# Interprocedural analysis: Sharir-Pnueli's functional approach

Deepak D'Souza

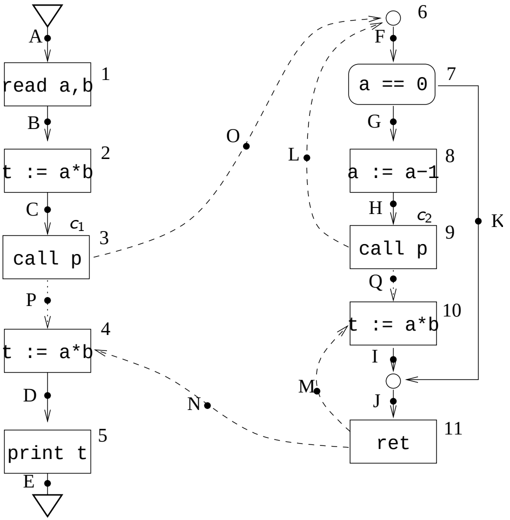
Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

24 September 2010

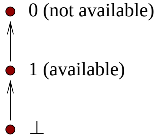
## Outline

# Example program with procedure calls

- We want join over all "valid" paths at each progam point.
- Simply taking "JOP" on extended CFG would lose precision.
- Can we compute "JVP" (Join over Valid Paths) values instead?

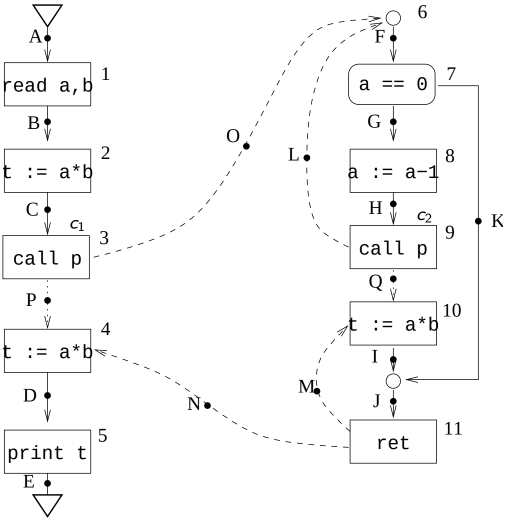



# Example program: Available expressions analysis

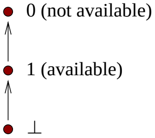



Lattice for Av-Exp analysis.

- Is a*b available at program point $N$?

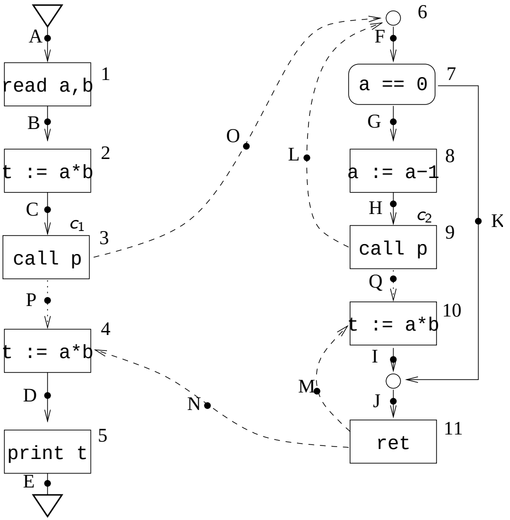


# Example program: Available expressions analysis

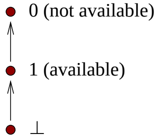



Lattice for Av-Exp analysis.

- Is a*b available at program point $N$?
- No if we consider all paths.

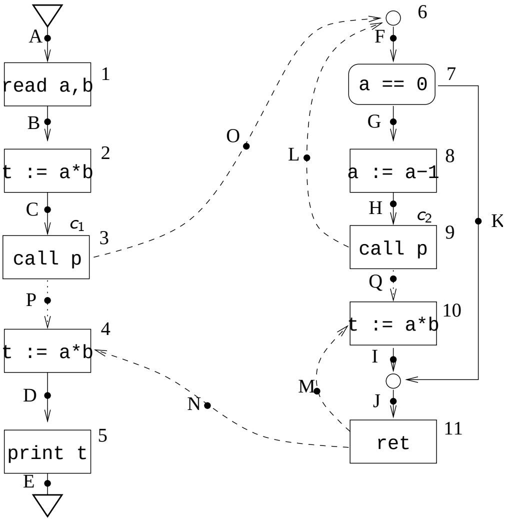


# Example program: Available expressions analysis



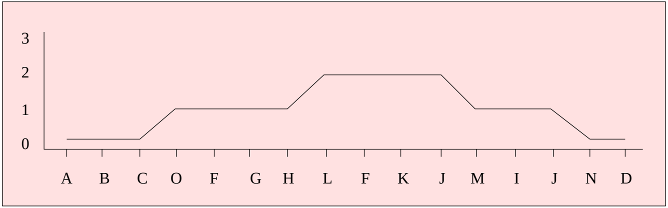

Lattice for Av-Exp analysis.

- Is a*b available at program point $N$?
- No if we consider all paths.
- Yes if we consider interprocedurally valid paths only.



## Interprocedurally valid and complete paths: $IVP_0(M, N)$

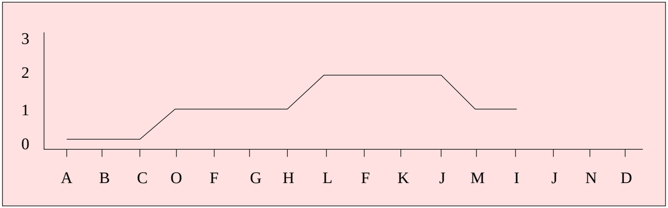
- Convention: $r_p$ and $e_p$ are respectively the root and return nodes of procedure $p$. Root of the main procedure is $r_1$.
- A path $\rho$ is interprocedurally valid and complete if the sequence of call nodes and return notes form a balanced parenthesis string.
- A path in $IVP_0(r_1, D)$ for example program:




- C ("call $p$") · O · · · H ("call $p$") · L · · · J ("ret") · M · · · J ("ret") · N · D.
- Note that "call $p$" must be matched by "$\text{ret}_p$."

# Interprocedurally valid paths: $IVP(M, N)$

- A path $\rho$ is interprocedurally valid if it is a prefix of a valid and complete path.
- A path in $IVP(r_1, I)$ for example program:



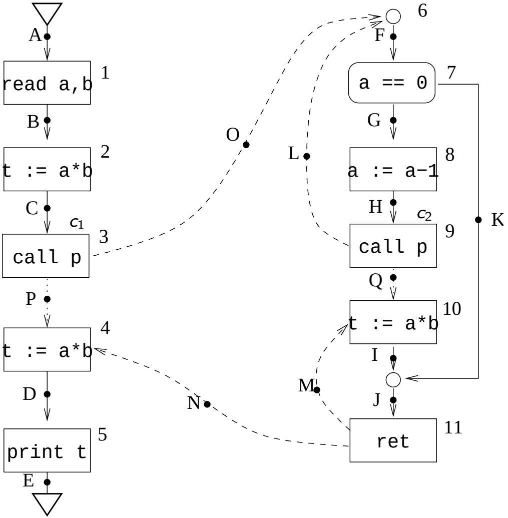
## Defining JVP

For a given program $P$ and analysis $((D, \leq), f_{MN}, d_0)$, the *join over all interprocedurally valid paths* (JVP) at point $N$ is defined to be:

$$\bigsqcup_{\rho \in IVP(r_1, N)} f_\rho(d_0).$$

# Equation solving: Problems with naive approach

- In non-procedural case, we setup equations to capture JOP assuming distributivity. Least solution to these equations gave us exact/over-approx JOP depending on distributive/monotonic framework.
- Try to set up similar equations for $x_N$ (JVP at program point $N$).

