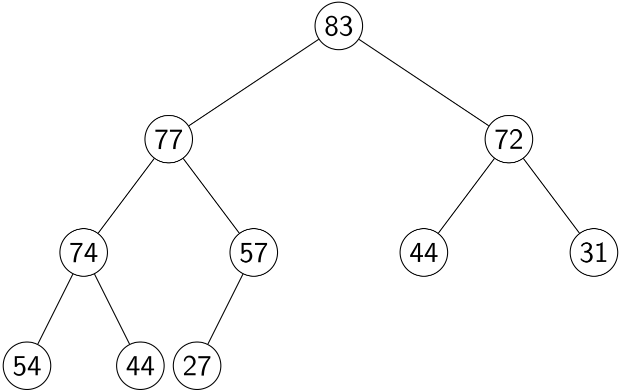
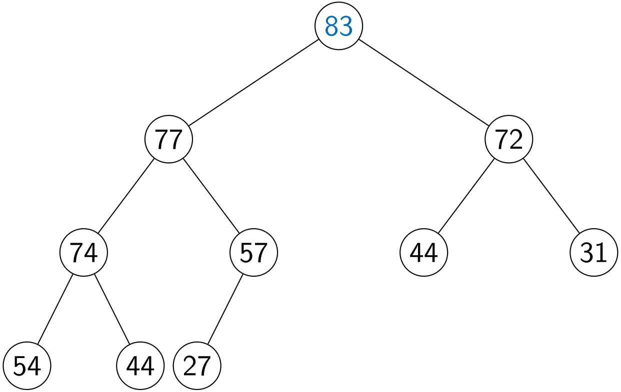
- If we have N nodes, at most $1 + \log N$ levels
- `insert()` is $O(\log N)$



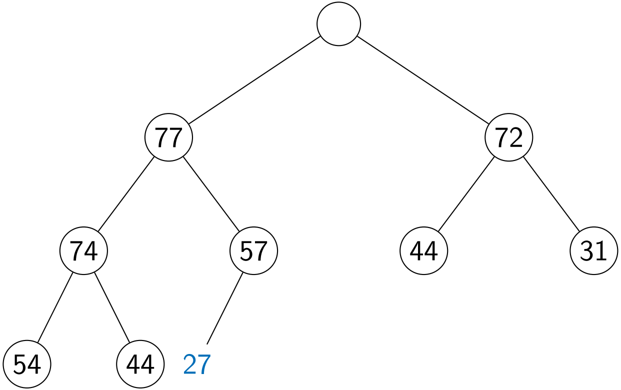
delete_max()

- Maximum value is always at the root



delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level



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- Maximum value is always at the root
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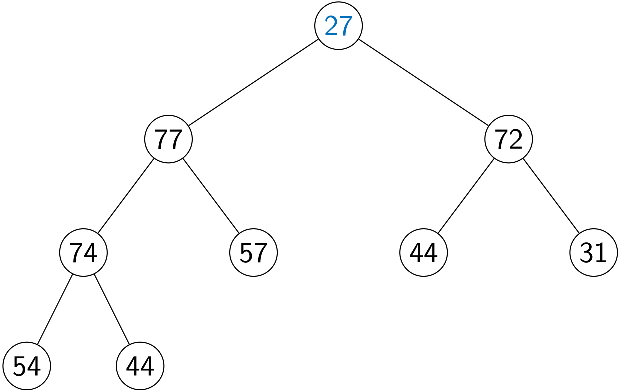
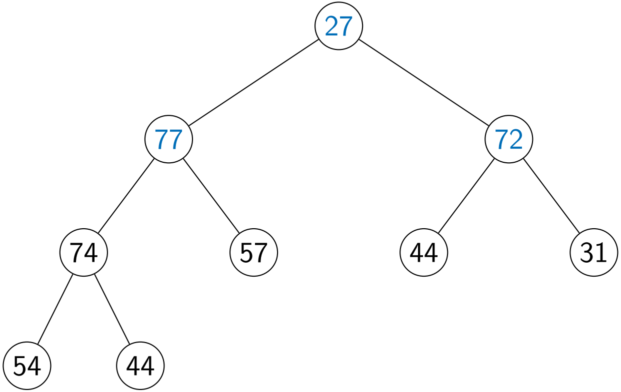
- Move "homeless" value to the root



delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level
- Move "homeless" value to the root
- Restore the heap property downwards



delete_max()

- Maximum value is always at the root
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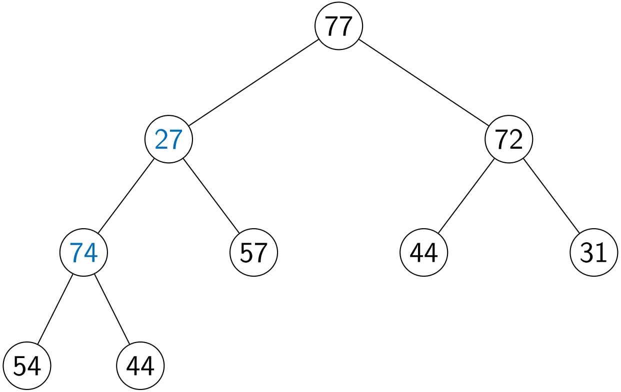

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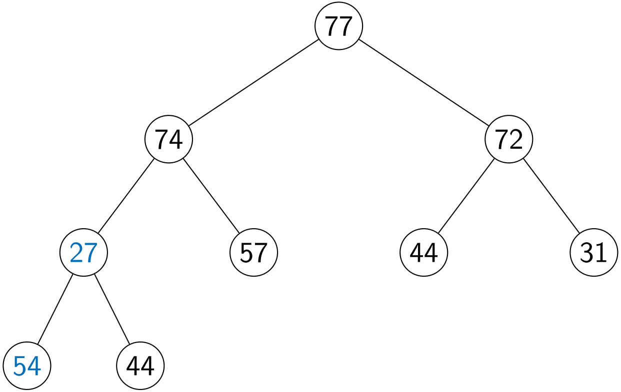

