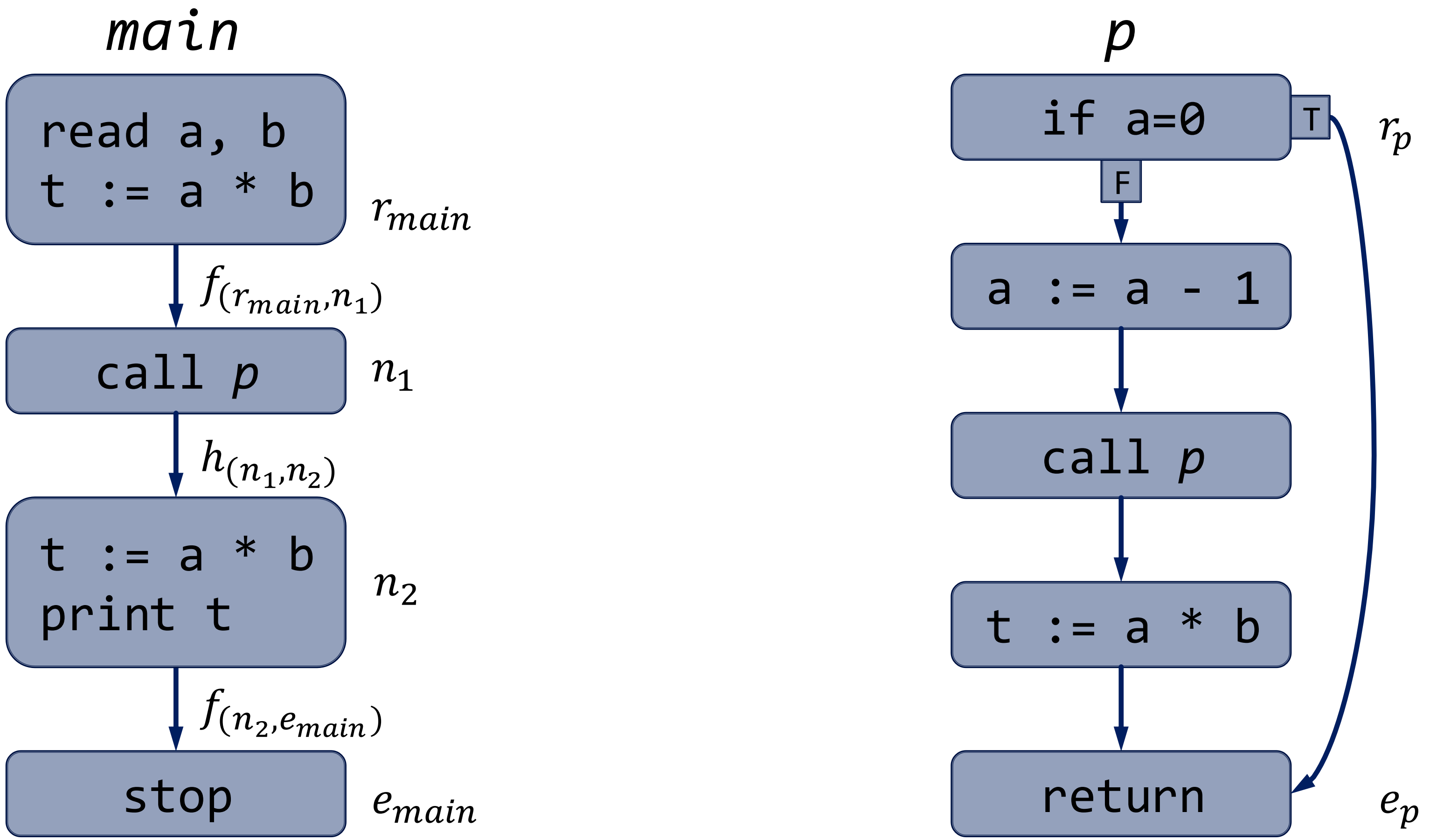
Two Approaches to Interprocedural Data Flow Analysis