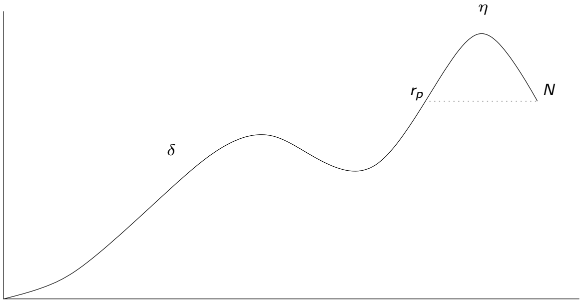


# Instead try to capture join over complete paths first

- Set up equations to capture join over complete paths.
- Now set up equations to capture JVP using join over complete path values.

## Basic idea: Why join over complete paths help

An IVP path $\rho$ from $r_1$ to $N$ in procedure $p$ can be written as $\delta \cdot \eta$ where $\delta$ is in $\mathrm{IVP}(r_1, r_p)$, and $\eta$ is in $\mathrm{IVP}_0(r_p, N)$.




Consider point where procedure $p$ was last entered.

## Valid and complete paths from $r_p$ to $N$

For a proecedure $p$ and node $N$ in $p$, define:

$$\phi_{r_p,N} : D \rightarrow D$$

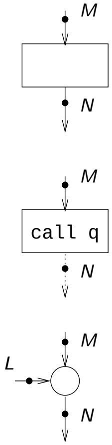
given by

$$\phi_{r_p,n}(d) = \bigsqcup_{\text{paths } \rho \in \mathrm{IVP}_0(r_p,N)} f_\rho(d).$$

$\phi_{r_p,N}$ is thus the join of all functions $f_\rho$ where $\rho$ is an interprocedurally valid and complete path from $r_p$ to $N$.

# Equations (1) to capture $\phi_{r_p,N}$

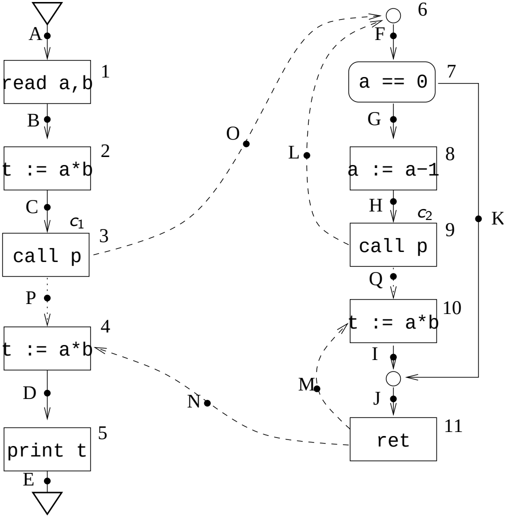
$$
\begin{aligned}
\psi_{r_p,r_p} &= id_D \\
\psi_{r_p,N} &= f_{MN} \circ \psi_{r_p,M} \\
\psi_{r_p,N} &= \psi_{r_q,e_q} \circ \psi_{r_p,M} \\
\psi_{r_p,N} &= \psi_{r_p,L} \sqcup \psi_{r_p,M}.
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Equations (2) to capture JVP

$$
\begin{array}{rcll}
x_1 & \geq & d_0 & \\
x_{r_p} & = & \bigsqcup_{\text{calls } c \text{ to } p \text{ in } q} \phi_{r_q,c}(x_{r_q}) & \\
x_n & = & \phi_{r_p,n}(x_{r_p}) & \text{for } n \in N_p - \{r_p\}.
\end{array}
$$

# Example: Equations for $x_N$'s (JVP)

$$
\begin{aligned}
x_A &\geq 0\\
x_B &= \mathbf{0}(x_A)\\
x_C &= \mathbf{1}(x_A)\\
x_P &= \mathbf{1}(x_A)\\
x_D &= \mathbf{1}(x_A)\\
x_E &= \mathbf{1}(x_A)
\end{aligned}
$$

$$
\begin{aligned}
x_F &= \mathbf{1}(x_A) \sqcup \mathbf{0}(x_F)\\
x_G &= id(x_F)\\
x_K &= id(x_F)\\
x_H &= \mathbf{0}(x_F)\\
x_Q &= \mathbf{0}(x_F)\\
x_I &= \mathbf{1}(x_F)\\
x_J &= id(x_F).
\end{aligned}
$$

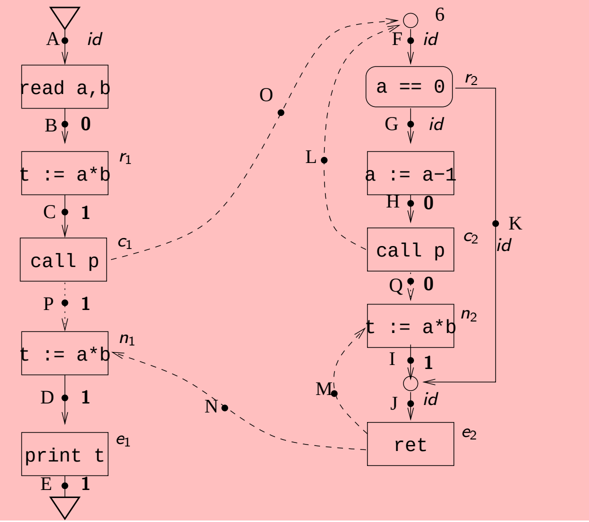



Fig. shows values of $\phi_{r_p,N}$'s in bold.

## Correctness and algo

- Consider lattice $(F, \leq)$ of functions from $D$ to $D$, obtained by closing the transfer functions, identity, and $f_{\perp} : d \mapsto \perp$ (denoted $f_{\Omega}$ by Sharir-Pnueil) under composition and join.
- Ordering is $f \leq g$ iff $f(d) \leq g(d)$ for each $d \in D$.
- $(F, \leq)$ is also a complete lattice.
- $\overline{f}$ induced by Eq (1) is a monotone function on the complete lattice $(\overline{F}, \overline{\leq})$.
- LFP / least solution exists.

**Claim**

$\phi_{r_p,N}$'s are the least solution to Eq (1) when $f_{MN}$'s are distributive. Otherwise $\phi_{r_p,N}$'s are dominated by the least solution to Eq (1).

Kleene/Kildall's algo will compute LFP (assuming $D$ finite).

## Correctness and algo - II

- $\overline{f}$ induced by Eq (2) is a monotone function on the complete lattice $(\overline{D}, \overline{\leq})$.
- LFP / least solution exists.

