Searching in a List

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Programming and Data Structures with Python

Lecture 15, 15 Nov 2021

Cost of brute force nested loops

How to quantify efficiency

- Asymptotic behaviour - $t(n)$ as n grows large
  ↑ input size

- Worst case (vs average)

- $O(\,)$ notation.

$$f(n) = O(g(n))$$

↑ Upper bound

$\exists c.\ \forall n > n_0$

$$f(n) \leq c\,g(n)$$

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def naivesearch(v,l):
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    if v == x:
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  return(False)
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- Is value `v` present in list `l`?
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- Input size n, the length of the list
- Worst case is when `v` is not present in `l`
- Worst case complexity is $O(n)$

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- Compare `v` with the midpoint of `l`

Searching a sorted list

- What if `l` is sorted in ascending order?
- Compare `v` with the midpoint of `l`
  - If midpoint is `v`, the value is found
  - If `v` less than midpoint, search the first half
  - If `v` greater than midpoint, search the second half
  - Stop when the interval to search becomes empty

```
def binarysearch(v,l):
  if l == []:
    return(False)

  m = len(l)//2

  if v == l[m]:
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  if v < l[m]:
    return(binarysearch(v,l[:m]))
  else:
    return(binarysearch(v,l[m+1:]))
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Searching a sorted list

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- Compare v with the midpoint of l
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Binary search

- How long does this take?
  - Each call halves the interval to search
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- $\log n$ — number of times to divide n by 2 to reach 1
  - $1 // 2 = 0$, so next call reaches empty interval

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len(l)=1
m=0
v == l[0]

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  - $1 // 2 = 0$, so next call reaches empty interval
- $O(\log n)$ steps

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Alternative calculation

- $T(n)$: the time to search a list of length n
  - If $n = 0$, we exit, so $T(n) = 1$
  - If $n > 0$, $T(n) = T(n // 2) + 1$

Any constant c is O(1)

c ≤ c · 1

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  $T(n//2^3) + 3$

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- For a sorted list, binary search takes time $O(\log n)$
  - Halve the interval to search each time
- In a sorted list, we can determine that v is absent by examining just $\log n$ values!

n	log n
10	3. . .
100	6. . .
1000	10
10^6	20
10^9	30

≈ 1000 values ≈ 2^{10} = 1024

10 probes tell you v is absent!

Selection Sort

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Sorting a list

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  - Binary search
  - Finding the median
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Sorting a list

74 32 89 55 21 64

Sorting a list

74 32 89 55 ~~21~~ 64

21

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Sorting a list

74	~~32~~	89	~~55~~	~~21~~	64

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Selection sort

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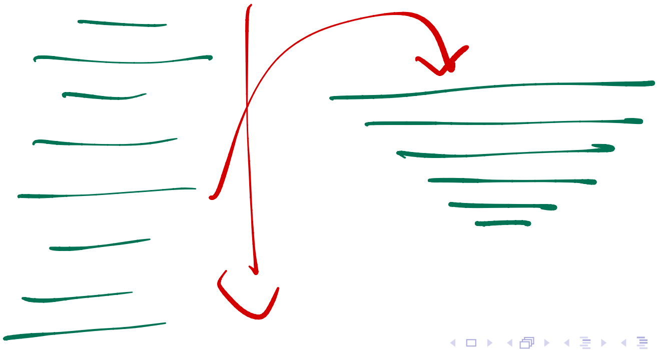

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  - Swap the minimum element into the first position
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  - ...

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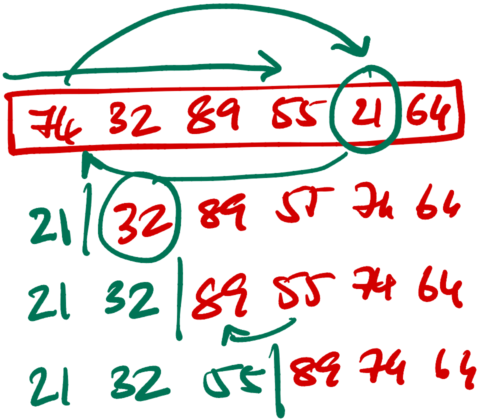

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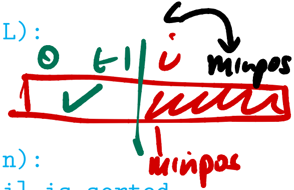


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  if n < 1:
    return(L)
  for i in range(n):
    # Assume L[:i] is sorted
    mpos = i
    # mpos: position of minimum in L[:i]
    for j in range(i+1,n):
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Analysis of selection sort

- Correctness follows from the invariant

relationship between values that holds at the start of every iteration

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- $T(n)$ is $O(n^2)$

$\frac{n^2+n}{2}$

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- Selection sort is an intuitive algorithm to sort a list
- Repeatedly find the minimum (or maximum) and append to sorted list
- Worst case complexity is $O(n^2)$ Not good!
  - Every input takes this much time
  - No advantage even if list is arranged carefully before sorting

SIM-Aadhaar

n^2 sort 300 years!

$n \log n$ search

Insertion Sort

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  - Your task is to arrange the papers in descending order of marks

Sorting a list

- You are the TA for a course
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Strategy 2

Sorting a list

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- Move the first paper to a new pile

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- Third paper
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- Do this for the remaining papers
  - Insert each one into correct position in the second pile

Sorting a list

74 32 89 55 21 64

Sorting a list

~~74~~ 32 89 55 21 64

74

Sorting a list

~~74~~	~~32~~	89	55	21	64

32	74

Sorting a list

~~74~~ ~~32~~ ~~89~~ 55 21 64

32 74 89

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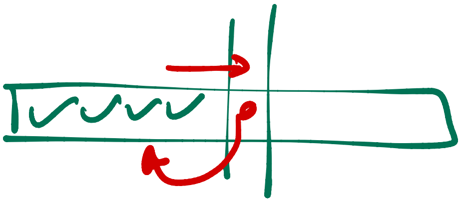

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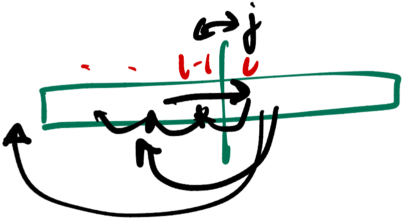