Micha Sharir Amir Pnueli

Part Two: The Call String Approach

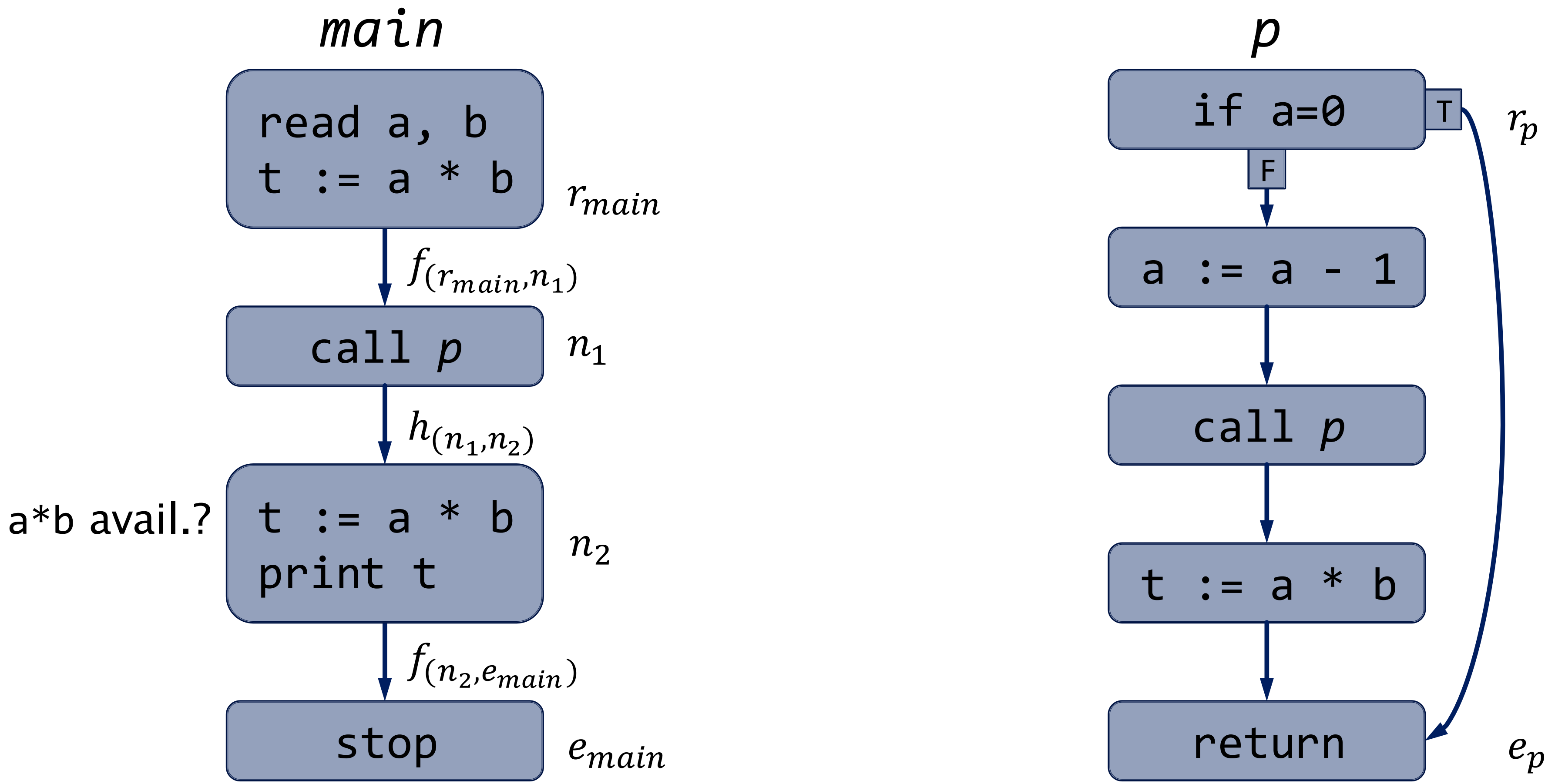
Recap

Functional approach



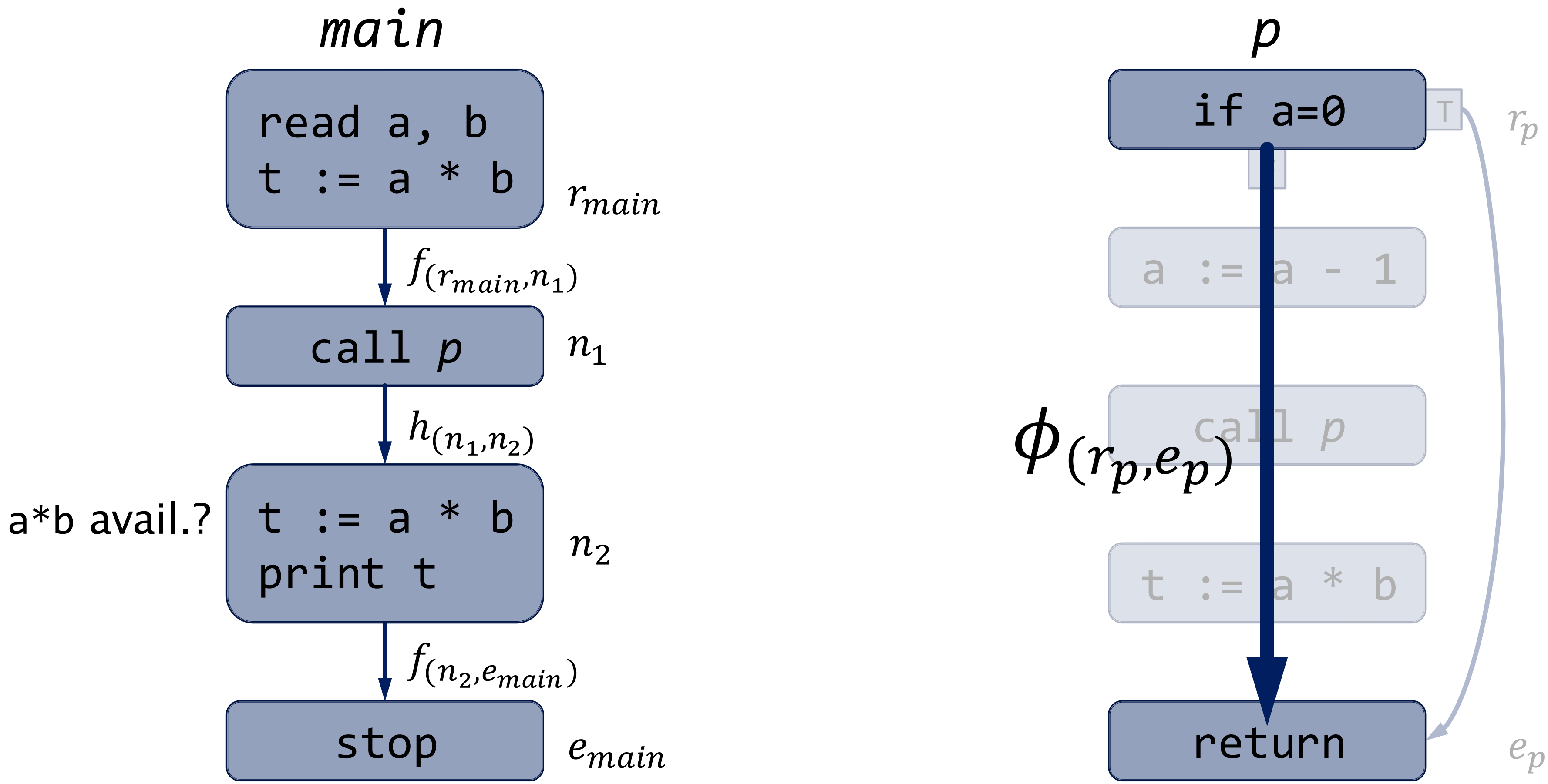
Recap

Functional approach



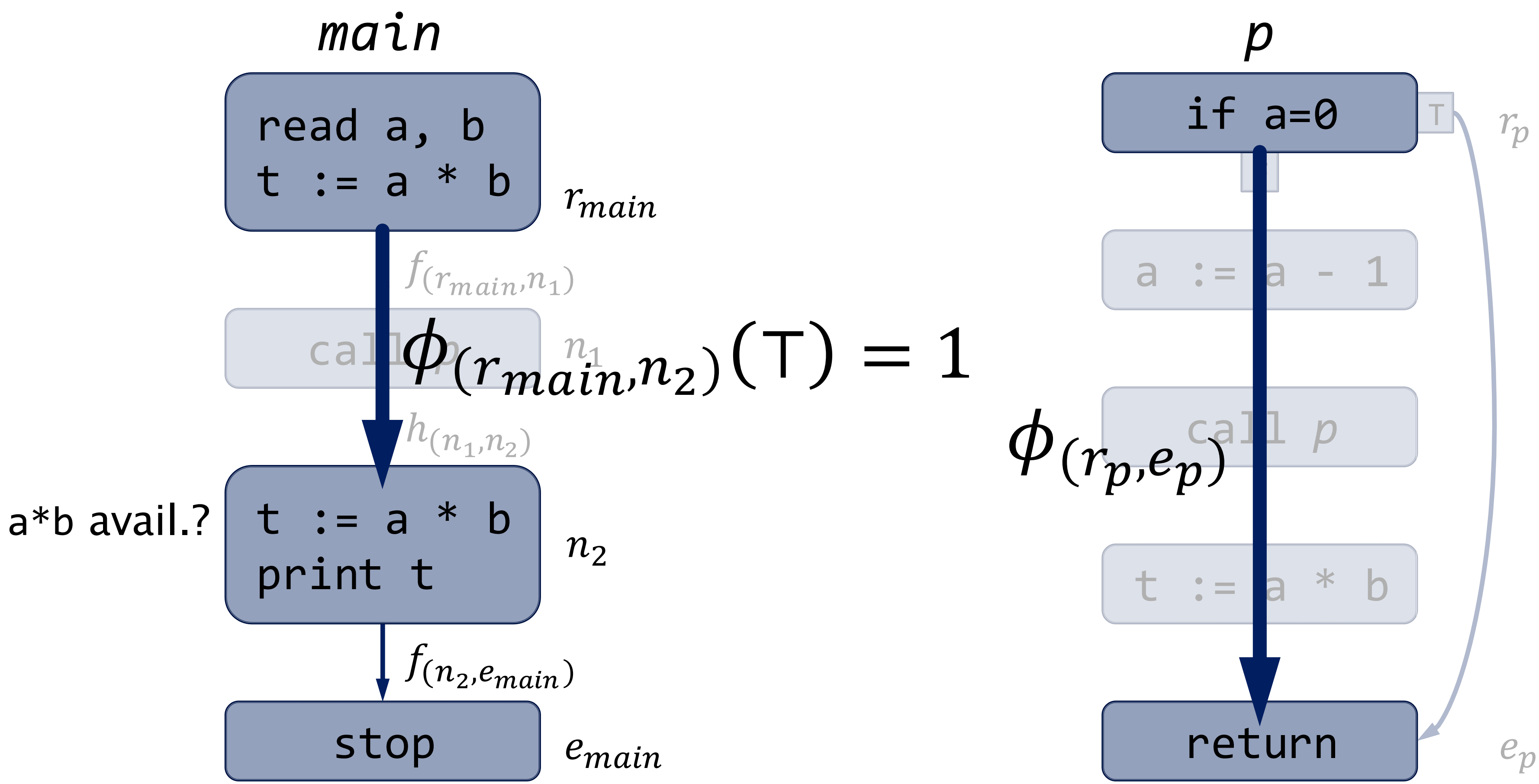
Recap

Functional approach



Recap

Functional approach



Motivation for a different approach

- Avoid expensive functional compositions

- Possibility to trade off precision vs. performance to reduce complexity

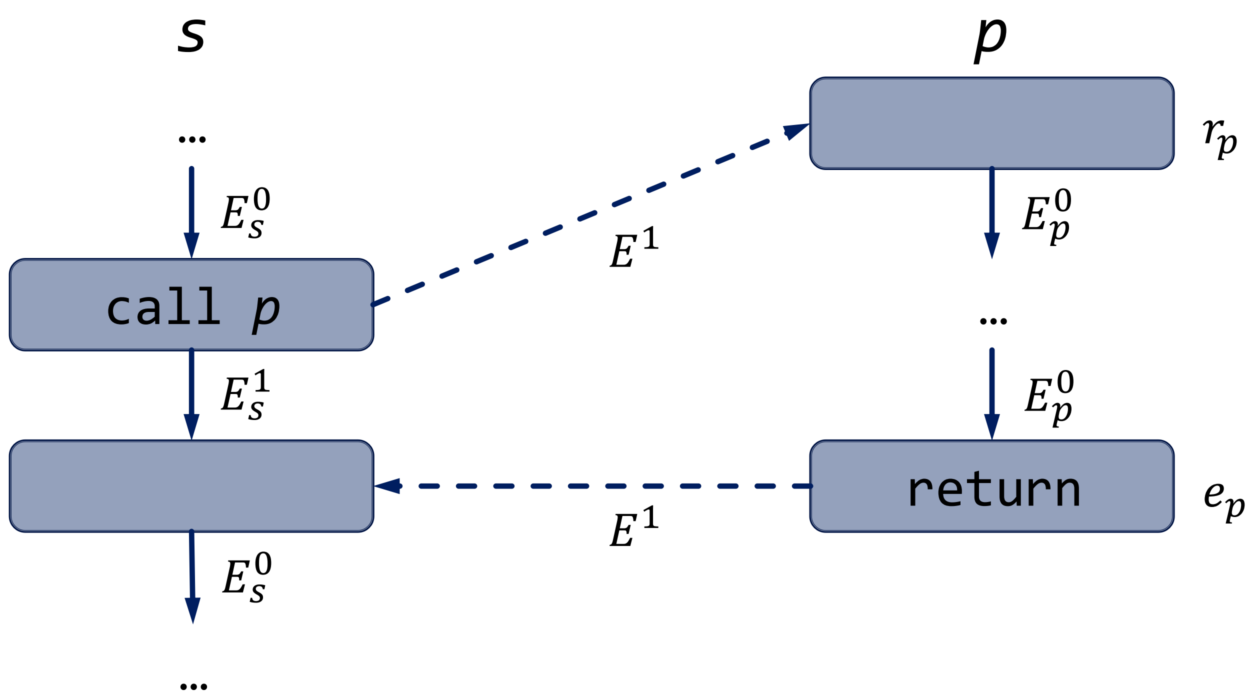
Structure

1. **Definition of a new DF problem $(\boldsymbol{L}^*, \boldsymbol{F}^*)$**
2. Proof: Solution to $(L^*, F^*) \equiv$ MOP solution
3. Feasibility and precise variants
4. Approximative solution

Recap: Interprocedural Graphs

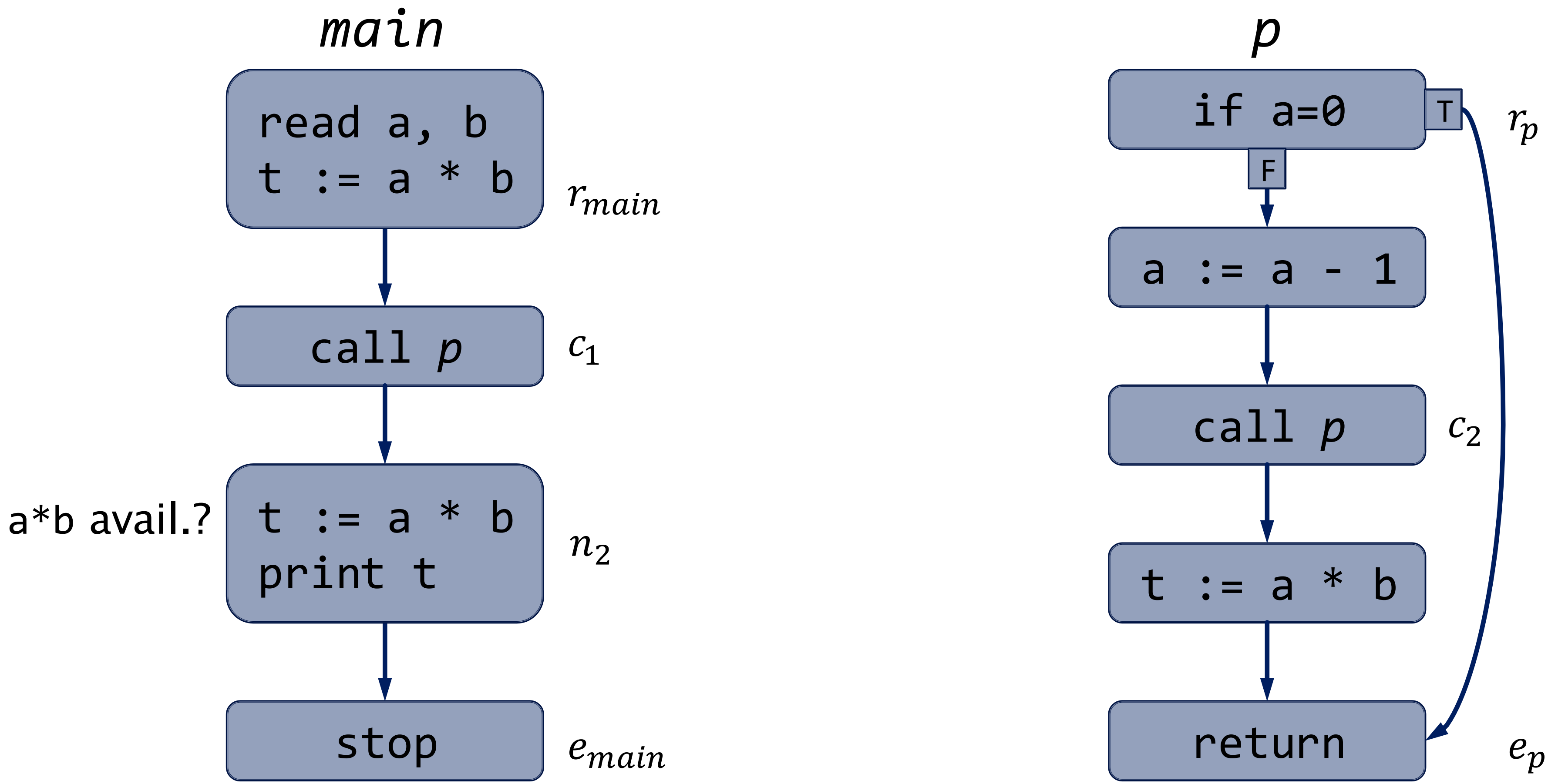
Two representations:

1. $G = \left(\bigcup_p N_p , \bigcup_p E_p\right)$ $\quad E_p = E_p^0 \cup E_p^1 \quad E^0 = \bigcup_p E_p^0$
2. $G^* = \left(\bigcup_p N_p , E^*\right)$ $\quad E^* = E^0 \cup E^1$



Basic idea

Call String (CS) approach



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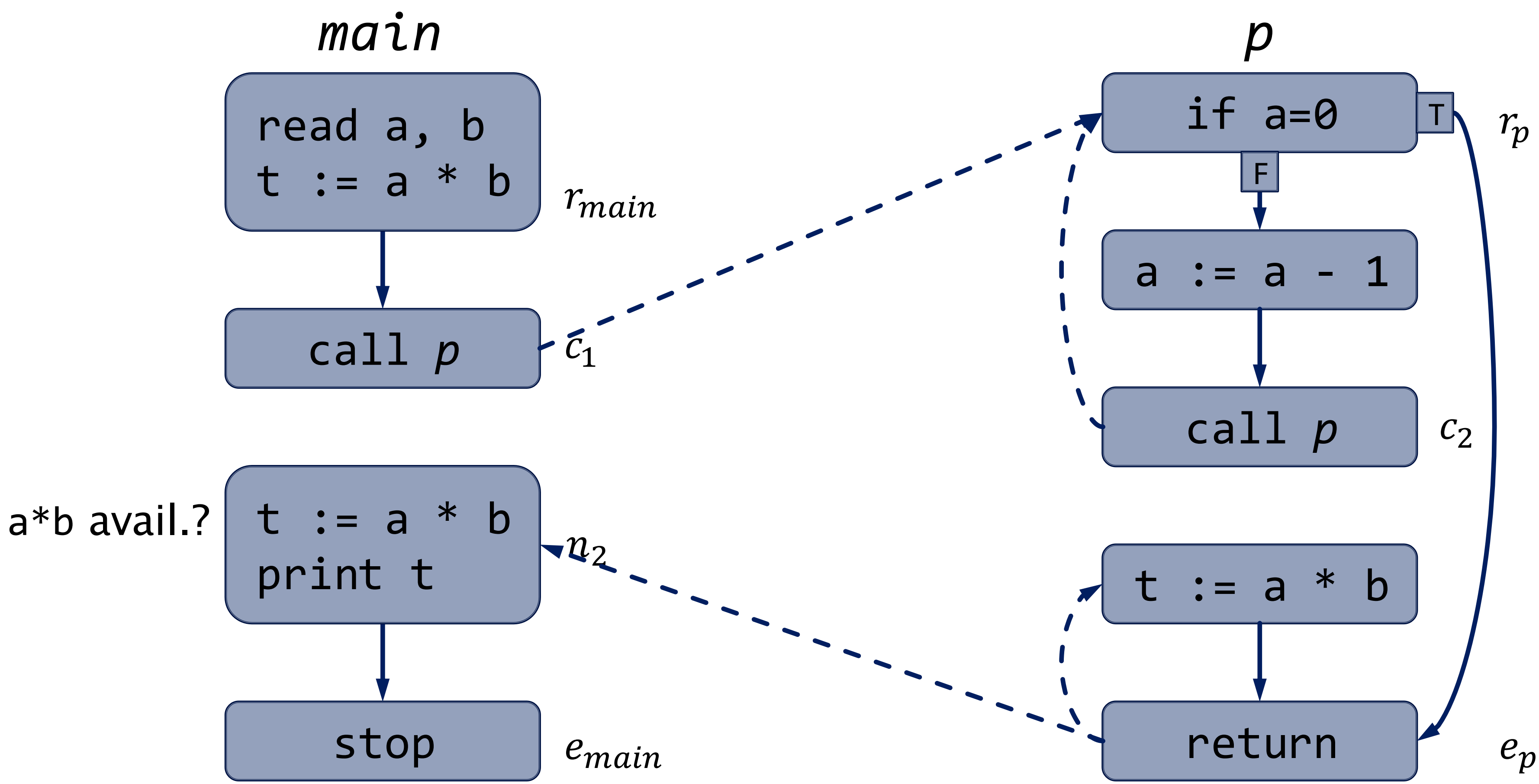

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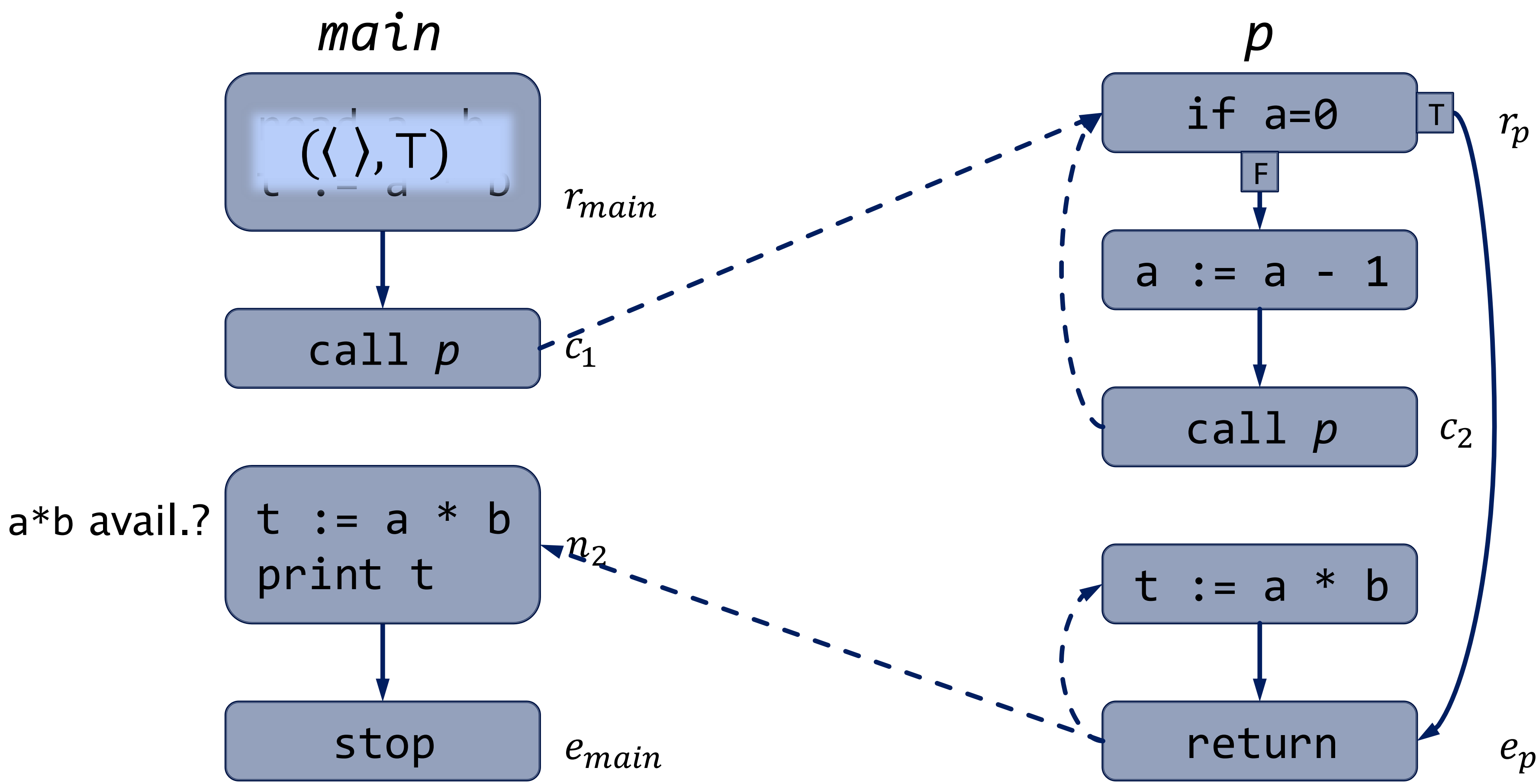

