

Analysis of Merge Sort

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Programming and Data Structures with Python

Lecture 16, 18 Nov 2021

Merge sort

- To sort A into B , both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
 - Sort $A[:n//2]$ into L
 - Sort $A[n//2:]$ into R
 - Merge L and R into B

Merging two sorted lists A and B into C

- If A is empty, copy B into C
- If B is empty, copy A into C
- Otherwise, compare first elements of A and B
 - Move the smaller of the two to C
- Repeat till all elements of A and B have been moved

Analysing merge

- Merge A of length m , B of length n

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def merge(A,B):
    (m,n) = (len(A),len(B))
    (C,i,j,k) = ([],0,0,0)
    while k < m+n:
        if i == m:
            C.extend(B[j:])
            k = k + (n-j)
        elif j == n:
            C.extend(A[i:])
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        elif A[i] < B[j]:
            C.append(A[i])
            (i,k) = (i+1,k+1)
        else:
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- If $m \approx n$, `merge` take time $O(n)$

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 - For simplicity, assume $n = 2^k$ for some k

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def mergesort(A):  
    n = len(A)  
  
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- Unwind the recurrence to solve

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 - Consider an input of the form [0,2,4,6,1,3,5,9]
- Can we divide the list so that everything on the left is smaller than everything on the right?
 - No need to merge!

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- How do we find the median?
 - Sort and pick up the middle element
 - But our aim is to sort the list!
 - Instead pick some value in L — **pivot**
 - Split L with respect to the pivot element

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- Input list

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 - If it is larger than the pivot, extend **Upper** to include this element
 - If it is less than or equal to the pivot, exchange with the first element in **Upper**. This extends **Lower** and shifts **Upper** by one position.

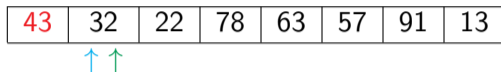
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43	32	22	78	63	57	91	13
----	----	----	----	----	----	----	----

Partitioning

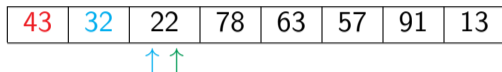
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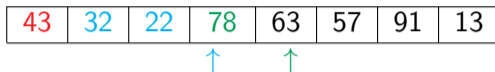
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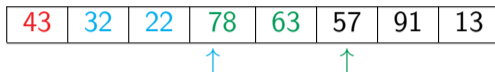
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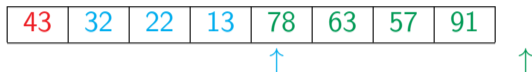
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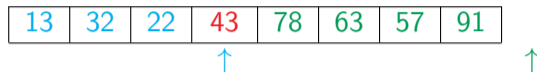
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- After partitioning, exchange the pivot with the last element of the **Lower** segment

Quicksort code

- Scan the list from left to right
- Four segments: **Pivot**, **Lower**, **Upper**, Unclassified
- Classify the first unclassified element
 - If it is larger than the pivot, extend **Upper** to include this element
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def quicksort(L,l,r): # Sort L[l:r]
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        return(L)
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    for i in range(l+1,r):
        if L[i] > pivot: # Extend upper segment
            upper = upper+1
        else: # Exchange L[i] with start of upper segment
            (L[i], L[lower]) = (L[lower], L[i])
            # Shift both segments
            (lower,upper) = (lower+1,upper+1)
    # Move pivot between lower and upper
    (L[l],L[lower-1]) = (L[lower-1],L[l])
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    # Recursive calls
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- Expected running time is again $O(n \log n)$

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- Very often the default algorithm used for in-built sort functions
 - Sorting a column in a spreadsheet
 - Library sort function in a programming language

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Summary

- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$
- Randomly choosing the pivot is a good strategy to beat worst case inputs
- Quicksort works in-place and can be implemented iteratively
- Very fast in practice, and often used for built-in sorting functions
 - Good example of a situation when the worst case upper bound is pessimistic

Sorting: Concluding Remarks

Madhavan Mukund

<https://www.cmi.ac.in/~madhavan>

Programming and Data Structures with Python

Lecture 16, 18 Nov 2021

Stable sorting

- Often list values are tuples
 - Rows from a table, with multiple columns / attributes
 - A list of students, each student entry has a roll number, name, marks, ...

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- Sorting on column B should not disturb sorting on column A

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- The quicksort implementation we described is not stable
 - Swapping values while partitioning can disturb existing sorted order
- Merge sort is stable if we merge carefully
 - Do not allow elements from the right to overtake elements on the left
 - While merging, prefer the left list while breaking ties

- Minimizing data movement
 - Imagine each element is a heavy carton
 - Reduce the effort of moving values around

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 - Database tables that are too large to store in memory all at once
 - Retrieve in parts from the disk and write back
- Other $O(n \log n)$ algorithms exist — heapsort
- Sometimes hybrid strategies are used
 - Use divide and conquer for large n
 - Switch to insertion sort when n becomes small (e.g., $n < 16$)