```
def InsertionSort(L):
  n = len(L)
  if n < 1:
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  for i in range(n):
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[L[-1]]

Analysis of iterative insertion sort

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- For input of size n, let
  - $TI(n)$ be the time taken by `Insert`
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  n = len(L)
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Analysis of recursive insertion sort

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  - Unwind to get $TI(n) = n$

$$TI(n) = TI(n-1) + 1$$
$$= TI(n-2) + 1 + 1$$
$$\vdots$$
$$= TI(0) + \underbrace{1 + 1 + 1 \cdots 1}_{n}$$

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$TS(n) = TS(n-1) + n-1$

$= TS(n-2) + n-2 + n-1$

Summary

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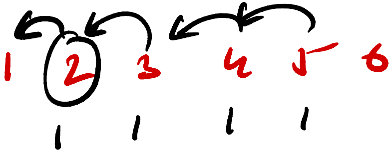
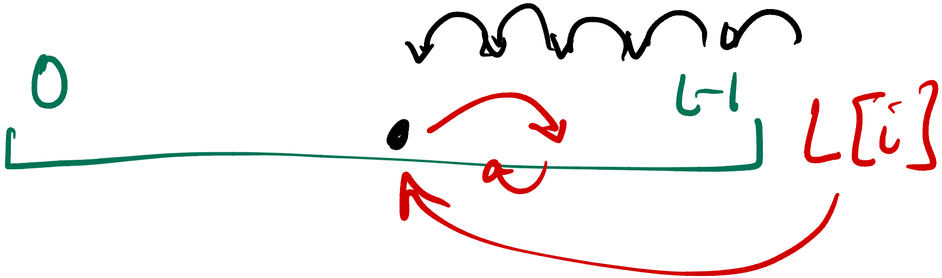
- Insertion sort is another intuitive algorithm to sort a list
- Create a new sorted list
- Repeatedly insert elements into the sorted list
- Worst case complexity is $O(n^2)$
  - Unlike selection sort, not all cases take time n^2
  - If list is already sorted, `Insert` stops in 1 step
  - Overall time can be close to $O(n)$



