Hoare Logic

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Outline

- Hoare triples as assertions of partial correctness.
- Hoare logic rules.
- Weakest Precondition calculus.

Hoare Logic

- A way of asserting properties of programs.
- Hoare triple: $\{A\}P\{B\}$ asserts that "If program P is started in a state satisfying condition A, if it terminates, it will terminate in a state satisfying condition B."
- A proof system for proving such assertions.
- A way of reasoning about such assertions using the notion of "Weakest Preconditions" (due to Dijkstra).

A simple programming language

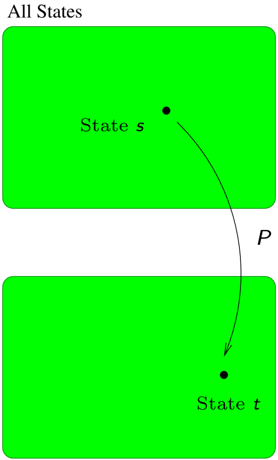
- `skip`
- `x := ` e (assignment)
- `if` b `then` S `else` T (if-then-else)
- `while` b `do` S (while)
- S `;` T (sequencing)

Example program

```
x := n;
a := 1;
while (x ≥ 1) {
   a := a * x;
   x := x - 1
 }
```

Programs as State Transformers

View program P as a partial map $[P] : \mathit{Stores} \rightarrow \mathit{Stores}$.

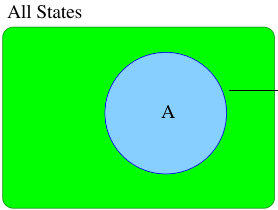


$\{x \mapsto 2,\ y \mapsto 10,\ z \mapsto 3\}$

```
y = y + 1;
z = x + y
```

$\{x \mapsto 2,\ y \mapsto 11,\ z \mapsto 12\}$

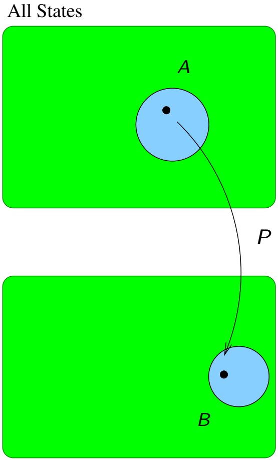
Predicates on States



States satisfying
Predicate A
Eg. $x \geq 0 \wedge x < y$

Assertion of "Partial Correctness" $\{A\}P\{B\}$

$\{A\}P\{B\}$ asserts that "If program P is started in a state satisfying condition A, either it will not terminate, or it will terminate in a state satisfying condition B."



$\{10 \leq y\}$

```
y = y + 1;
z = x + y
```

$\{x < z\}$

Give “weakest” preconditions

1. $\{?\}$ `x := x + 2` $\{x \geq 5\}$
2. $\{?\}$ `if` $(y < 0)$ `then x:=x+1 else x:=y` $\{x > 0\}$
3. $\{?\}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

Proof rules of Hoare Logic

Skip:

$$\frac{}{\{A\}\ \texttt{skip}\ \{A\}}$$

Assignment

$$\frac{}{\{A[e/x]\}\ \texttt{x := e}\ \{A\}}$$

Proof rules of Hoare Logic

If-then-else:

$$\frac{\{P \wedge b\}\ S\ \{Q\},\ \{P \wedge \neg b\}\ T\ \{Q\}}{\{P\}\ \texttt{if}\ \ b\ \ \texttt{then}\ \ S\ \ \texttt{else}\ \ T\ \{Q\}}$$

While (here P is called a *loop invariant*)

$$\frac{\{P \wedge b\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ \ b\ \ \texttt{do}\ \ S\ \{P \wedge \neg b\}}$$

Sequencing:

$$\frac{\{P\}\ S\ \{Q\},\ \{Q\}\ T\ \{R\}}{\{P\}\ S\,;T\ \{R\}}$$

Weakening:

$$\frac{P \implies Q,\ \{Q\}\ S\ \{R\},\ R \implies T}{\{P\}\ S\ \{T\}}$$

Some examples to work on

1. $\{x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$
2. $\{(y < 0 \wedge x > -1) \vee (y > 0)\}$ `if` $(y < 0)$ `then x:=x+1 else x:=y` $\{x > 0\}$
3. $\{x \leq 6\}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

Exercise

Prove using Hoare logic $\{x \geq 1 \wedge x = n \wedge a = 1\}\ P\ \ \{a = n!\}$, where P is:

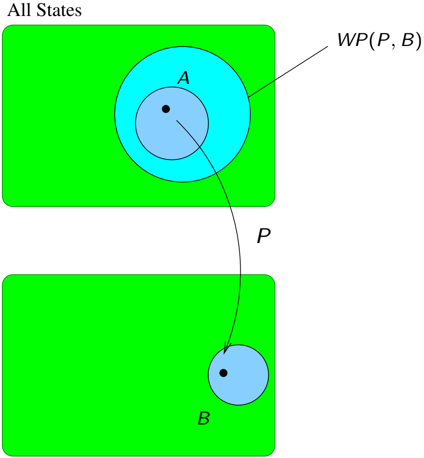
```
while (x ≥ 1) {
   a := a * x;
   x := x - 1
 }
```

Relative completeness of Hoare rules

- Does $\{A\}P\{B\}$ mean there exists a proof tree for the same using the rules mentioned above?
- *Yes*, provided the underlying logic is complete.
    - That is, whenever $A \Rightarrow B$ there ought to exist a proof for the same using the rules of the underlying logic.
    - For example, (plain) first-order logic, and presburger arithmetic (first-order logic, plus natural numbers with addition) are complete. Peano arithmetic (which includes multiplication) is not complete.