**Claim**

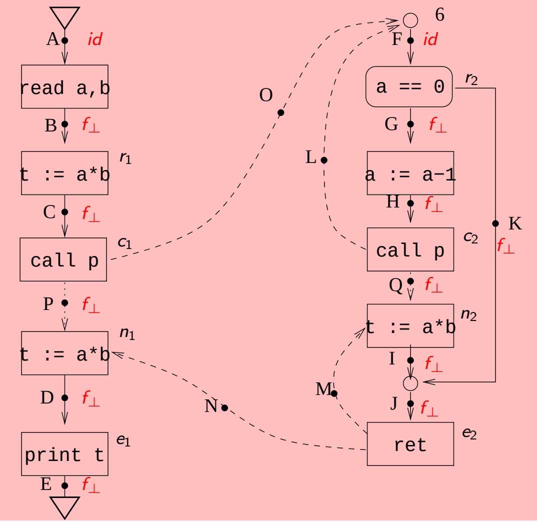
$\mathrm{JVP}_N$'s are the least solution to Eq (2) when $f_{MN}$'s are distributive. Otherwise $\mathrm{JVP}_N$'s are dominated by the least solution to Eq (2).

Kleene/Kildall's algo will compute LFP (assuming $D$ finite).

# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

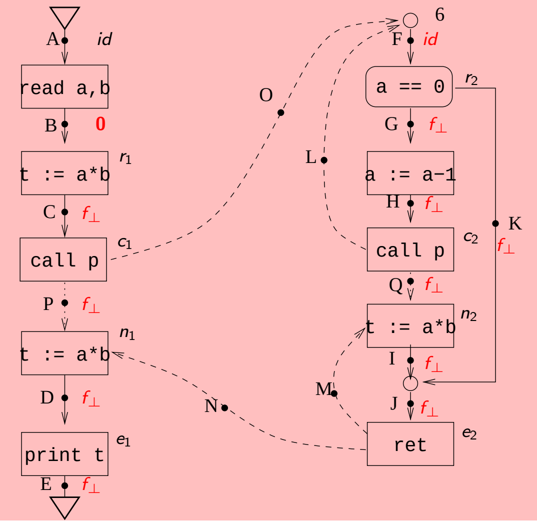
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$\psi_{A,A} = id$$
$$\psi_{A,B} = \mathbf{0} \circ \psi_{A,A}$$
$$\psi_{A,C} = \mathbf{1} \circ \psi_{A,B}$$
$$\psi_{A,P} = \phi_{F,J} \circ \psi_{A,C}$$
$$\psi_{A,D} = \mathbf{1} \circ \psi_{A,P}$$
$$\psi_{A,E} = id \circ \psi_{A,D}$$

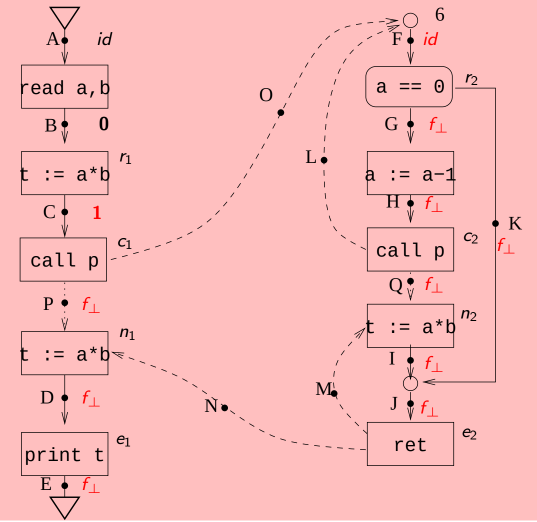
$$\phi_{F,F} = id$$
$$\phi_{F,G} = id \circ \psi_{F,F}$$
$$\phi_{F,K} = id \circ \psi_{F,F}$$
$$\phi_{F,H} = \mathbf{0} \circ \psi_{F,G}$$
$$\phi_{F,Q} = \psi_{F,J} \circ \psi_{F,H}$$
$$\phi_{F,I} = \mathbf{1} \circ \psi_{F,Q}$$
$$\phi_{F,J} = \psi_{F,I} \sqcup \psi_{F,K}$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

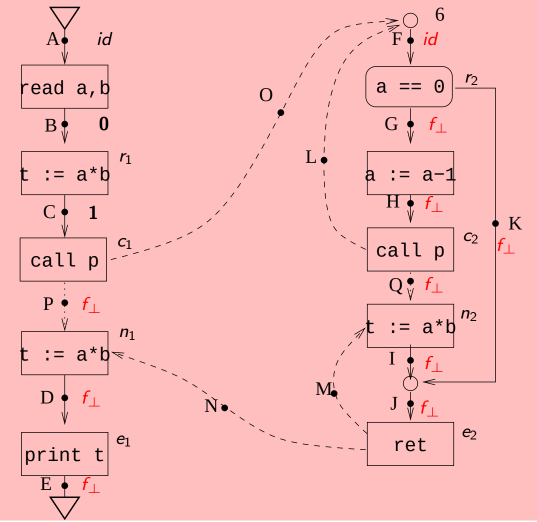
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id\\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A}\\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B}\\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C}\\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P}\\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

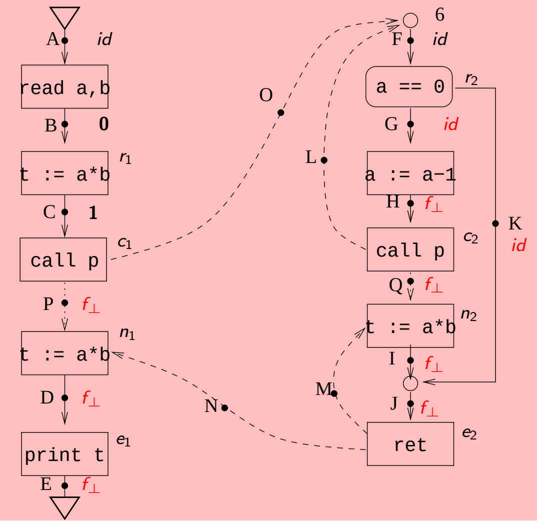
$$
\begin{aligned}
\phi_{F,F} &= id\\
\phi_{F,G} &= id \circ \psi_{F,F}\\
\phi_{F,K} &= id \circ \psi_{F,F}\\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G}\\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H}\\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q}\\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



## Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

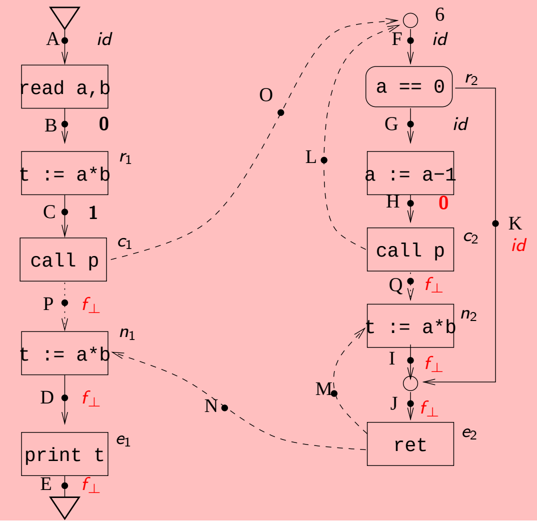
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