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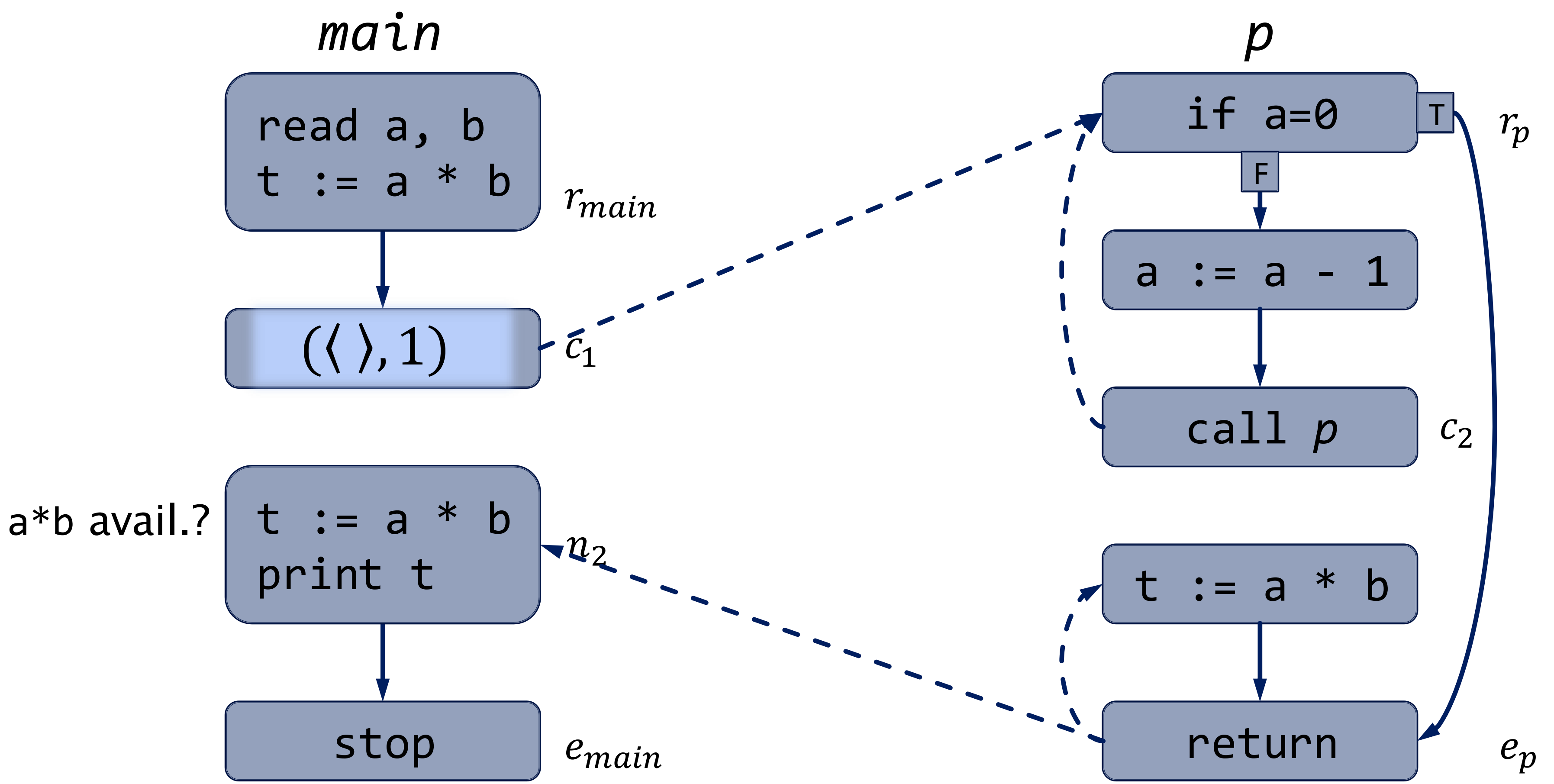

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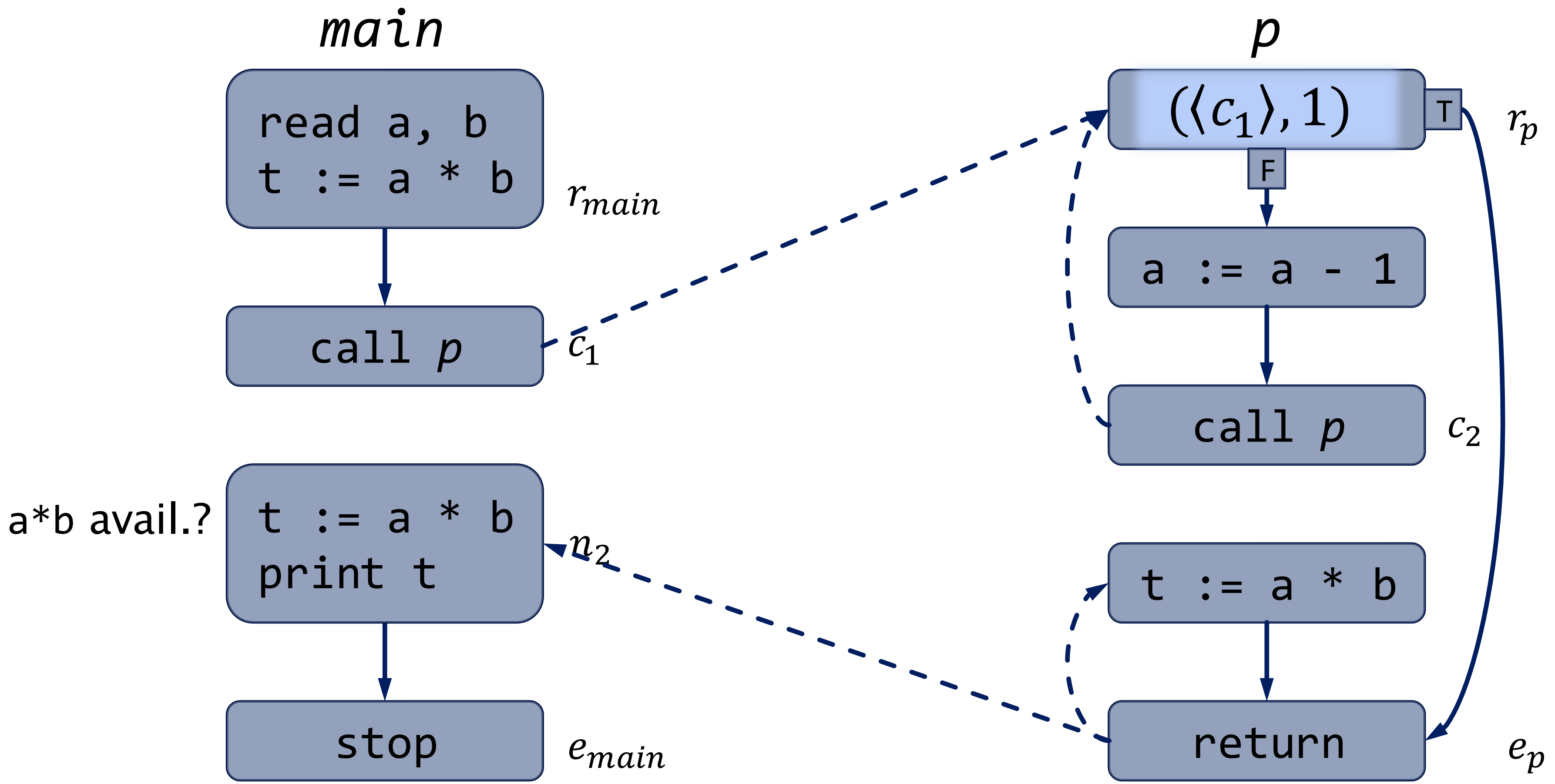

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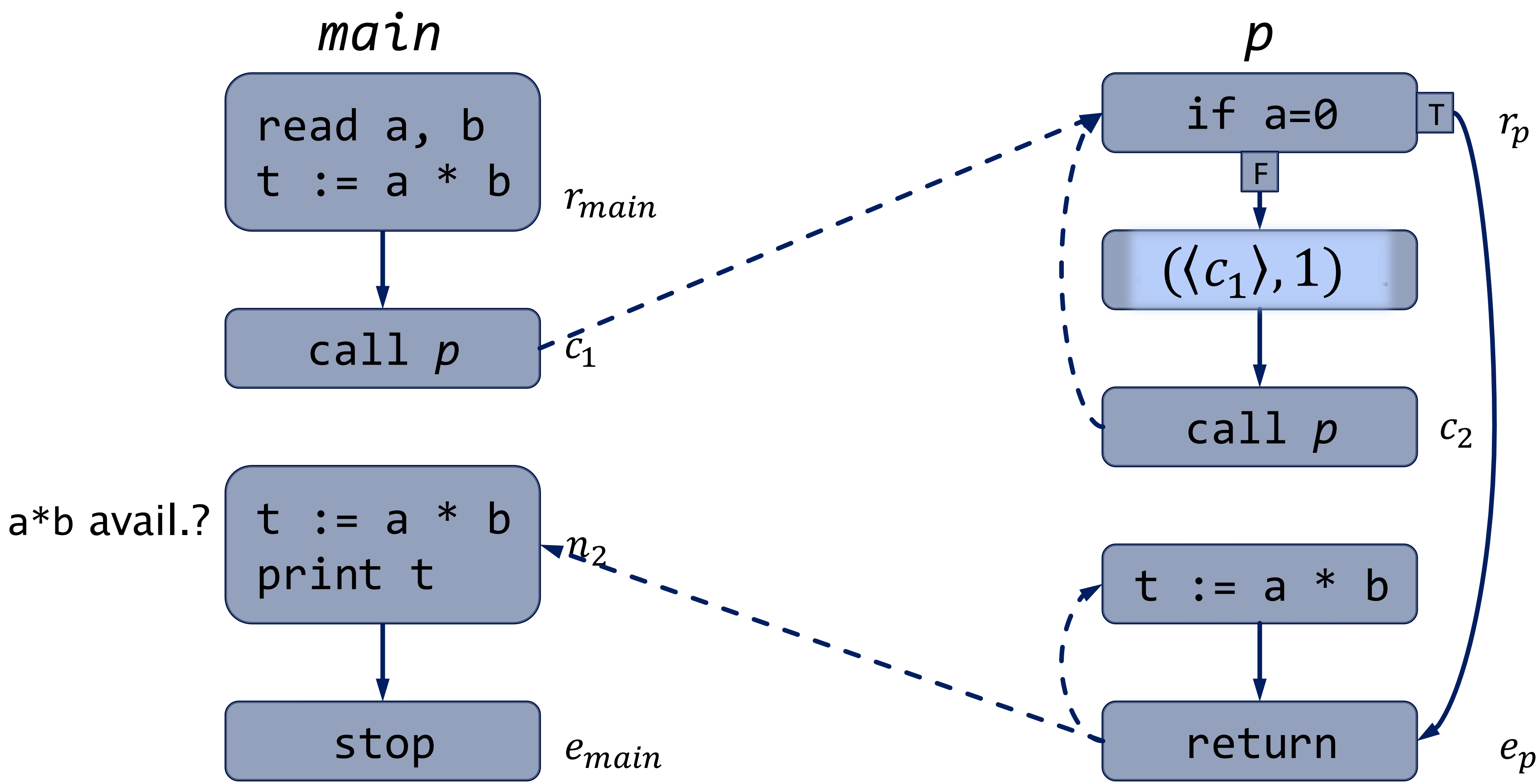

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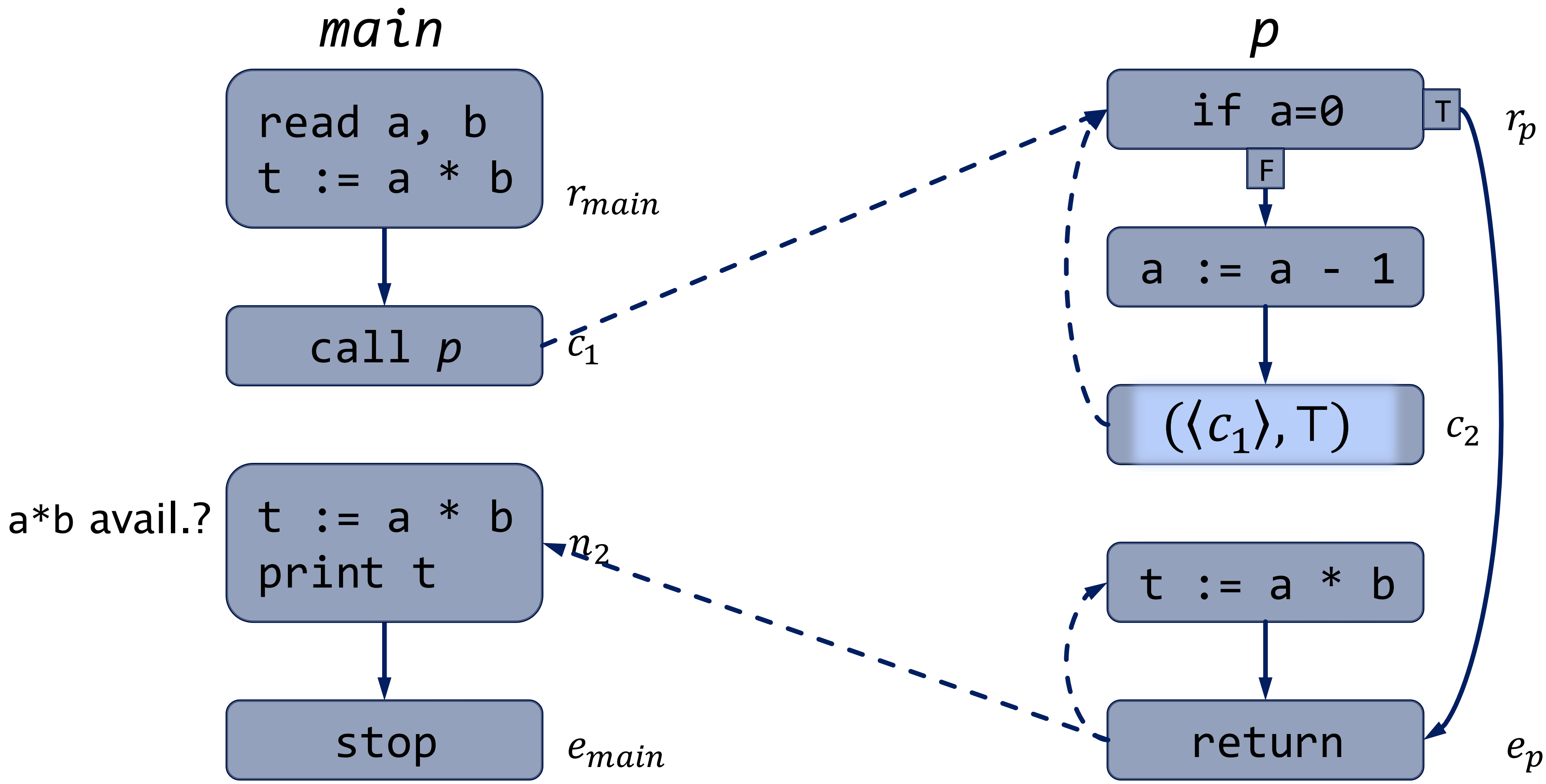