Also Use binary search to insert



Suppose in $\log(i)$ steps we find $L[i] < L[0]$

- Move $L[i]$ to $L[0]$
- Push each $L[j]$ to $L[j+1]$

} Takes $O(i)$ time

Merge Sort

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming and Data Structures with Python

Lecture 15, 15 Nov 2021

Beating the $O(n^2)$ barrier

- Both selection sort and insertion sort take time $O(n^2)$
- This is infeasible for $n > 10000$

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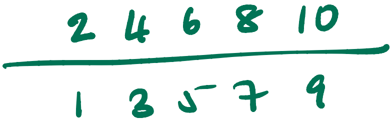
Strategy 3

- Divide the list into two halves
- Separately sort the left and right half
- Combine the two sorted halves to get a fully sorted list



Combining two sorted lists

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32	74	89
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32 74 89

~~21~~ 55 64

21

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Merge sort

- Let n be the length of L

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- Let `n` be the length of L
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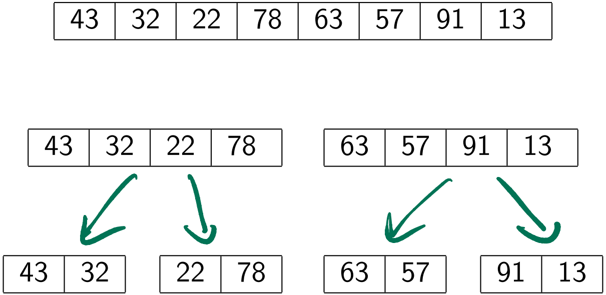

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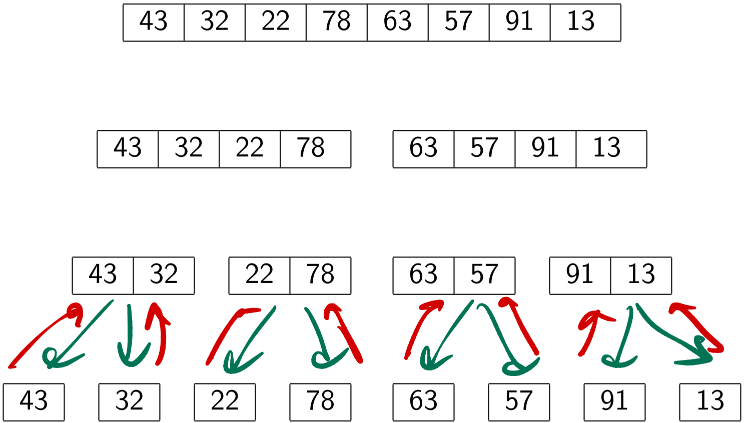

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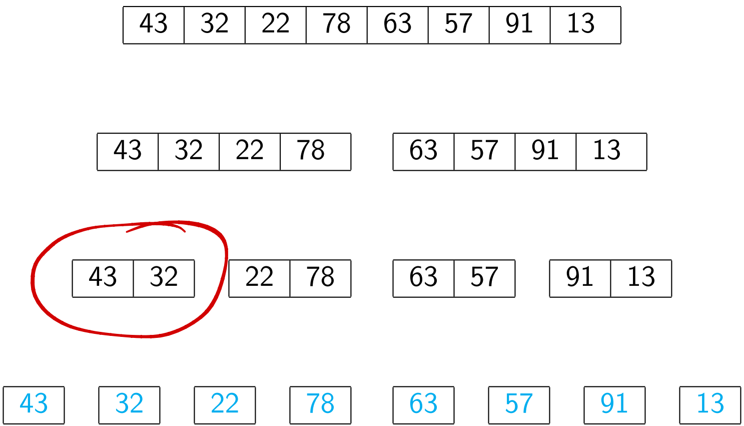