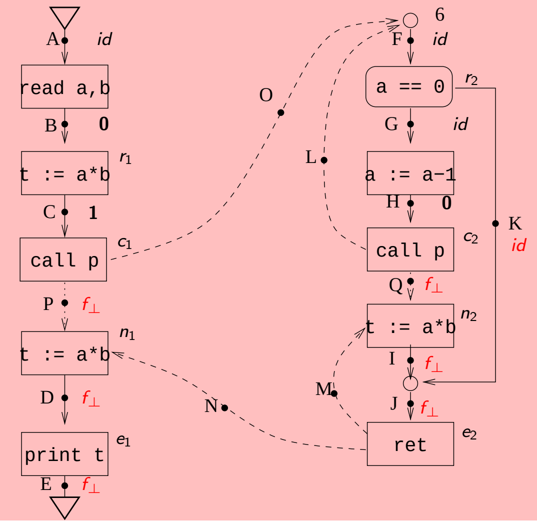
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$\psi_{A,A} = id$$
$$\psi_{A,B} = \mathbf{0} \circ \psi_{A,A}$$
$$\psi_{A,C} = \mathbf{1} \circ \psi_{A,B}$$
$$\psi_{A,P} = \phi_{F,J} \circ \psi_{A,C}$$
$$\psi_{A,D} = \mathbf{1} \circ \psi_{A,P}$$
$$\psi_{A,E} = id \circ \psi_{A,D}$$

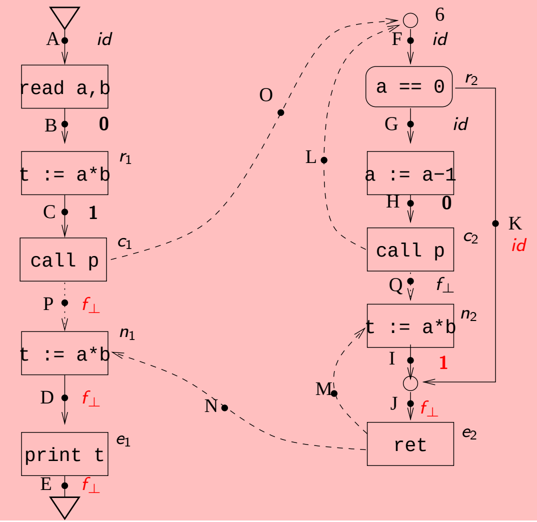
$$\phi_{F,F} = id$$
$$\phi_{F,G} = id \circ \psi_{F,F}$$
$$\phi_{F,K} = id \circ \psi_{F,F}$$
$$\phi_{F,H} = \mathbf{0} \circ \psi_{F,G}$$
$$\phi_{F,Q} = \psi_{F,J} \circ \psi_{F,H}$$
$$\phi_{F,I} = \mathbf{1} \circ \psi_{F,Q}$$
$$\phi_{F,J} = \psi_{F,I} \sqcup \psi_{F,K}$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

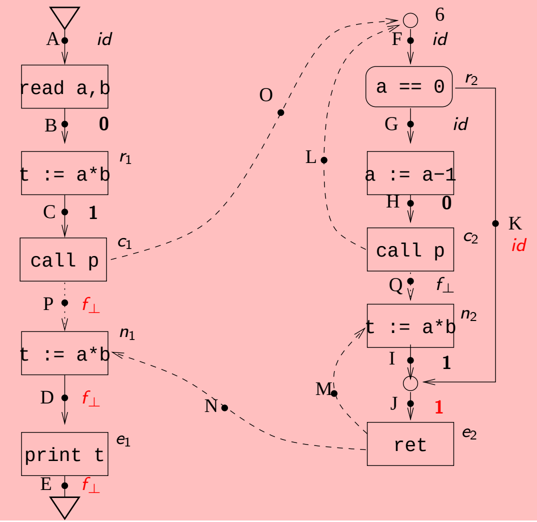
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$\psi_{A,A} = id$$
$$\psi_{A,B} = \mathbf{0} \circ \psi_{A,A}$$
$$\psi_{A,C} = \mathbf{1} \circ \psi_{A,B}$$
$$\psi_{A,P} = \phi_{F,J} \circ \psi_{A,C}$$
$$\psi_{A,D} = \mathbf{1} \circ \psi_{A,P}$$
$$\psi_{A,E} = id \circ \psi_{A,D}$$

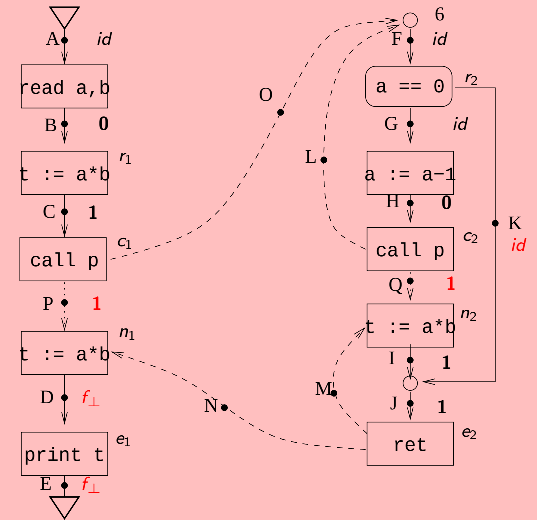
$$\phi_{F,F} = id$$
$$\phi_{F,G} = id \circ \psi_{F,F}$$
$$\phi_{F,K} = id \circ \psi_{F,F}$$
$$\phi_{F,H} = \mathbf{0} \circ \psi_{F,G}$$
$$\phi_{F,Q} = \psi_{F,J} \circ \psi_{F,H}$$
$$\phi_{F,I} = \mathbf{1} \circ \psi_{F,Q}$$
$$\phi_{F,J} = \psi_{F,I} \sqcup \psi_{F,K}$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

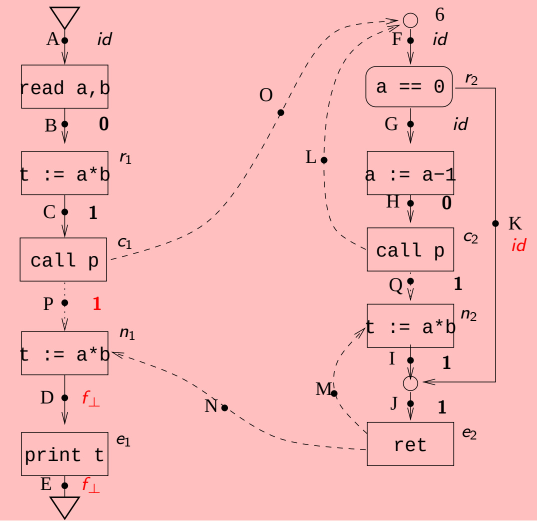
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

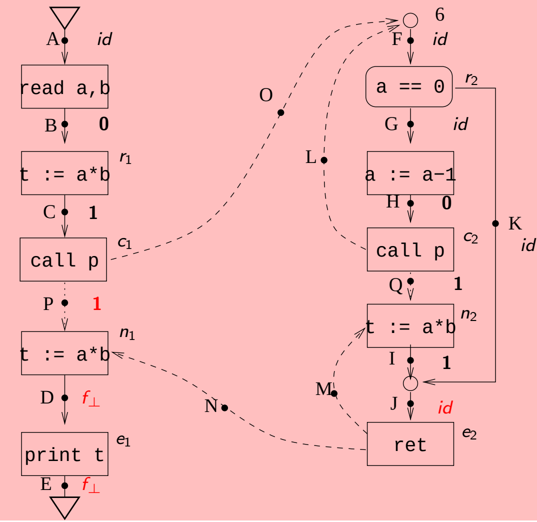
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