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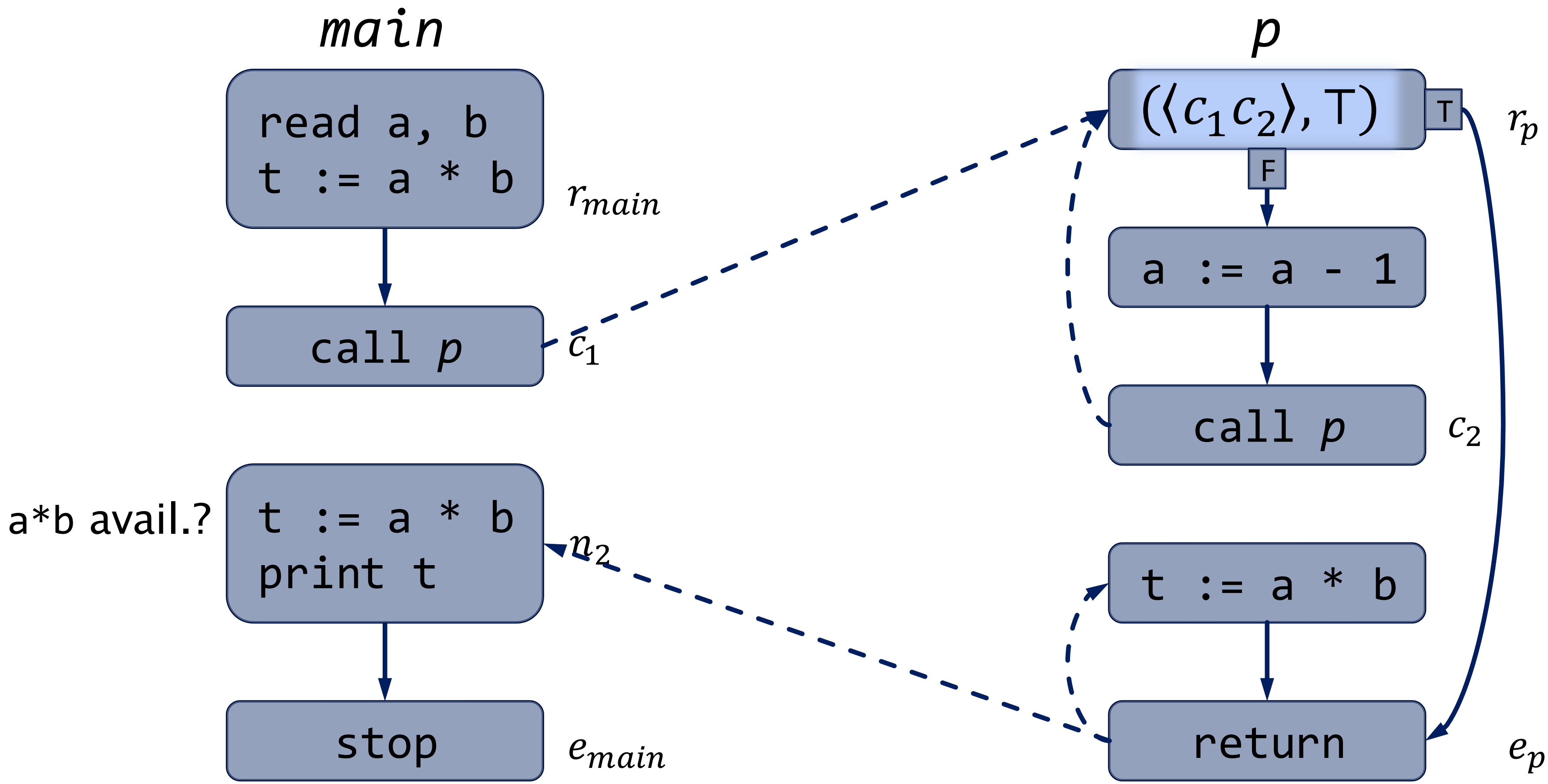

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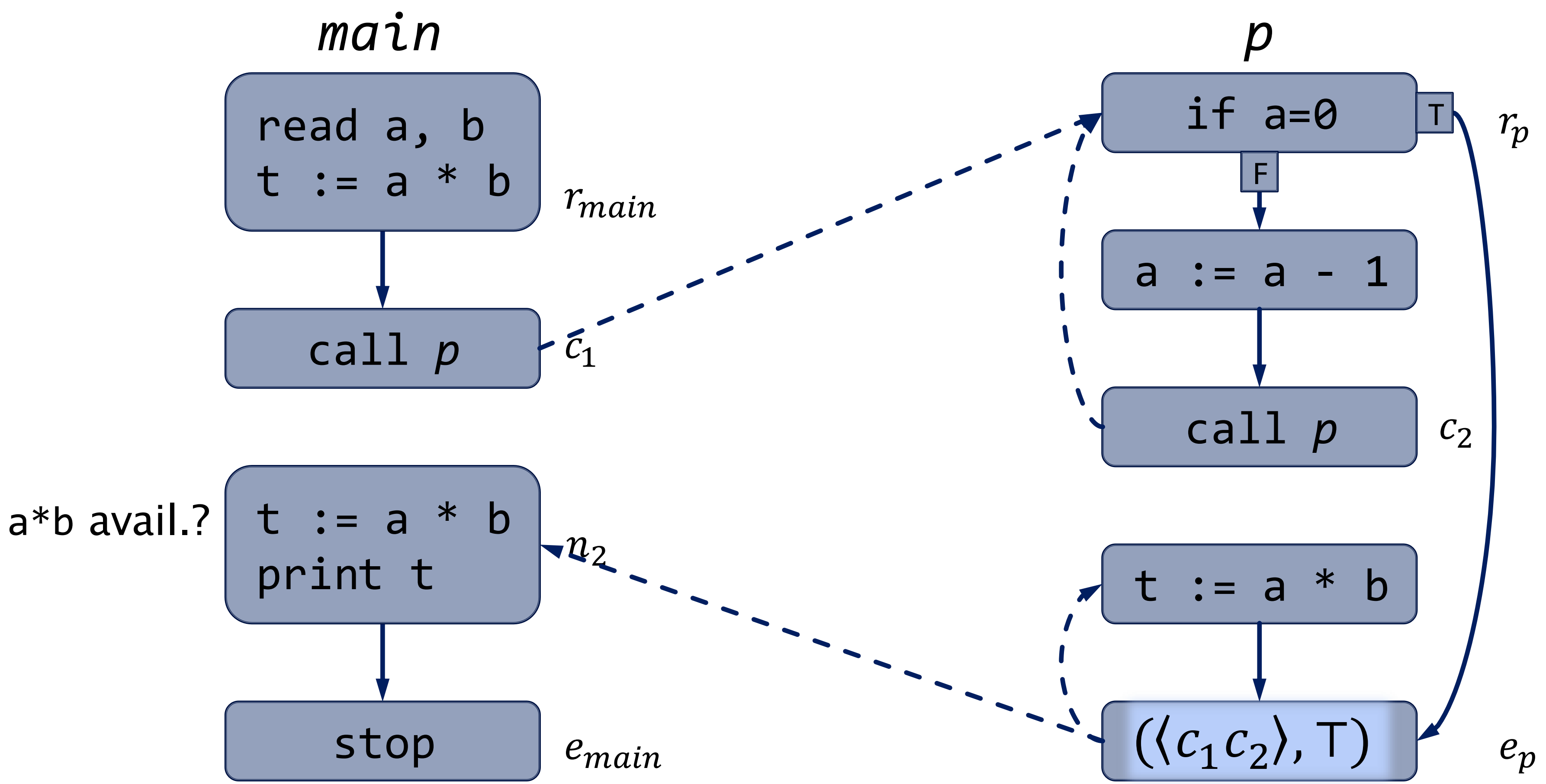

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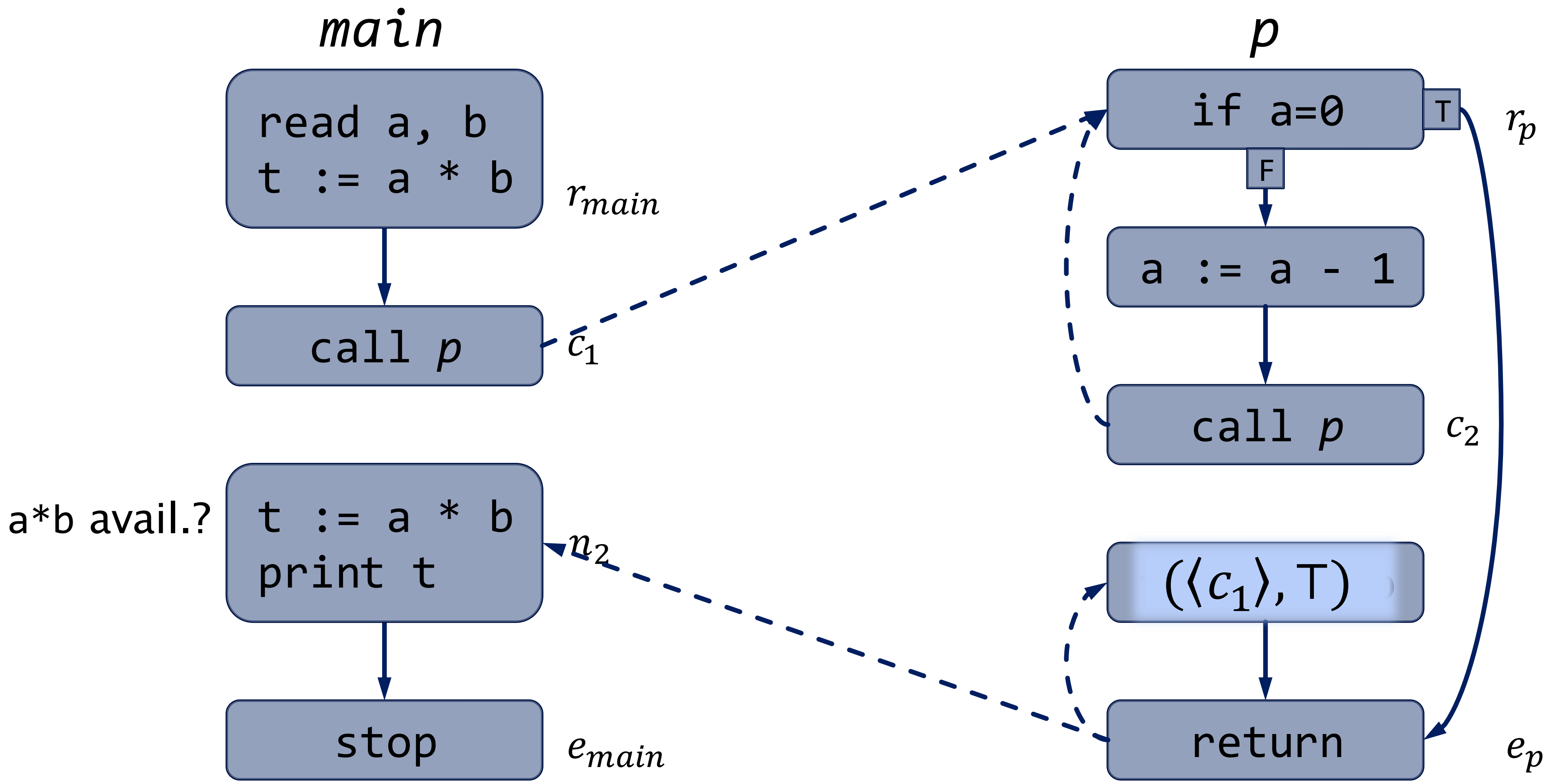

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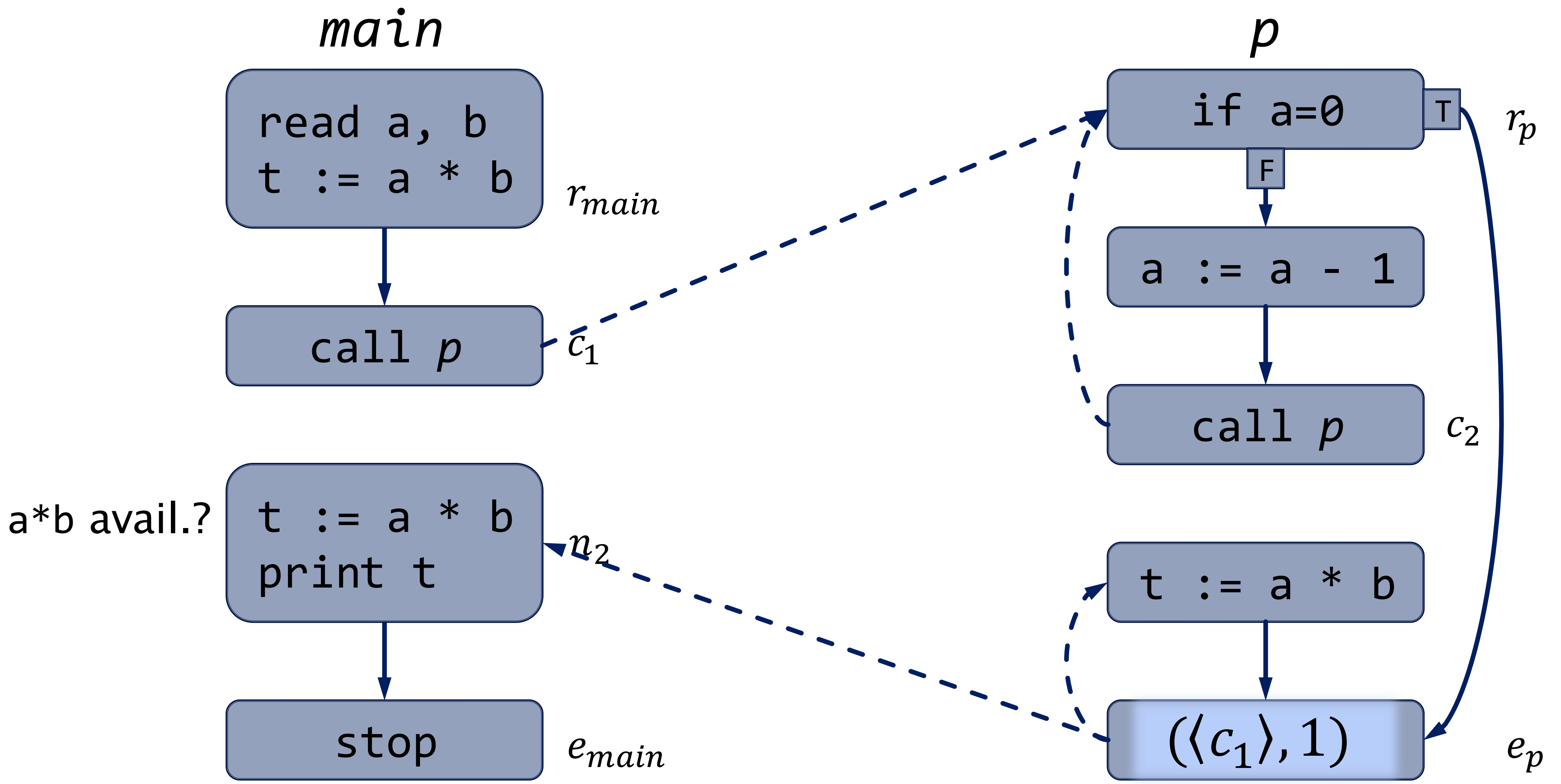