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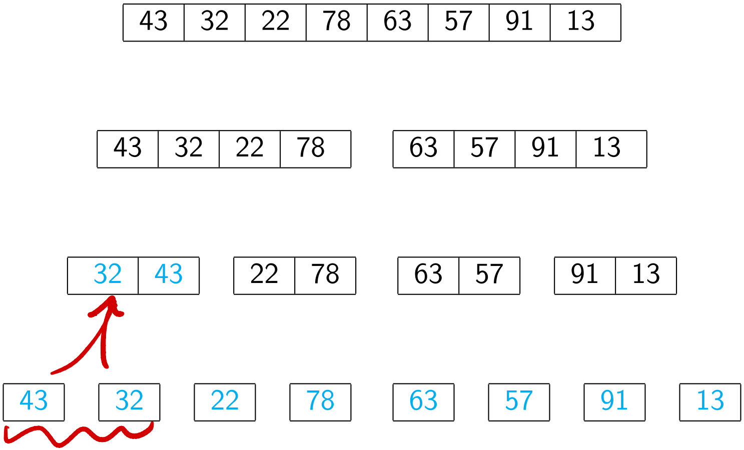

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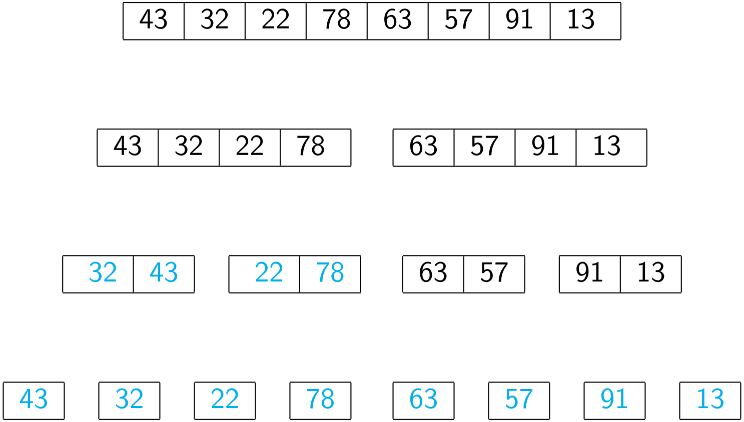

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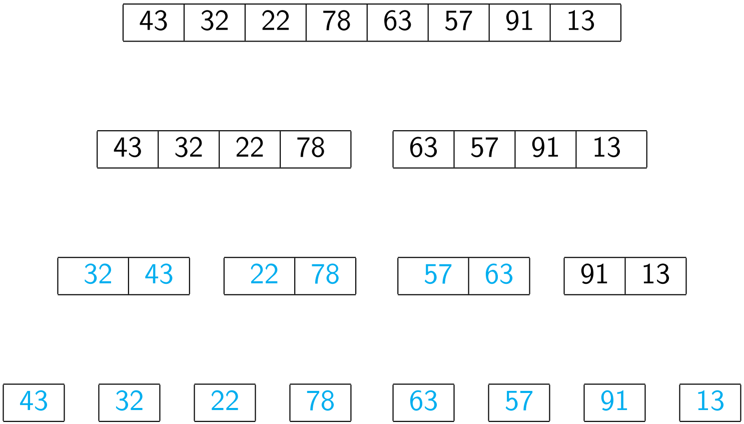

