Analysis of Merge Sort

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Programming and Data Structures with Python

Lecture 16, 18 Nov 2021

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`
  - Sort `A[n//2:]` into `R`
  - Merge `L` and `R` into `B`

Merging two sorted lists `A` and `B` into `C`

- If `A` is empty, copy `B` into `C`
- If `B` is empty, copy `A` into `C`
- Otherwise, compare first elements of `A` and `B`
  - Move the smaller of the two to `C`
- Repeat till all elements of `A` and `B` have been moved

Analysing merge

- Merge `A` of length m, `B` of length n

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def merge(A,B):
  (m,n) = (len(A),len(B))
  (C,i,j,k) = ([],0,0,0)
  while k < m+n:
    if i == m:
      C.extend(B[j:])
      k = k + (n-j)
    elif j == n:
      C.extend(A[i:])
      k = k + (n-i)
    elif A[i] < B[j]:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    else:
      C.append(B[j])
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- If $m \approx n$, merge take time $O(n)$

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Analysing mergesort

- Let $T(n)$ be the time taken for input of size n
  - For simplicity, assume $n = 2^k$ for some k

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def mergesort(A):
  n = len(A)

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  - $T(0) = T(1) = 1$
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- Unwind the recurrence to solve

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Quicksort

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  - Consider an input of the form `[0,2,4,6,1,3,5,9]`
- Can we divide the list so that everything on the left is smaller than everything on the right?
  - No need to merge!

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  - Sort and pick up the middle element
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- Instead pick some value in `L` — pivot
  - Split `L` with respect to the pivot element

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High level view of quicksort

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- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

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- After partitioning, exchange the pivot with the last element of the Lower segment

Quicksort code

- Scan the list from left to right
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- Classify the first unclassified element
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  - Partitions are of size 0, $n-1$
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- Expected running time is again $O(n \log n)$

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  - Recursive calls — disjoint segments, no recombination of results required
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  - Explicitly track endpoints of each segment to be sorted
- In practice, quicksort is very fast
- Very often the default algorithm used for in-built sort functions
  - Sorting a column in a spreadsheet
  - Library sort function in a programming language

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Summary

- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$
- Randomly choosing the pivot is a good strategy to beat worst case inputs
- Quicksort works in-place and can be implemented iteratively
- Very fast in practice, and often used for built-in sorting functions
  - Good example of a situation when the worst case upper bound is pessimistic

Sorting: Concluding Remarks

Madhavan Mukund

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Programming and Data Structures with Python

Lecture 16, 18 Nov 2021

Stable sorting

- Often list values are tuples
  - Rows from a table, with multiple columns / attributes
  - A list of students, each student entry has a roll number, name, marks, ...

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  - Swapping values while partitioning can disturb existing sorted order
- Merge sort is stable if we merge carefully
  - Do not allow elements from the right to overtake elements on the left
  - While merging, prefer the left list while breaking ties

Other criteria

- Minimizing data movement
  - Imagine each element is a heavy carton
  - Reduce the effort of moving values around

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- Merge sort is typically used for "external" sorting
  - Database tables that are too large to store in memory all at once
  - Retrieve in parts from the disk and write back
- Other $O(n \log n)$ algorithms exist — heapsort
- Sometimes hybrid strategies are used
  - Use divide and conquer for large n
  - Switch to insertion sort when n becomes small (e.g., $n < 16$)