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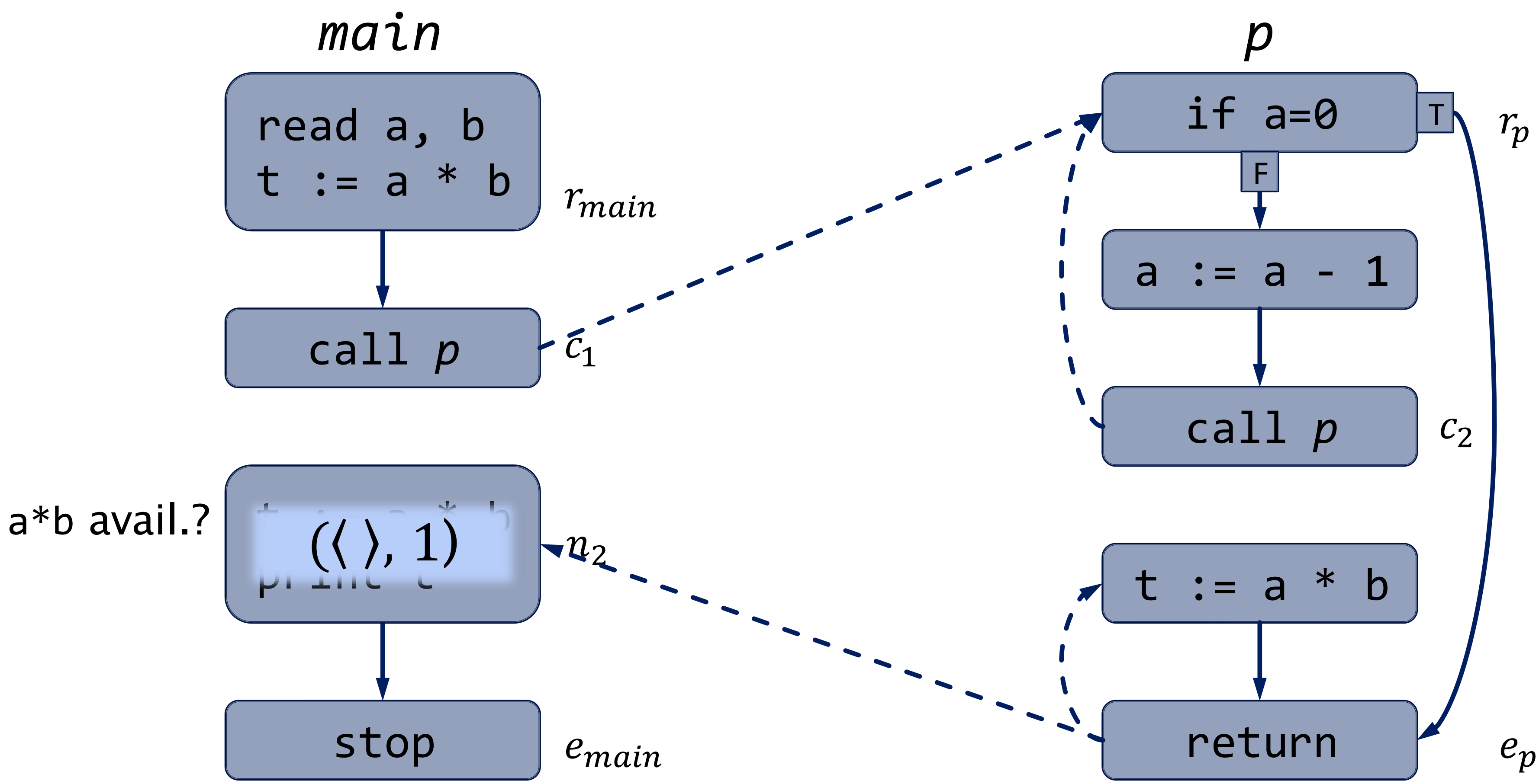
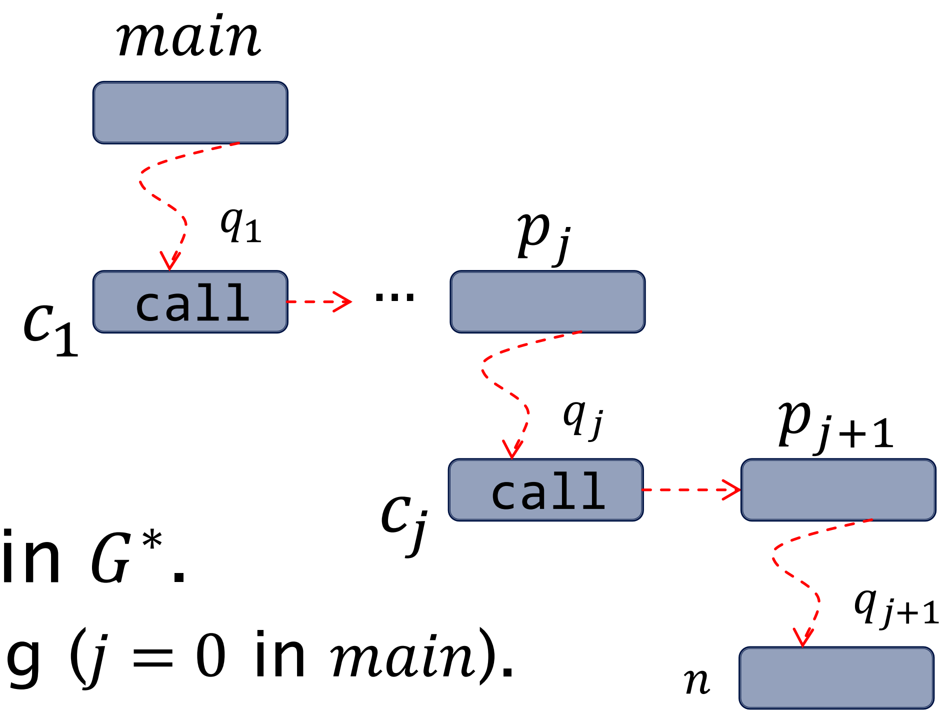


Definition: Call strings

- Let $q \in IVP(r_{main}, n)$ decomposed as:



- $\langle c_1 c_2 \ldots c_j \rangle =: \gamma \in \Gamma$ **call string (CS)** to q in G^*.
  $\lambda \in \Gamma$ is empty call string ($j = 0$ in $main$).

- $\Gamma =$ space of valid call strings in G^*

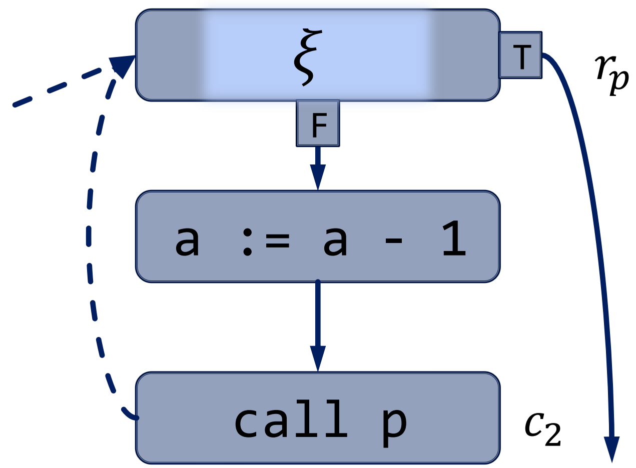
- $CM: IVP \to \Gamma$ with $CM(q) = \gamma$

New DF framework

- (L^*, F^*) := "Extended" version of (L,F)
  - Uses **interprocedural** flow graph G^*
  - L^*: Values $\in L$ tagged with call strings $\in \Gamma$
  - F^*: For data set $\xi \in \mathrm{L}^*$ apply edge effect to call string and value

Value domain L^*

- $L^* := \Gamma \to L$ semi–lattice
  - $\top^* = \gamma \mapsto \top$
  - $\bot^* = \gamma \mapsto \bot$
  - $\wedge$ pointwise in L:
    $(\xi_1 \wedge \xi_2)(\gamma) = \xi_1(\gamma) \wedge \xi_2(\gamma)$
- "$\xi \in L^*$ maps call strings γ to values in L propagated via all paths $q \in CM^{-1}(\gamma)$"



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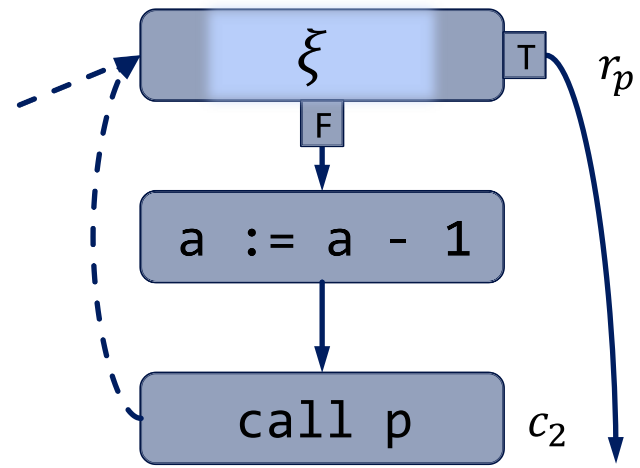


- "$\xi \in L^*$ maps call strings γ to values in L propagated via all paths $q \in CM^{-1}(\gamma)$"

$$\xi = \left\{ \begin{array}{c} (\langle c_1 \rangle, 1), \\ (\langle c_1 c_2 \rangle, \top), \\ (\langle c_1 c_2 c_2 \rangle, \top) \end{array} \right\}$$

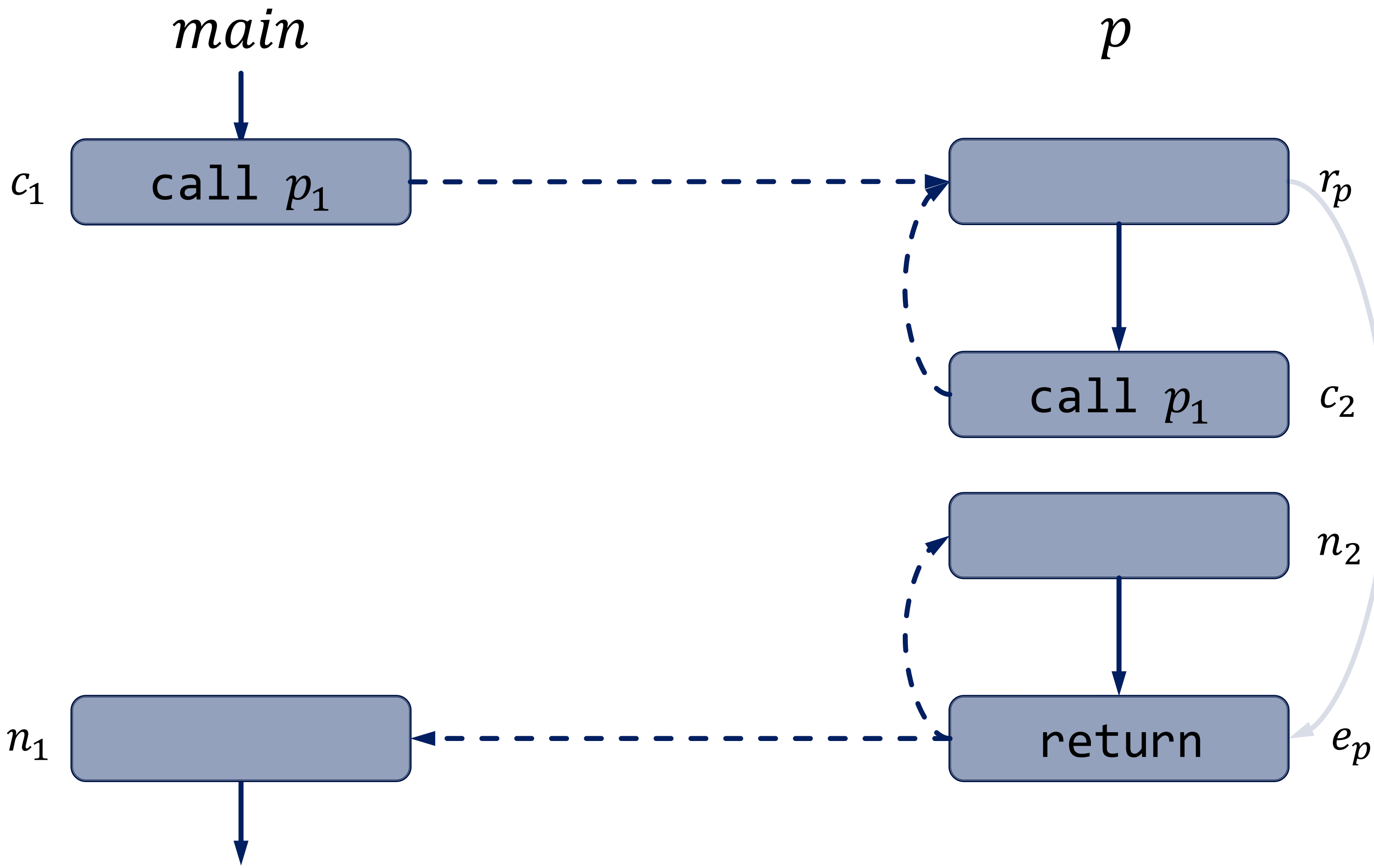
$$\xi(\langle c_1 \rangle) = 1$$
$$\xi(\langle c_2 \rangle) = \bot$$

Edge effects domain F^*

- $F^* := L^* \to L^*$ embeds edge effects from F into L^* domain
- For $(m, n) \in E^*$ and data $\xi = \{.., (\gamma', v'), ..\} \in L^*$ at m, data at n is $f^*_{(m,n)}(\xi) = \{.., (\gamma, v), ..\}$
  - update each γ' to reflect (m, n) taken: $\gamma = \gamma' \circ (m, n)$
  - propagate effect $f_{(m,n)} \in L$ applied to each v': $v = f_{(m,n)}(v')$