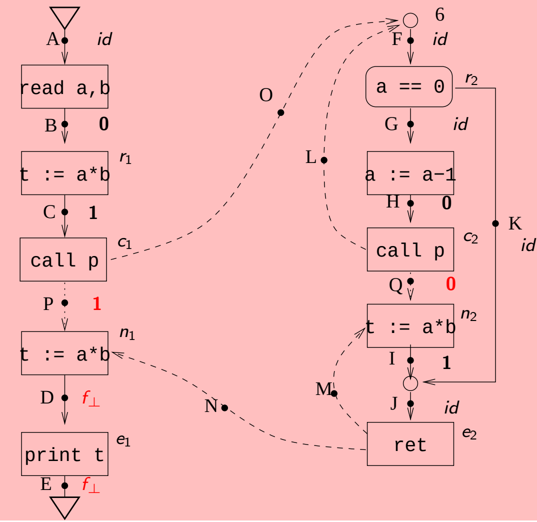
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

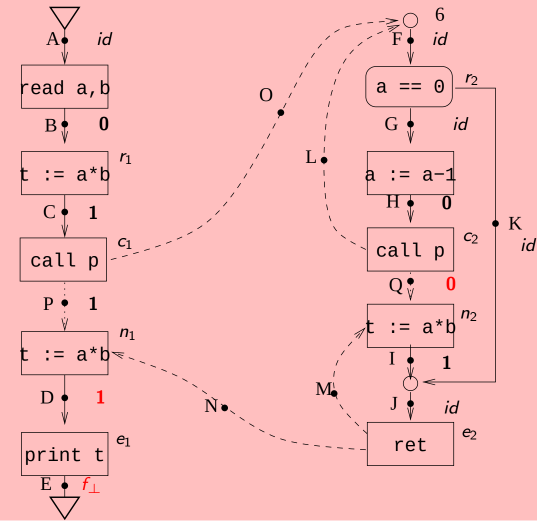
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

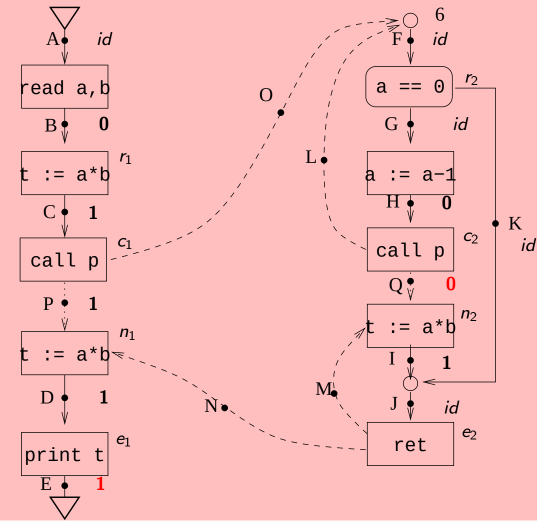
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

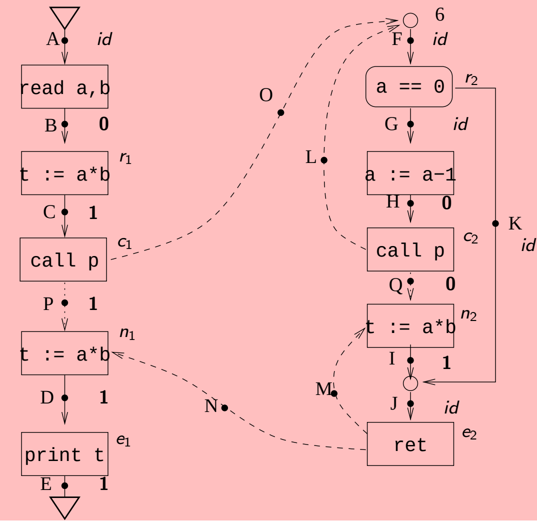
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $\phi$'s

$$
\begin{aligned}
\psi_{A,A} &= id \\
\psi_{A,B} &= \mathbf{0} \circ \psi_{A,A} \\
\psi_{A,C} &= \mathbf{1} \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= \mathbf{1} \circ \psi_{A,P} \\
\psi_{A,E} &= id \circ \psi_{A,D}
\end{aligned}
$$

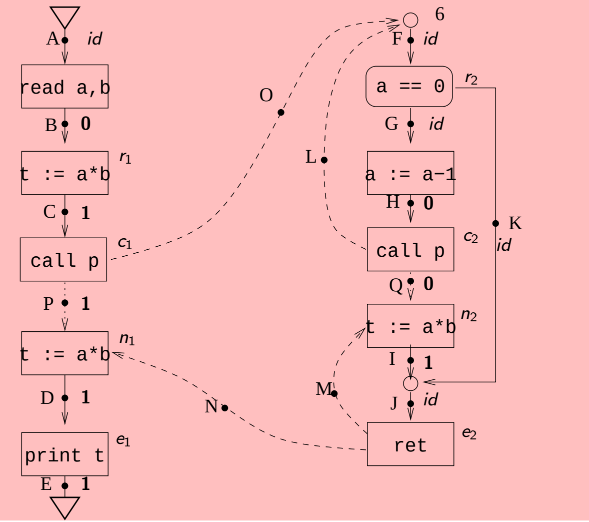
$$
\begin{aligned}
\phi_{F,F} &= id \\
\phi_{F,G} &= id \circ \psi_{F,F} \\
\phi_{F,K} &= id \circ \psi_{F,F} \\
\phi_{F,H} &= \mathbf{0} \circ \psi_{F,G} \\
\phi_{F,Q} &= \psi_{F,J} \circ \psi_{F,H} \\
\phi_{F,I} &= \mathbf{1} \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \sqcup \psi_{F,K}
\end{aligned}
$$



# Example: Equations for $x_N$'s (JVP)

$$
\begin{aligned}
x_A &\geq 0 \\
x_B &= \mathbf{0}(x_A) \\
x_C &= \mathbf{1}(x_A) \\
x_P &= \mathbf{1}(x_A) \\
x_D &= \mathbf{1}(x_A) \\
x_E &= \mathbf{1}(x_A)
\end{aligned}
$$

$$
\begin{aligned}
x_F &= \mathbf{1}(x_A) \sqcup \mathbf{0}(x_F) \\
x_G &= id(x_F) \\
x_K &= id(x_F) \\
x_H &= \mathbf{0}(x_F) \\
x_Q &= \mathbf{0}(x_F) \\
x_I &= \mathbf{1}(x_F) \\
x_J &= id(x_F).
\end{aligned}
$$



# Example: Equations for $x_N$'s (JVP)

$$
\begin{aligned}
x_A &\geq 0 \\
x_B &= \mathbf{0}(x_A) \\
x_C &= \mathbf{1}(x_A) \\
x_P &= \mathbf{1}(x_A) \\
x_D &= \mathbf{1}(x_A) \\
x_E &= \mathbf{1}(x_A)
\end{aligned}
$$

$$
\begin{aligned}
x_F &= \mathbf{1}(x_A) \sqcup \mathbf{0}(x_F) \\
x_G &= id(x_F) \\
x_K &= id(x_F) \\
x_H &= \mathbf{0}(x_F) \\
x_Q &= \mathbf{0}(x_F) \\
x_I &= \mathbf{1}(x_F) \\
x_J &= id(x_F).
\end{aligned}
$$