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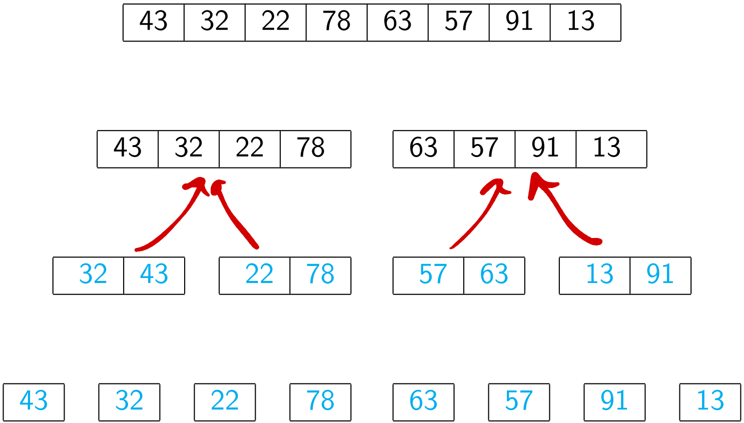

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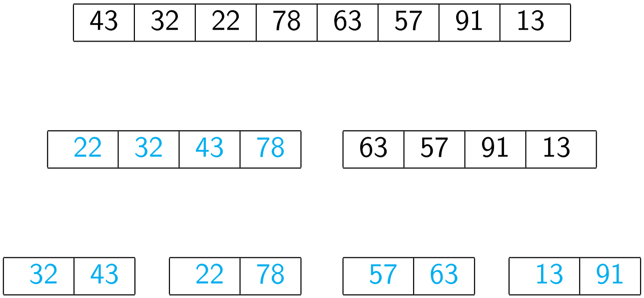

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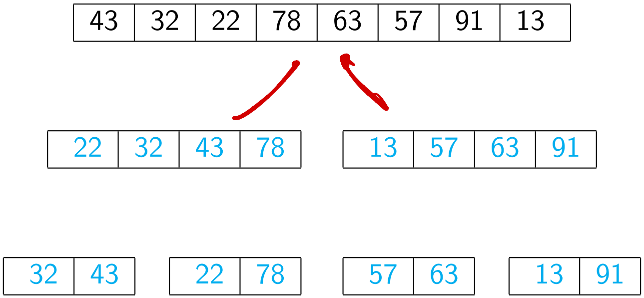

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Divide and Conquer

- Break up the problem into disjoint parts
- Solve each part separately
- Combine the solutions efficiently

Merging sorted lists

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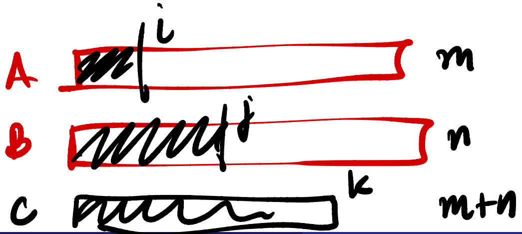


```
def merge(A,B):
  (m,n) = (len(A),len(B))
  (C,i,j,k) = ([],0,0,0)
  while k < m+n:
    if i == m:
      C.extend(B[j:])
      k = k + (n-j)
    elif j == n:
      C.extend(A[i:])
      k = k + (n-i)
    elif A[i] < B[j]:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    else:
      C.append(B[j])
      (j,k) = (j+1,k+1)
  return(C)
```

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- To sort `A` into `B`, both of length n

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- To sort `A` into `B`, both of length n
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  - Sort `A[:n//2]` into `L`

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`
  - Sort `A[n//2:]` into `R`

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`
  - Sort `A[n//2:]` into `R`
  - Merge `L` and `R` into `B`

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`
  - Sort `A[n//2:]` into `R`
  - Merge `L` and `R` into `B`

```
def mergesort(A):
  n = len(A)

  if n <= 1:
     return(A)

  L = mergesort(A[:n//2])
  R = mergesort(A[n//2:])

  B = merge(L,R)

  return(B)
```

Summary

- Merge sort using divide and conquer to sort a list

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- Merge sort using divide and conquer to sort a list
- Divide the list into two halves
- Sort each half
- Merge the sorted halves
- Next, we have to check that the complexity is less than $O(n^2)$