CS update operation $\circ$

- $\circ\colon \Gamma \times E^* \to \Gamma$ updates call strings along edges



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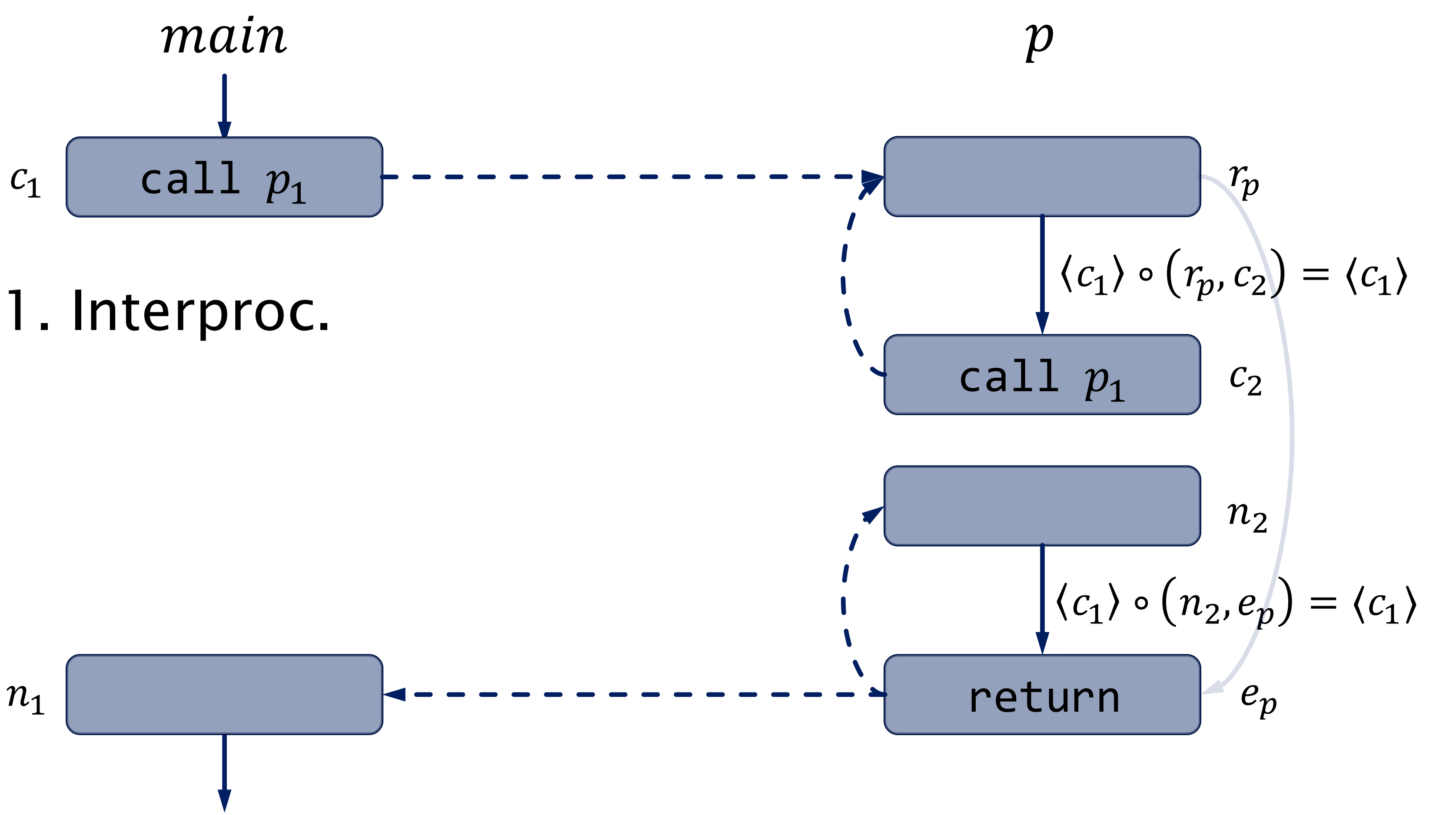
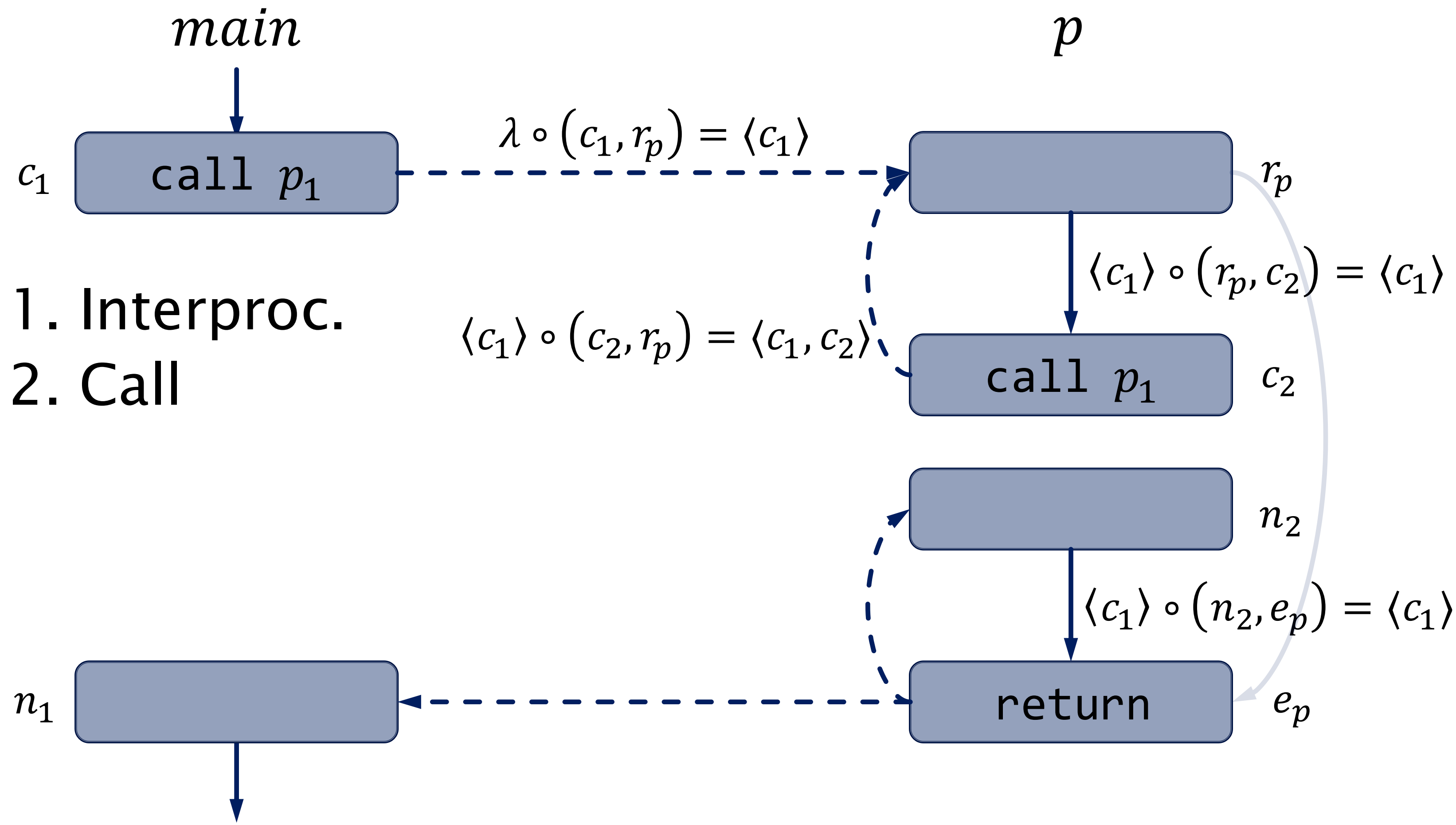


CS update operation ∘

- $\circ: \Gamma \times E^* \to \Gamma$ updates call strings along edges

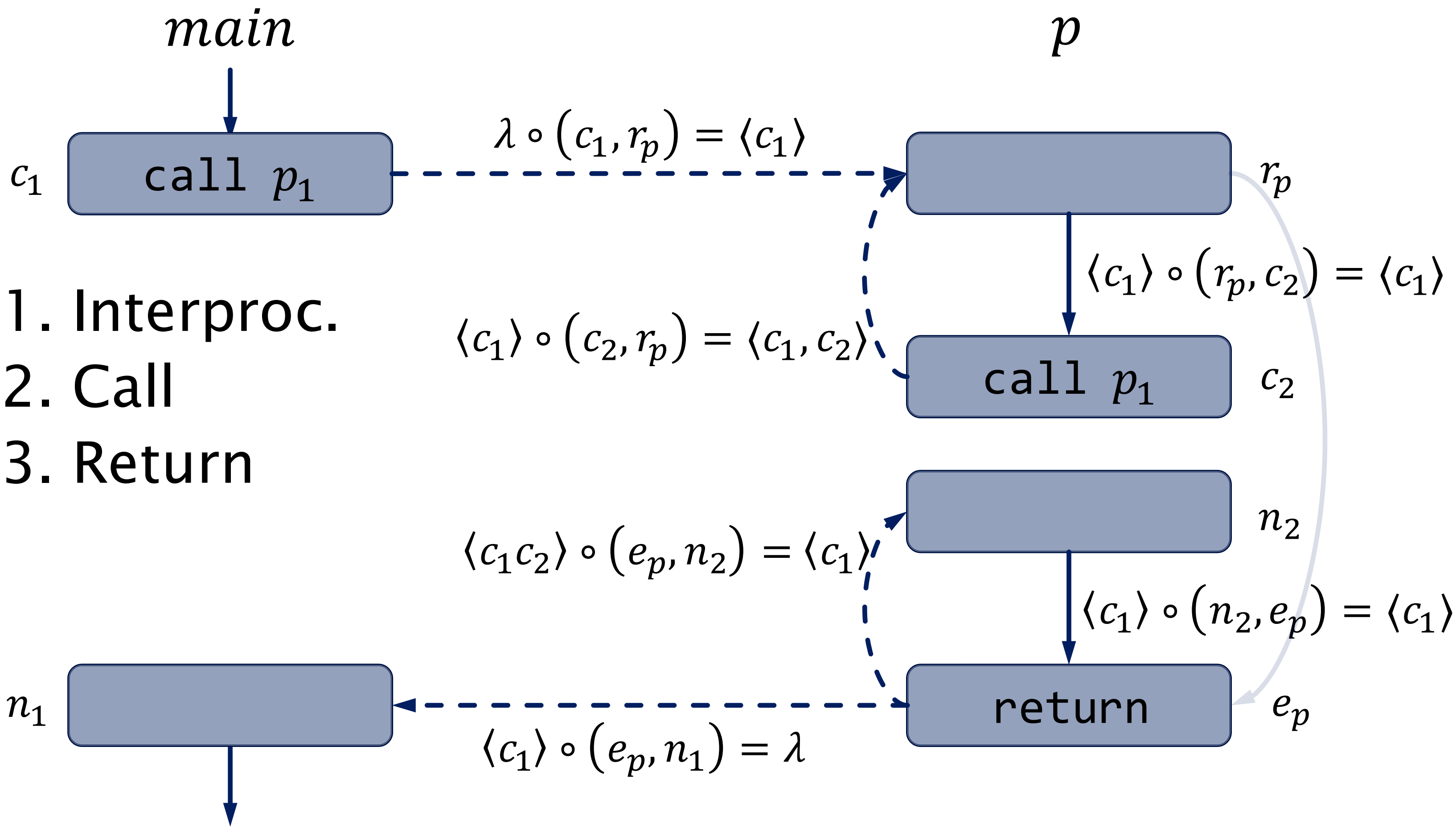


1. Interproc.
2. Call

CS update operation ∘

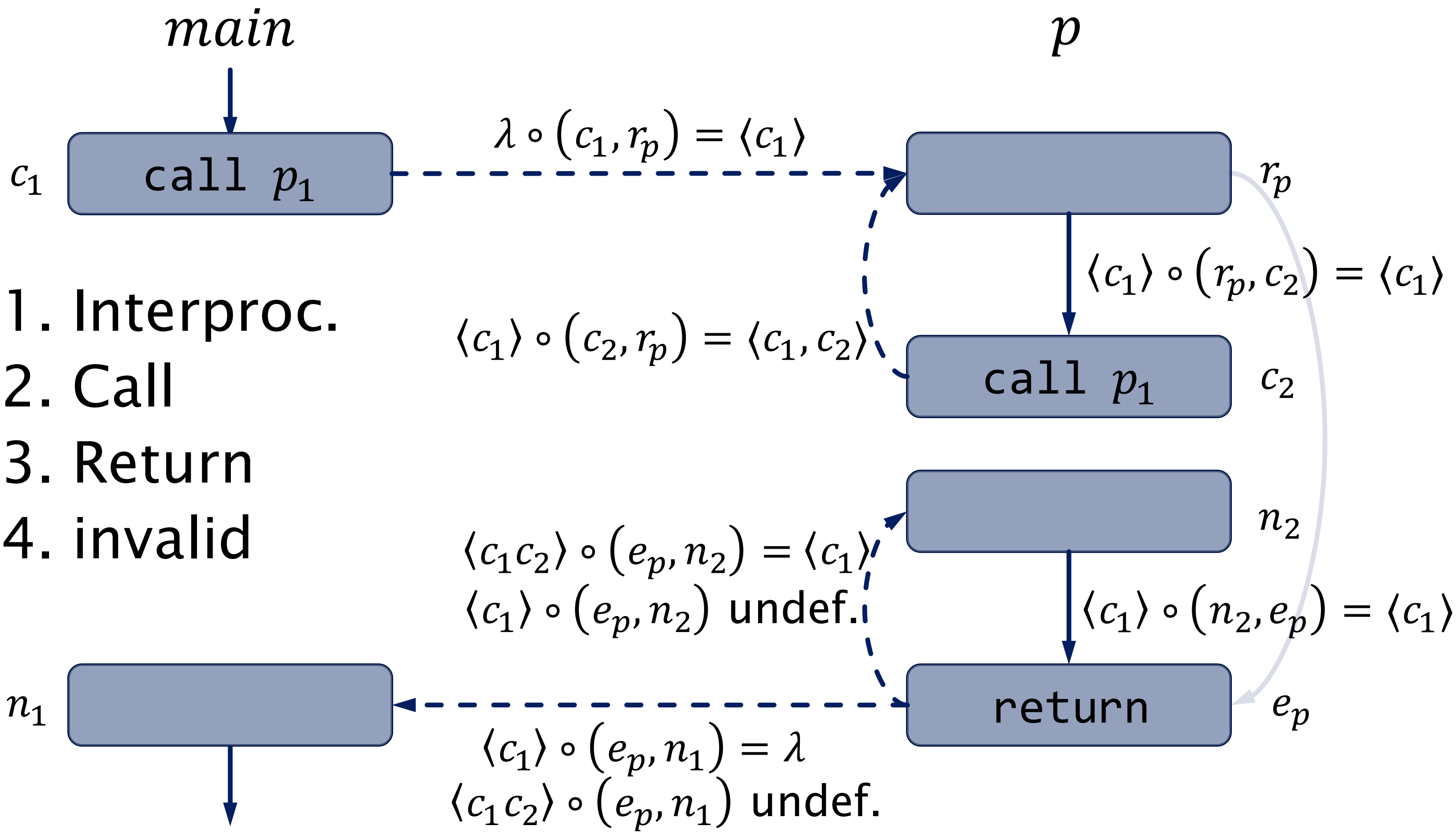
- $\circ: \Gamma \times E^* \rightarrow \Gamma$ updates call strings along edges