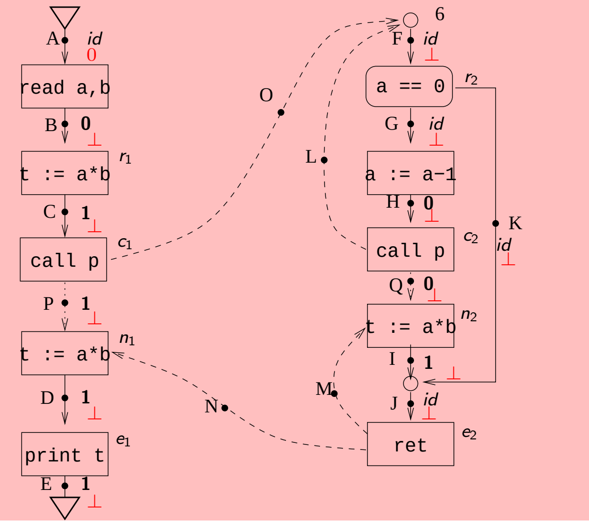



Fig shows initial (red) and final (blue) values.

# Example: Equations for $x_N$'s (JVP)

$$
\begin{aligned}
x_A &\geq 0 \\
x_B &= \mathbf{0}(x_A) \\
x_C &= \mathbf{1}(x_A) \\
x_P &= \mathbf{1}(x_A) \\
x_D &= \mathbf{1}(x_A) \\
x_E &= \mathbf{1}(x_A)
\end{aligned}
$$

$$
\begin{aligned}
x_F &= \mathbf{1}(x_A) \sqcup \mathbf{0}(x_F) \\
x_G &= id(x_F) \\
x_K &= id(x_F) \\
x_H &= \mathbf{0}(x_F) \\
x_Q &= \mathbf{0}(x_F) \\
x_I &= \mathbf{1}(x_F) \\
x_J &= id(x_F).
\end{aligned}
$$

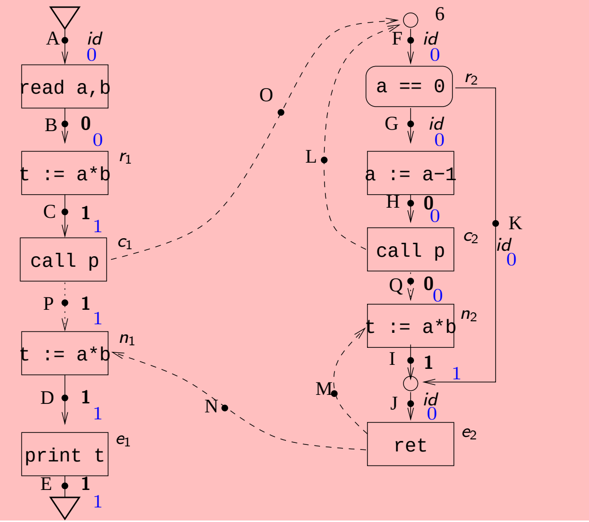
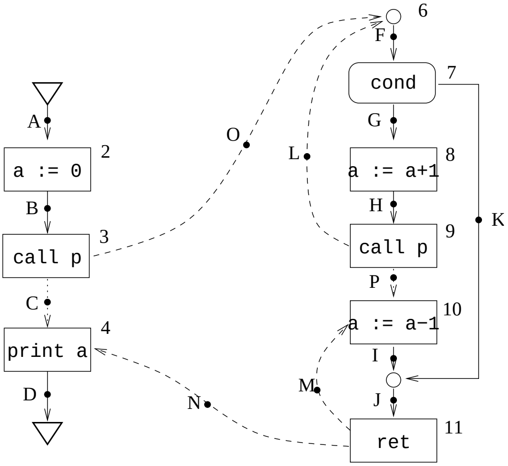



Fig shows initial (red) and final (blue) values.

# Exercise

Exercise: Use the functional method to do interprocedural constant propagation analysis for the program below, with initial value ∅.



## Summary of functional approach

- Uses a two step approach
  1. Compute $\phi_{r_p,N}$'s.
  2. Compute $x_n$'s (JVP's) at each point.

Summary of conditions: For each property (column heading), the conjunction of the ticked conditions (row headings) are sufficient to ensure the property.

| | Termination | Least Sol of Eq(2) $\geq$ JVP | Least Sol of Eq(2)= JVP |
|---|---|---|---|
| $f_{MN}$'s monotonic | $\surd$ | $\surd$ | |
| Finite underlying lattice | $\surd$ | | |
| Distributive | | | $\surd$ |

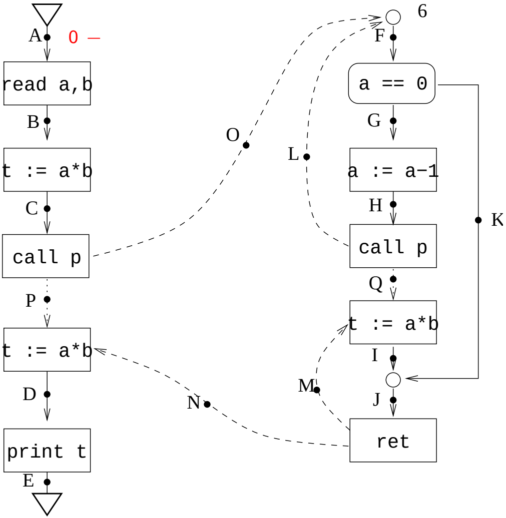
## Iterative/Tabulation Approach

- Maintain a table of values representing the current value of $\phi_{r_p,N}$ for each program point $N$ in procedure $p$.
- Informally, at $N$ in procedure $p$, the table has an entry $d \mapsto d'$ if we have seen valid paths $\rho$ from $r_1$ to $r_p$ with $\bigsqcup_\rho f_\rho(d_0) = d$, and valid and complete paths $\delta$ from $r_p$ to $N$ with $\bigsqcup_\delta f_\delta(d) = d'$.
- Apply Kildall's algo with initial value of $d_0 \mapsto d_0$ at $r_1$.

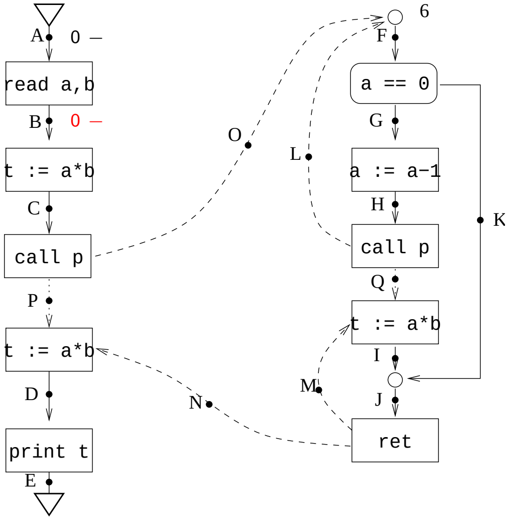
## Propogation rules

- If $d \mapsto d'$ at point $M$, and statement corresponding to $MN$ is not a `call` or `ret`, then propogate $d \mapsto f_{MN}(d')$ to point $N$.
- If $d \mapsto d'$ at point $M$, and statement after $M$ is `call q`, then
  - propogate $d \mapsto d'$ to point $r_q$,
  - propogate $d \mapsto d''$ to return site of $N$ of $M$, provided we have $d' \mapsto d''$ at point $e_q$.
- If $d \mapsto d'$ at point $e_q$ (i.e before `ret` in procedure q), then
  - If $LN$ corresponds to a `call q` and $(d'' \mapsto d)$ at $L$, then propogate $d'' \mapsto d'$ to point $N$. (Do this for all such $N$).