Weakest Precondition $WP(P, B)$

$WP(P, B)$ is "a predicate that describes the exact set of states s such that when program P is started in s, if it terminates it will terminate in a state satisfying condition B."



$\{-1 < y\}$

```
y = y + 1;
z = x + y;
```

$\{x < z\}$

Using weakest pre-conditions for verification

- Note that $\{A\}\ P\ \ \{B\}$ iff $A \implies WP(P, B)$.
- Therefore, if we have an algorithm for WP we can verify Hoare triples automatically.
- Tools such as Spec# verify Hoare triples, using the above approach.

Illustration

To check:

$\{y > 10\}$

```
y = y + 1;
z = x + y;
```

$\{x < z\}$

Check verification condition:

$$(y > 10) \implies (y > -1).$$

Rules for Computing Weakest Precondition

For assignment statement `x = e`:

$\{B[e/x]\}$

```
x = e;
```

$\{B\}$

Rules for Computing Weakest Precondition

For assignment statement `x = e`:

$\{B[e/x]\}$

```
x = e;
```

$\{B\}$

$\{(x + y) > 0 \wedge y = 0\}$

```
z = x + y;
```

$\{z > 0 \wedge y = 0\}$

Rules for Computing Weakest Precondition

If-the-else statement `if c then` S_1 `else` S_2:

$$\{(c \wedge WP(S_1, B)) \vee$$
$$(\neg c \wedge WP(S_2, B))\}$$

```
if (c)
   S1;
else
   S2;
```

$$\{B\}$$

Rules for Computing Weakest Precondition

If-the-else statement `if c then` S_1 `else` S_2:

$\{(c \wedge WP(S_1, B)) \vee$
$(\neg c \wedge WP(S_2, B))\}$

```
if (c)
   S1;
else
   S2;
```

$\{B\}$

$\{((x < y) \wedge (y > w)) \vee$
$((x \geq y) \wedge (x > w))\}$

```
if (x < y)
   z = y;
else
   z = x;
```

$\{z > w\}$

WP rule for sequencing

$$WP(S;T,\ B) = WP(S, WP(T, B)).$$

Weakest Precondition for `while` statements

- Let W = "`while` b `do` S".
- In general it is not possible to compute the precise $WP(W, B)$.
- It is possible to compute an under-approximating condition $WP'(W, B)$ such that $WP'(W, B) \implies WP(W, B)$.
  - Unroll the loop k times, for some chosen value $k \geq 0$, and let W' be the thus unrolled loop.
  - For e.g., for $k = 0$
    $W' = skip$
    for $k = 2$,
    W' = "`if` (b) $\{$ S; `if` (b) S $\}$".
  - Now, $WP'(W, B) \equiv WP(W', B \wedge (\neg b))$.
  - Higher value of k gives a better $WP'(W, B)$.
- Using this, one can verify a hoare triple $\{A\}$ P $\{B\}$ conservatively.
  - That is, the above triple is true if $A \implies WP'(W, B)$ (the converse is not necessarily true).

Another approach: under-approximating weakest pre-conditions given loop invariants

`while` loops

- i is said to be a correct loop invariant in $W =$ "$\texttt{while}\ b\ \texttt{invariant}\ i\ \texttt{do}\ S$" iff $(i \wedge b) \implies WP(S, i)$.
- $WP'(W, B) \equiv (B \wedge \neg b) \vee (((i \wedge \neg b) \implies B) \wedge i)$.

Illustration

Consider the example loop W below

`while (i < n) invariant` i
`  i++;`

- Let B = "`i == n`".
  - $i \equiv$ "`i < n`", is not a correct loop invariant.
  - $i \equiv$ "`i <= n`" is correct, and is sufficient to imply the post-condition B. In this case $WP'(W, B) = WP(W, B) =$ "`i <= n`".
  - $i \equiv$ "`i <= n+1`" is a correct (but weak) loop invariant, and is not sufficient to imply the post-condition. In this case $WP'(W, B)$ is *false*.
- Let B = "`n == 10`".
  - $i \equiv$ "`n == 10`" is a correct loop invariant, and is necessary to imply the post-condition B.

Conclusion

- Hoare logic can be extended to reason about programs with arrays, pointers [Separation Logic], function calls, etc.
- Finds application in recent program analysis techniques like finding "path conditions" in automated directed testing, and null-deference analysis.