1. Interproc.
2. Call
3. Return



CS update operation $\circ$

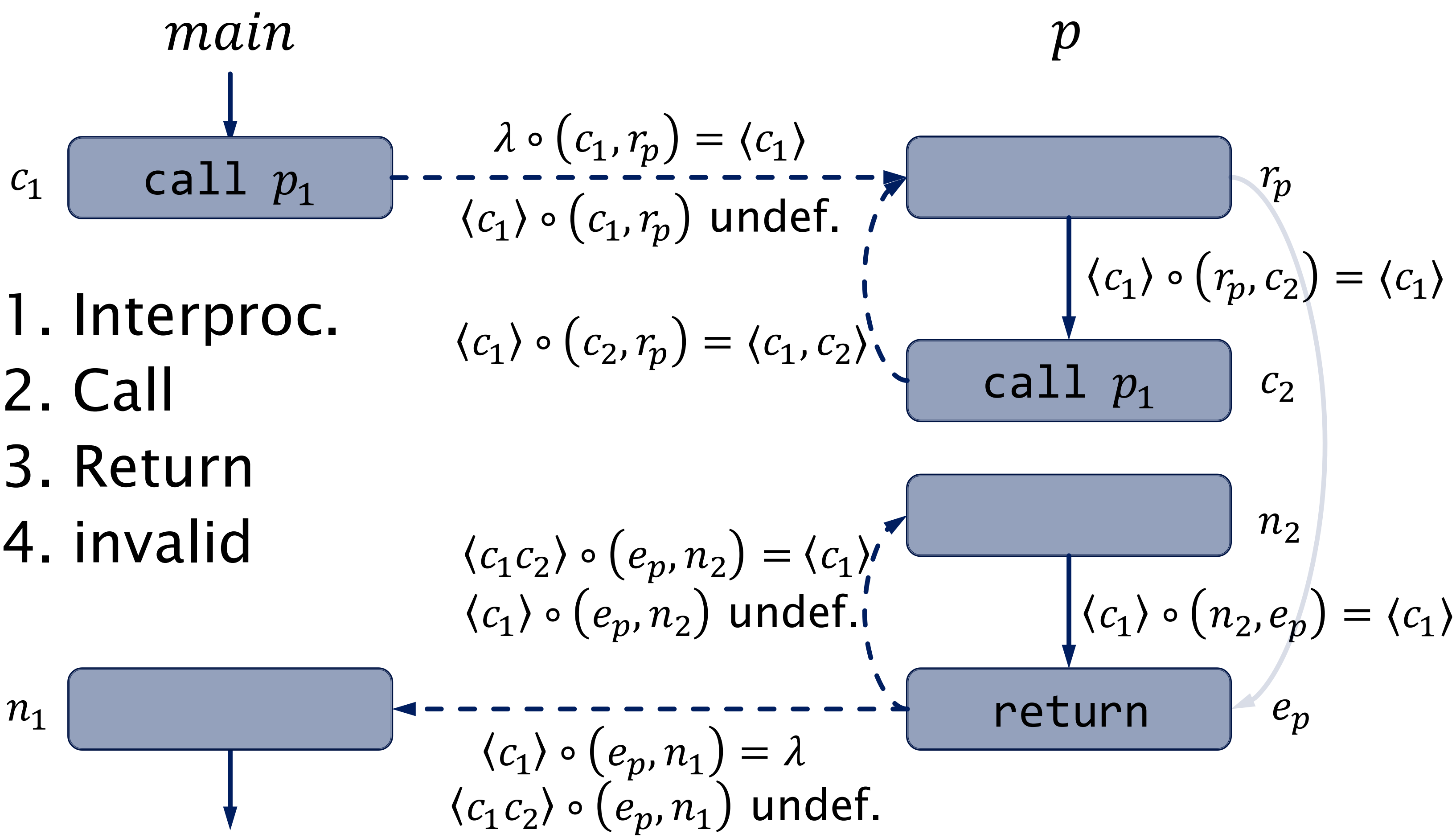
- $\circ: \Gamma \times E^* \rightarrow \Gamma$ updates call strings along edges



1. Interproc.
2. Call
3. Return
4. invalid

CS update operation $\circ$

- $\circ\colon \Gamma \times E^* \to \Gamma$ updates call strings along edges



1. Interproc.
2. Call
3. Return
4. invalid

$\circ$ consistent with IVP

- $\gamma' \circ (m, n) = \gamma$ only defined iff:

  1. (By definition)
     A path $q' \in IVP(r_{main}, m)$ exists with $CM(q') = \gamma'$

  2. (Lemma)
     $q = q' || (m, n)$ lies in $IVP(r_{main}, n)$ and $CM(q) = \gamma$

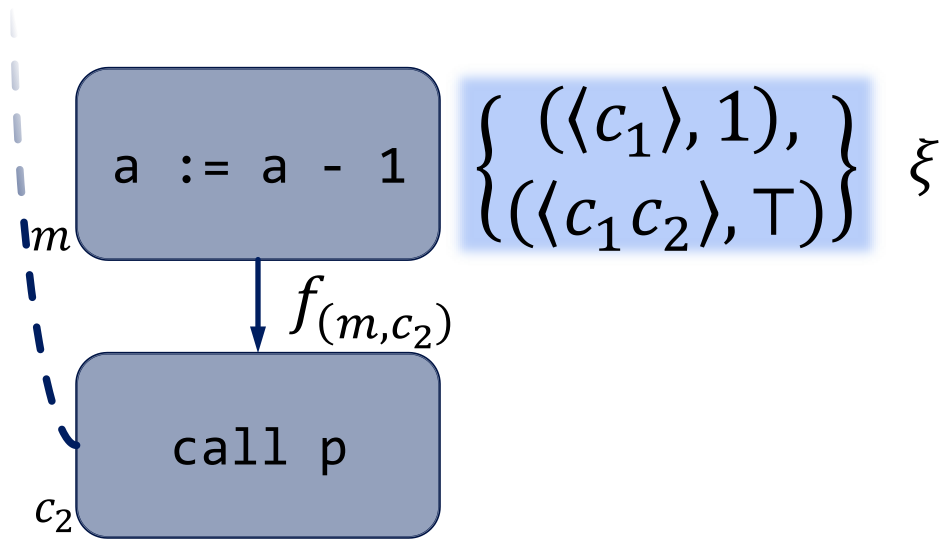
$\Rightarrow$ "$\circ$ only updates and yields call strings for interprocedurally valid paths"

Edge effects domain F^*

- $F^* := L^* \to L^*$ embeds edge effects from F into L^* domain
- For $(m,n) \in E^*$ and data $\xi = \{..,(\gamma',v'),..\} \in L^*$ at m, data at n is $f^*_{(m,n)}(\xi) = \{..,(\gamma,v),..\}$
  - update each γ' to reflect (m,n) taken: $\gamma = \gamma' \circ (m,n)$
  - propagate effect $f_{(m,n)} \in L$ applied to each v': $v = f_{(m,n)}(v')$

Edge effects $f^*_{(m,n)}$

Definition: Let $(m, n) \in E^*,\ f_{(m,n)} \in F, \xi \in L^*$.



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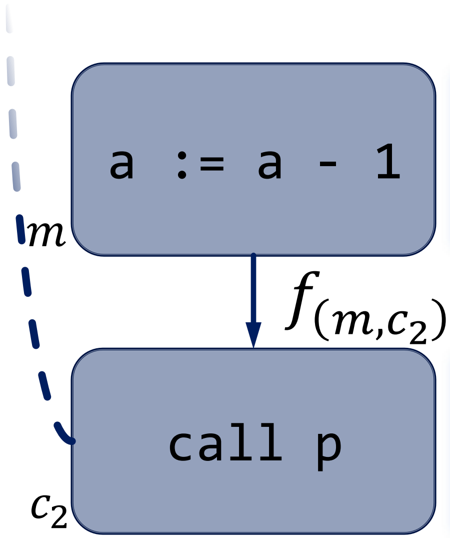
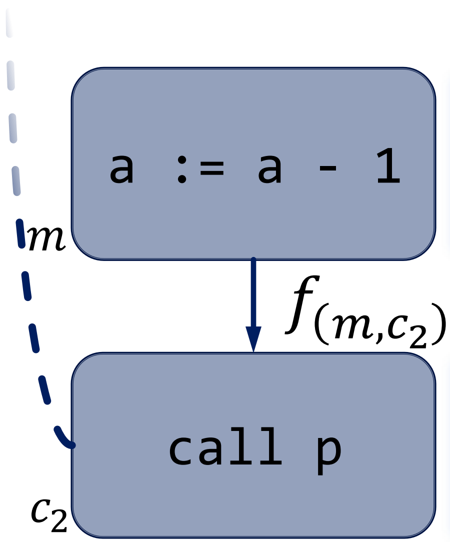


$$\left\{\begin{matrix} ((\langle c_1 \rangle, 1), \\ ((\langle c_1 c_2 \rangle, \top) \end{matrix}\right\} \quad \xi$$

$$\left\{\begin{matrix} ((\langle c_1 \rangle, \top), \\ ((\langle c_1 c_2 \rangle, \top) \end{matrix}\right\} \quad f^*_{(m,c_2)}(\xi)$$

Edge effects $f^*_{(m,n)}$

Definition: Let $(m,n) \in E^*,\ f_{(m,n)} \in F, \xi \in L^*$.



$$\xi = \left\{ \begin{array}{l} (\langle c_1 \rangle, 1), \\ (\langle c_1 c_2 \rangle, \top) \end{array} \right\}$$

$$f^*_{(m,c_2)}(\xi) = \left\{ \begin{array}{l} (\langle c_1 \rangle, \top), \\ (\langle c_1 c_2 \rangle, \top) \end{array} \right\}$$

$\xi(\langle c_1 \rangle)$

$f^*_{(m,c_2)}(\xi)(\langle c_1 \rangle)$

$\forall\, \gamma \in \Gamma$

$$f^*_{(m,n)}(\xi)(\gamma) := \begin{cases} f_{(m,n)}\big(\xi(\gamma')\big) & \text{if } \exists \gamma' \text{ with } \gamma = \gamma' \circ (m,n) \\ \bot & \text{else} \end{cases}$$

Edge effects domain F^*

Define $F^* \subseteq L^* \to L^*$ such that

1. F^* smallest set containing
$$\left\{f^*_{(m,n)} \middle| (m,n) \in E^*\right\} \cup \{id_{L^*}\}$$

2. F^* closed under functional composition and $\wedge$

Lemma: Properties of F^*

1. $F \begin{Bmatrix} \text{monotone} \\ \text{distributive} \end{Bmatrix}$ in $L \Rightarrow F^* \begin{Bmatrix} \text{monotone} \\ \text{distributive} \end{Bmatrix}$ in L^*

2. F distributive in $L \Rightarrow F^*$ **continuous** in L^*:

   For an infinite collection $\{\xi_k\}_{k\geq 1} \subseteq L^*$ and $(m,n) \in E^*$:

$$f^*_{(m,n)}\left(\bigwedge_k \xi_k\right) = \bigwedge_k f^*_{(m,n)}(\xi_k)$$

Dataflow equations x_n^*

- The dataflow problem for (F^*, L^*):
  - $x_{r_{main}}^* = \{(\lambda, \top)\}$
  - $x_n^* = \bigwedge_{(m,n)\in E^*} \left(f_{(m,n)}^*(x_m^*) \right) \qquad n \in N^* - \{r_{main}\}$
- **Claim:**
  ∃ maximum fixed-point solution (***MFP***$_{CS}$) for $\{x_n^*\}_{n\in N^*}$

Proof: Existence of max. FP solution

- Iteratively solving the equations yields decreasing (F^* monotone) chains
$$x_n^{*(i)} \geq x_n^{*(i+1)} \quad \forall n \in N^*.$$
Then $\forall \gamma \in \Gamma$
$$x_n^{*(i)}(\gamma) \geq x_n^{*(i+1)}(\gamma)$$
a decreasing chain in L with limit $\lim x_n^*(\gamma)$ (exists as L bounded).

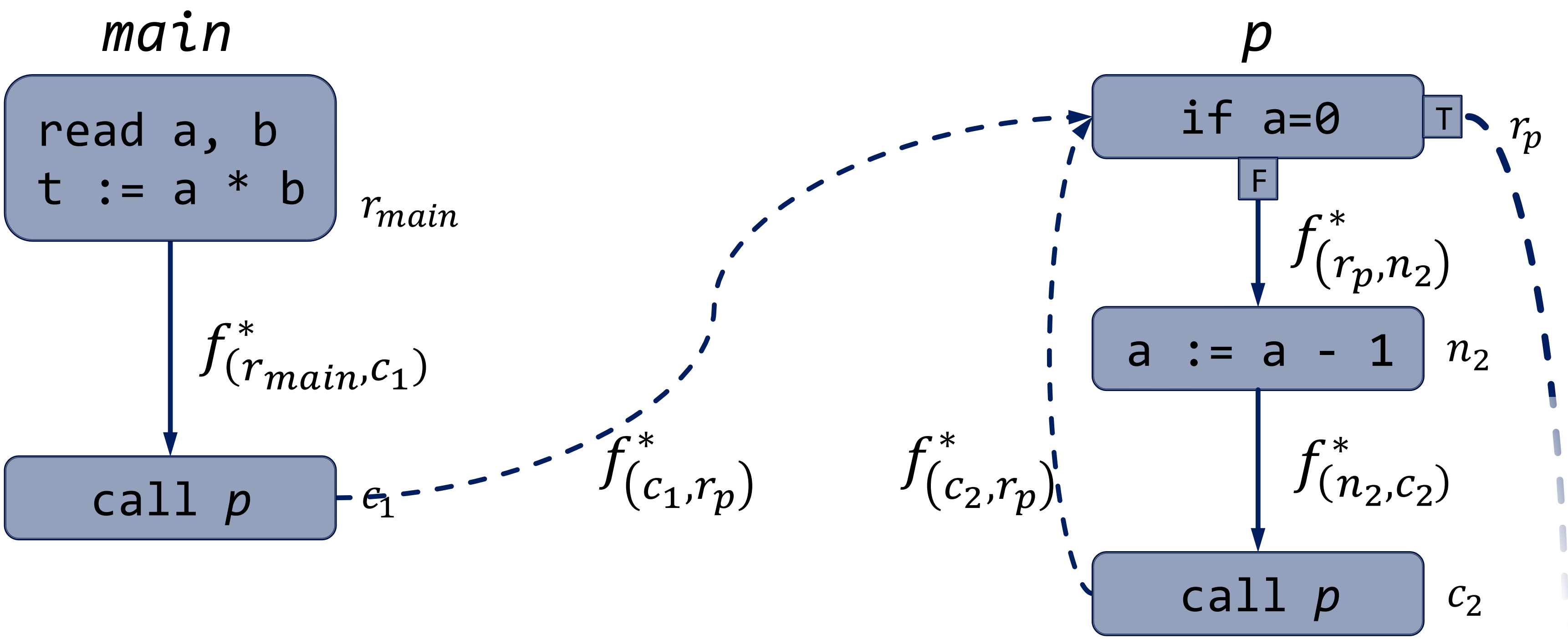
- $x_n^* \coloneqq \lim_{i \to \infty} x_n^{*(i)}$ possibly infinite but well-defined

$\Rightarrow \{x_n^*\}_{n \in N^*}$ is maximal FP solution to DF problem.