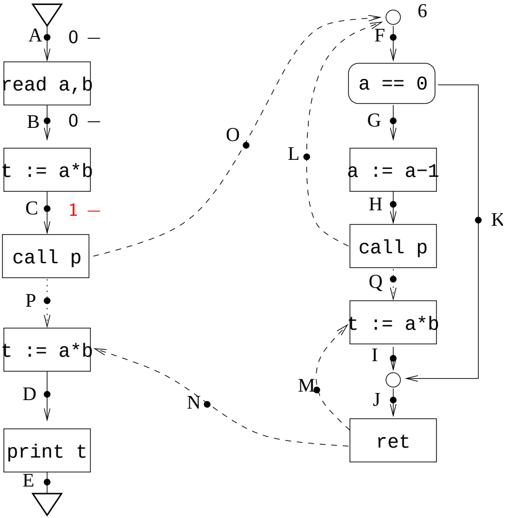
## Example: Computing $\phi$'s iteratively: 1



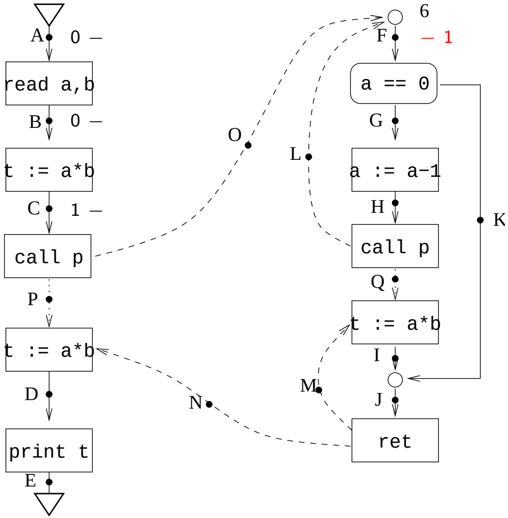
# Example: Computing $\phi$'s iteratively: 2



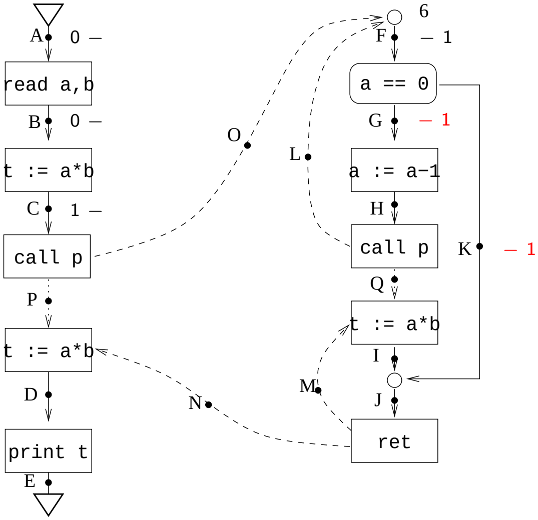
# Example: Computing $\phi$'s iteratively: 3



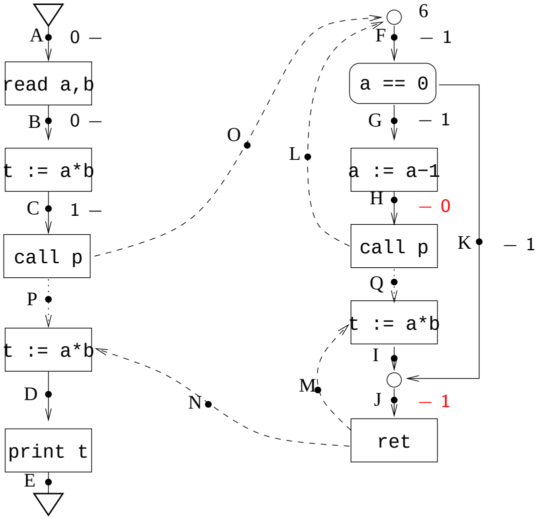
# Example: Computing $\phi$'s iteratively: 4



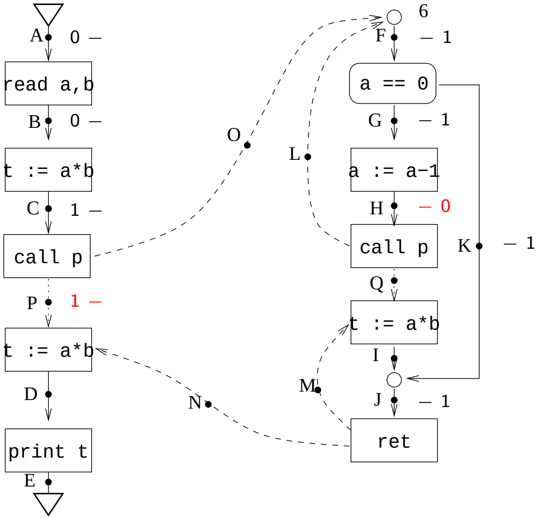
# Example: Computing $\phi$'s iteratively: 5



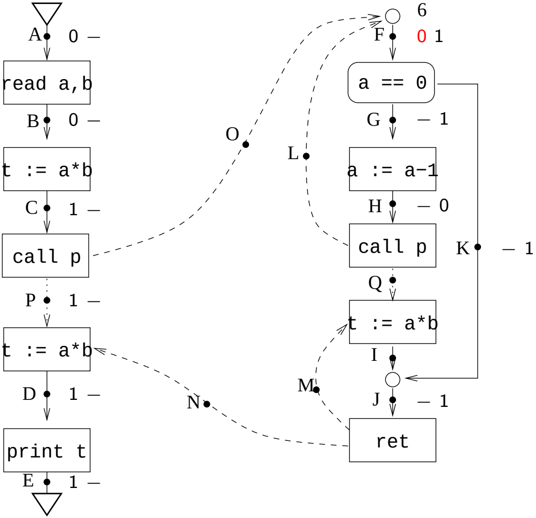
# Example: Computing $\phi$'s iteratively: 6



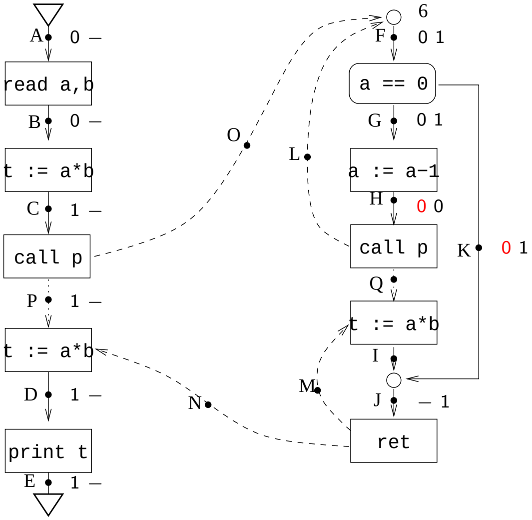
## Example: Computing $\phi$'s iteratively: 7