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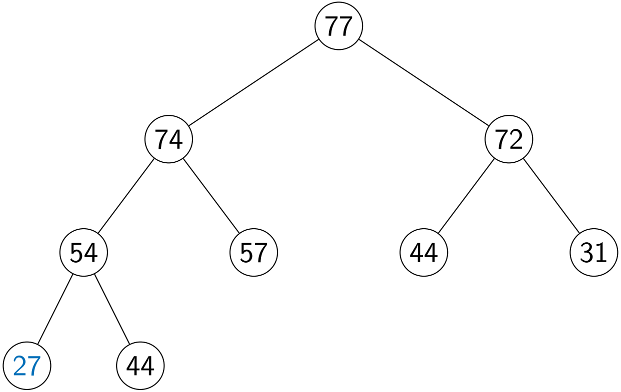
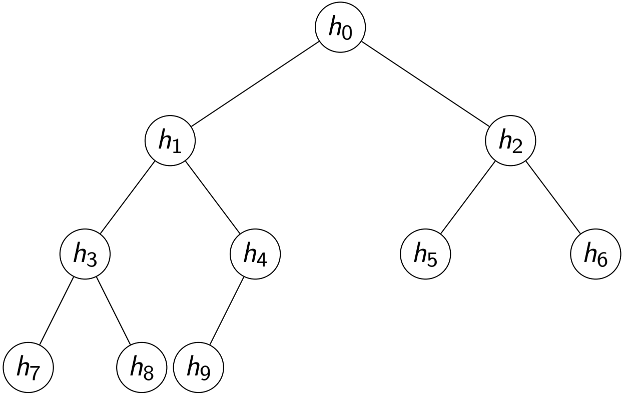


Implementation

- Number the nodes top to bottom left right
- Store as a list `H = [h0,h1,h2,...,h9]`
- Children of `H[i]` are at `H[2*i+1]`, `H[2*i+2]`
- Parent of `H[i]` is at `H[(i-1)//2]`, for `i > 0`



Building a heap — heapify()

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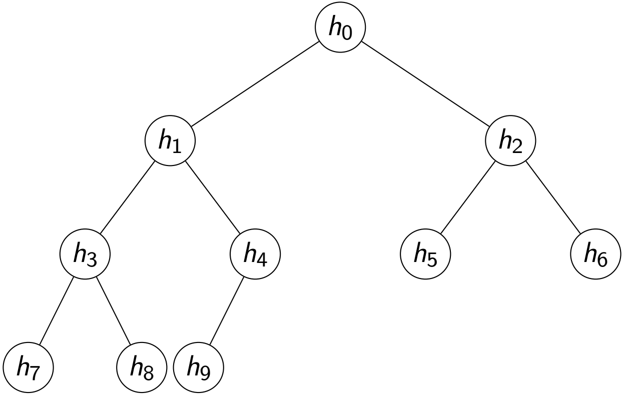
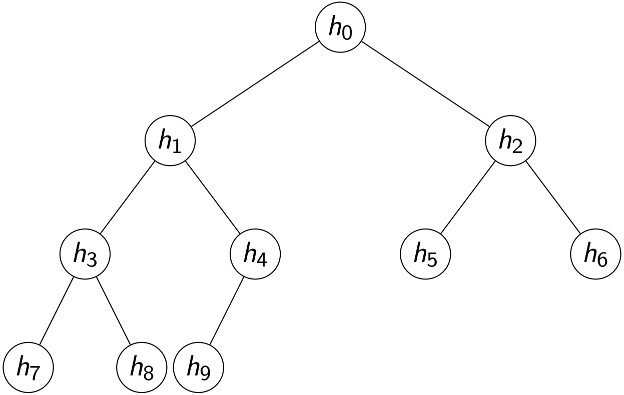


Building a heap — `heapify()`

- Convert a list `[v0,v1,...,vN]` into a heap
- Simple strategy
  - Start with an empty heap
  - Repeatedly apply `insert(vj)`
  - Total time is $O(N \log N)$



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- Cost turns out to be $O(n)$

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- Call `delete_max()` n times to extract elements in descending order — $O(n \log n)$
- After each `delete_max()`, heap shrinks by 1
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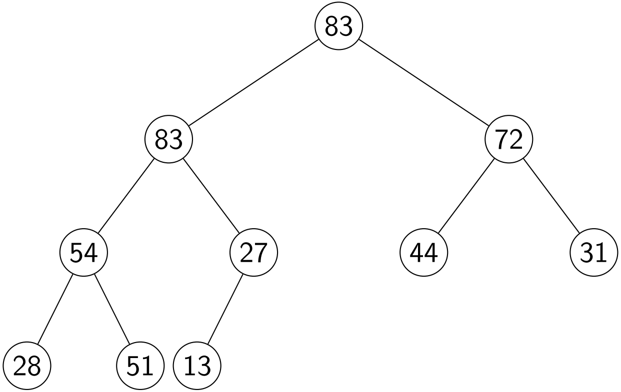
- In place $O(n \log n)$ sort

Summary

- Heaps are a tree implementation of priority queues
  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$



Summary

- Heaps are a tree implementation of priority queues
  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$
- Can invert the heap condition
  - Each node is smaller than its children
  - min-heap
  - `delete_min()` rather than `delete_max()`