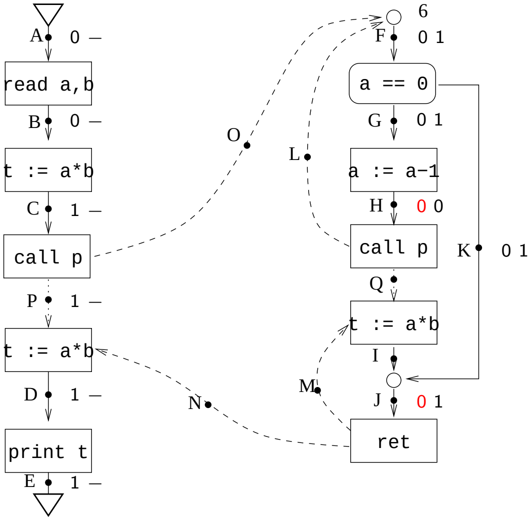
# Example: Computing $\phi$'s iteratively: 8



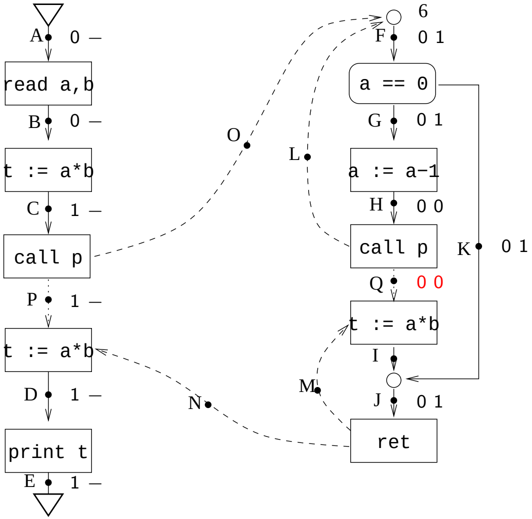
# Example: Computing $\phi$'s iteratively: 9



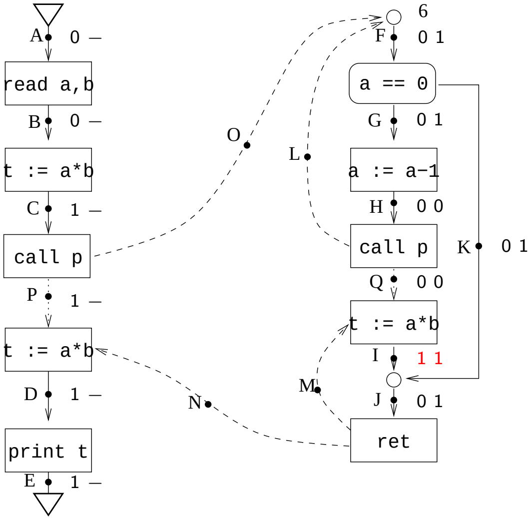
# Example: Computing $\phi$'s iteratively: 10



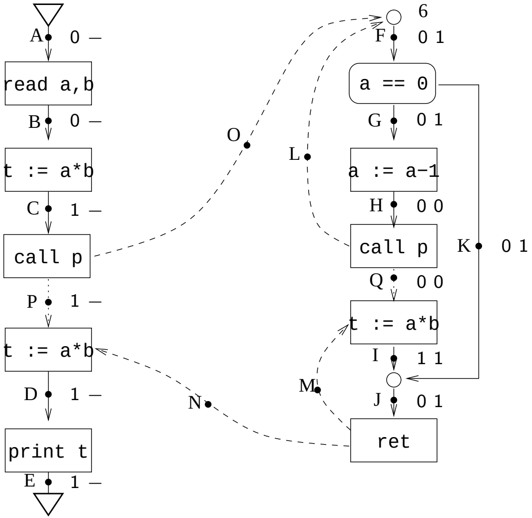
# Example: Computing $\phi$'s iteratively: 11



# Example: Computing $\phi$'s iteratively: 12

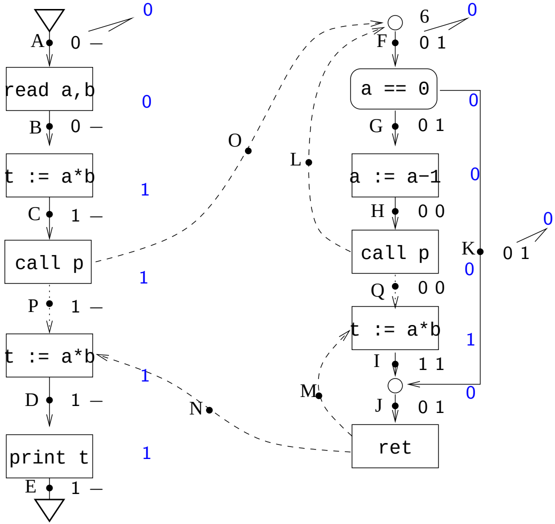


# Example: Computing $\phi$'s iteratively: 13



## Example: Finally compute $x_N$'s from $\phi$ values

At each point $N$ take join of reachable $\phi_{r_p,N}$ values.



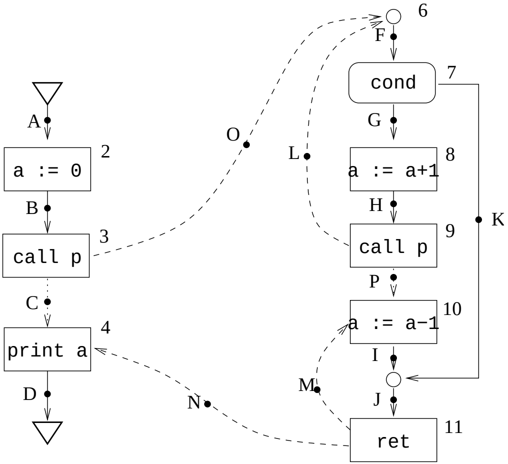
## Correctness of iterative algo

$$
\begin{array}{lcll}
x_1 & \geq & d_0 & \\
x_{r_p} & = & \bigsqcup_{\text{calls } c \text{ to } p \text{ in } q} \psi^*_{r_q,c}(x_{r_q}) & \\
x_n & = & \psi^*_{r_p,n}(x_{r_p}) & \text{for } n \in N_p - \{r_p\}.
\end{array}
$$

- Iterative algo terminates provided underlying lattice is finite.
- It computes the least solution to the equations above, where $\psi^*(rp_N)$'s are the least solution to Eq (1).
- It thus computes an overapproximation of JVP for monotonic transfer functions, and exact JVP when transfer functions are distributive.

# Exercise 2: Iterative algo

Exercise: Use the iterative algo to do constant propagation analysis for the program below with initial value $\emptyset$:



## Comparing functional vs iterative approach

- Functional algo can terminate even when underlying lattice is infinite, provided we can represent and compose/join functions "symbolically".
- Iterative is typically more efficient than functional since it only computes $\phi_{r_p,N}$'s for values reachable at start of procedure.