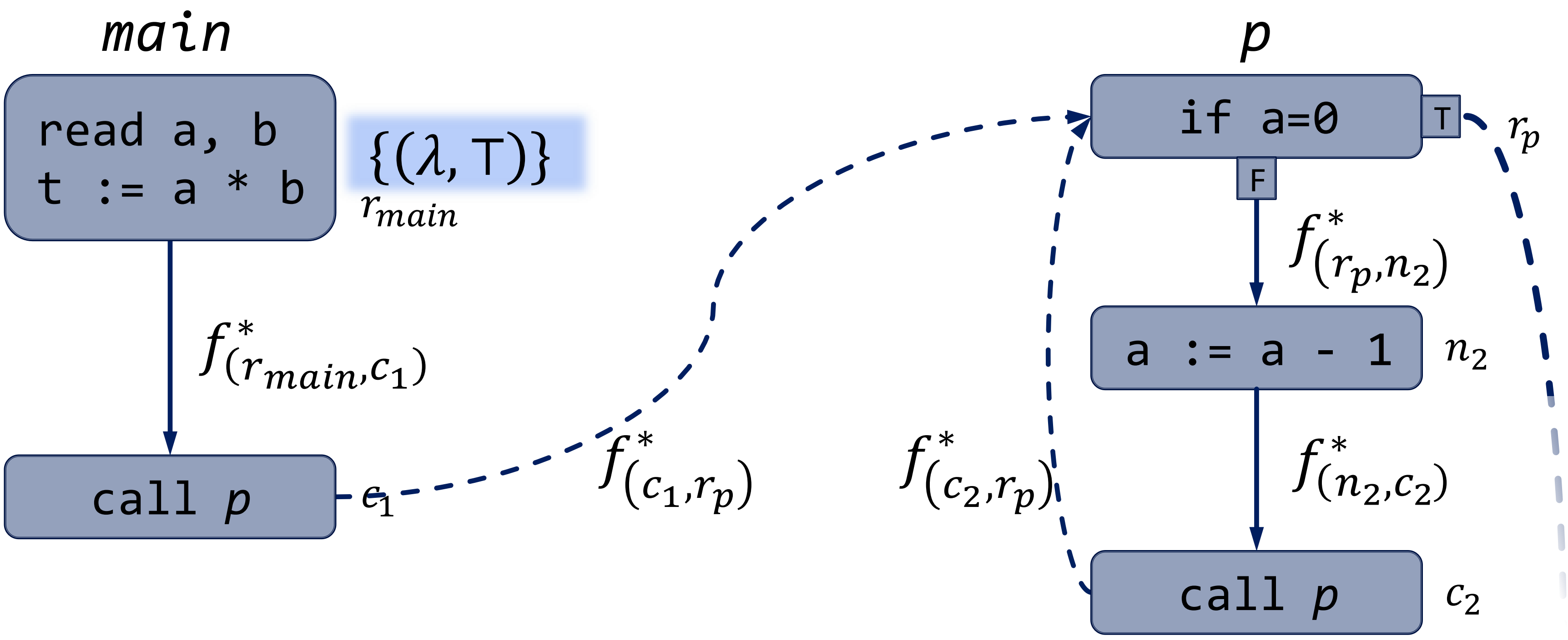
Maximum FP solution of (L^*, F^*)

- Initialise $\{x_n^*\}_{n \in N^*}$ with
  - $x_{r_{main}}^{*(0)} = \{(\lambda, \top)\}$
  - $x_n^{*(0)} = \bot^*$ $\quad n \in N^* - \{r_{main}\}$
- Result if Γ
  - **finite**: Any iterative algorithm terminates and reaches MFP.
  - **infinite**: Iteration may diverge, intermediate steps *unsafe* approx. of MFP

Example


main
read a, b
t := a * b
r_{main}
$f^*_{(r_{main},c_1)}$
call p
c_1
$f^*_{(c_1,r_p)}$
p
if a=0
T
F
r_p
$f^*_{(r_p,n_2)}$
a := a - 1
n_2
$f^*_{(c_2,r_p)}$
$f^*_{(n_2,c_2)}$
call p
c_2

Example



main
read a, b
t := a * b
{(λ, T)}
r_{main}
$f^*_{(r_{main},c_1)}$
call p
c_1
$f^*_{(c_1,r_p)}$
p
if a=0
T
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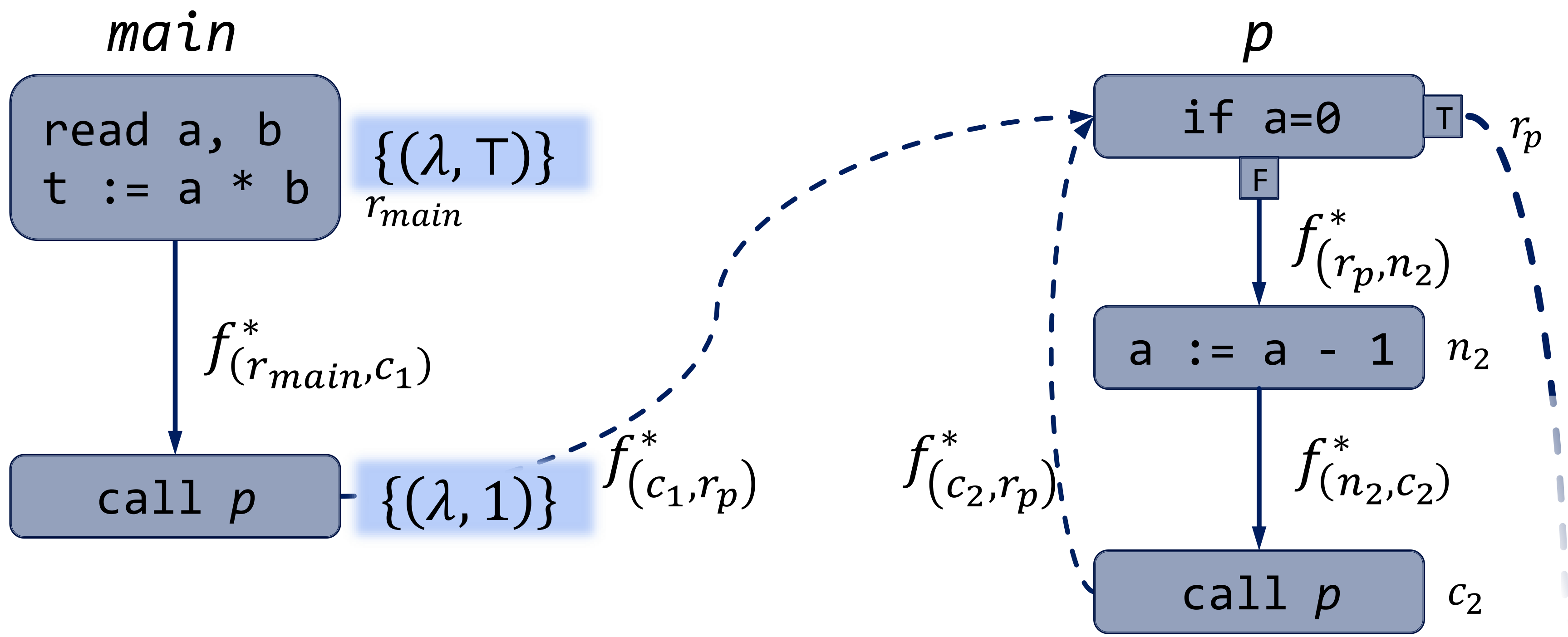
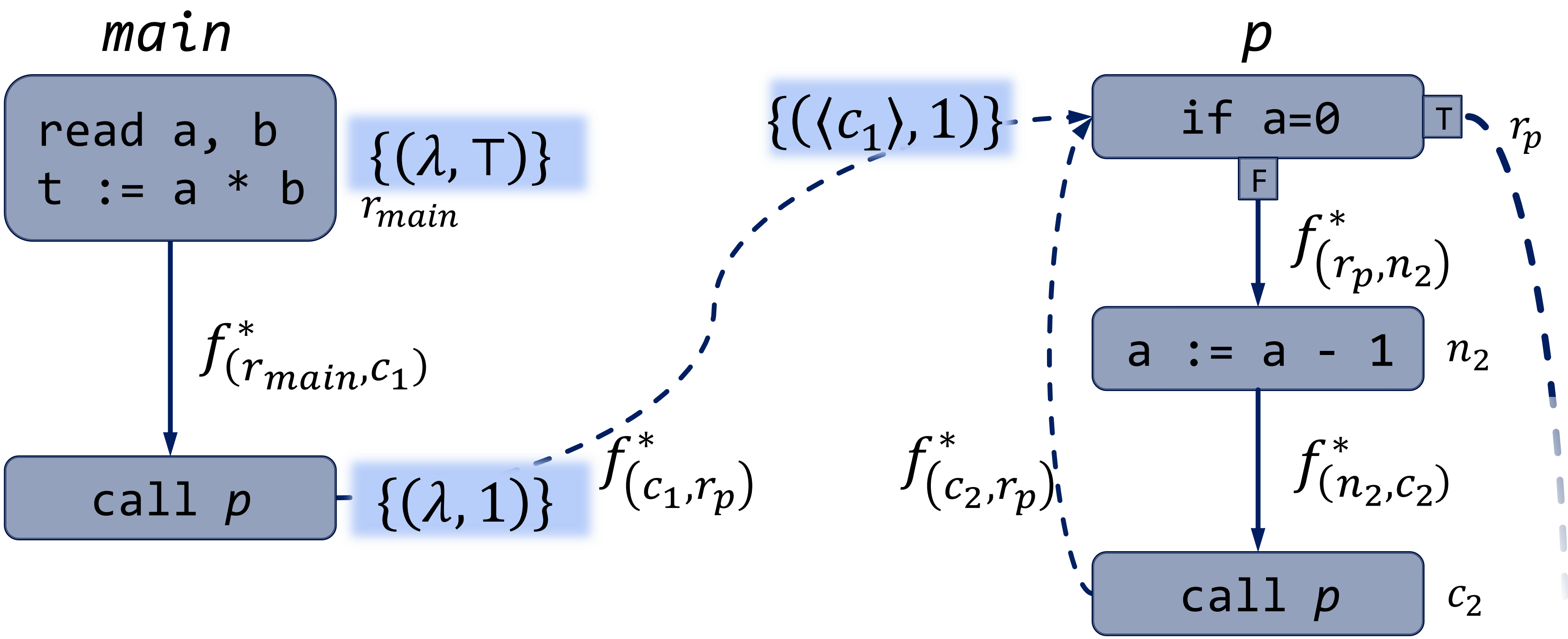


main
read a, b
t := a * b
{(λ, T)}
r_{main}
$f^*_{(r_{main},c_1)}$
call p
{(λ, 1)}
$f^*_{(c_1,r_p)}$
p
if a=0
T
r_p
F
$f^*_{(r_p,n_2)}$
a := a - 1
n_2
$f^*_{(c_2,r_p)}$
$f^*_{(n_2,c_2)}$
call p
c_2

Example



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{(λ, T)}
r_{main}
$f^*_{(r_{main},c_1)}$
call p
{(λ, 1)}
$f^*_{(c_1,r_p)}$
{((⟨c₁⟩, 1)}
p
if a=0
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F
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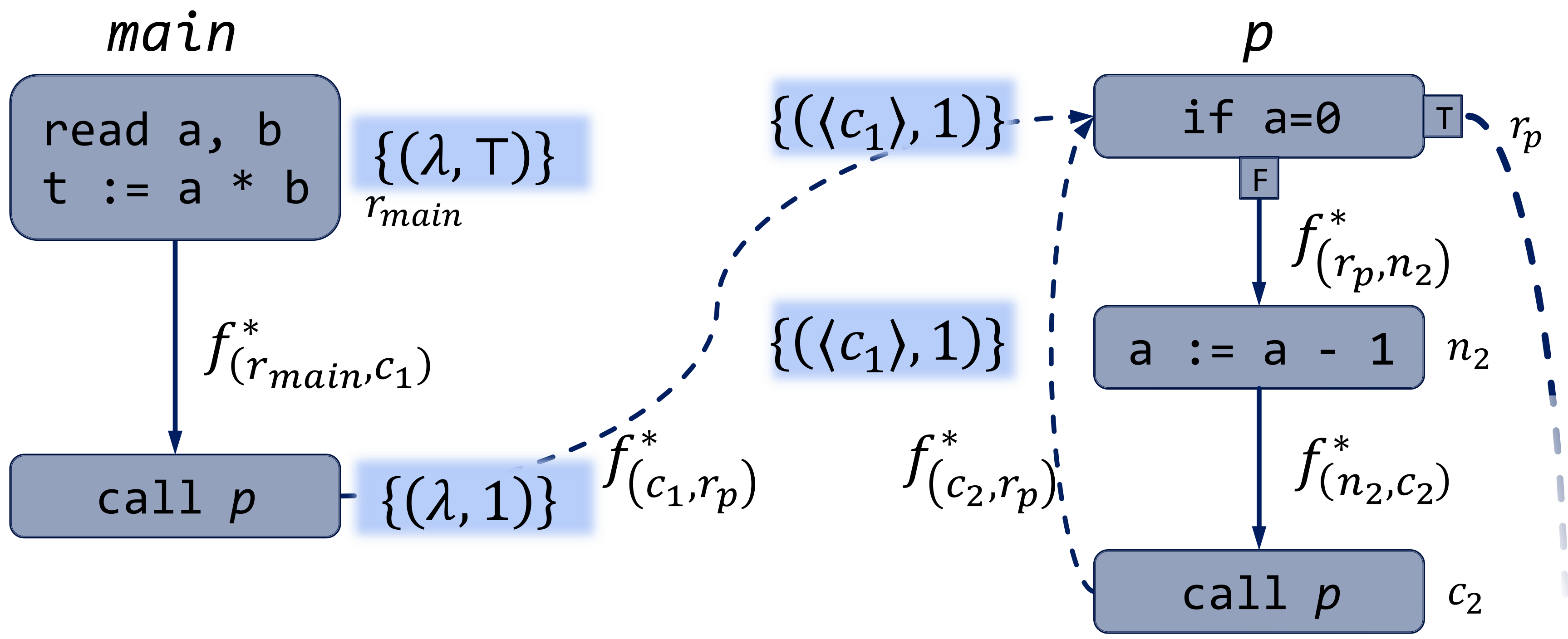

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read a, b
t := a * b
$\{(\lambda, \top)\}$
r_{main}
$f^*_{(r_{main},c_1)}$
call p
$\{(\lambda, 1)\}$
$f^*_{(c_1,r_p)}$
$\{(\langle c_1 \rangle, 1)\}$
$\{(\langle c_1 \rangle, 1)\}$
$f^*_{(c_2,r_p)}$
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if a=0
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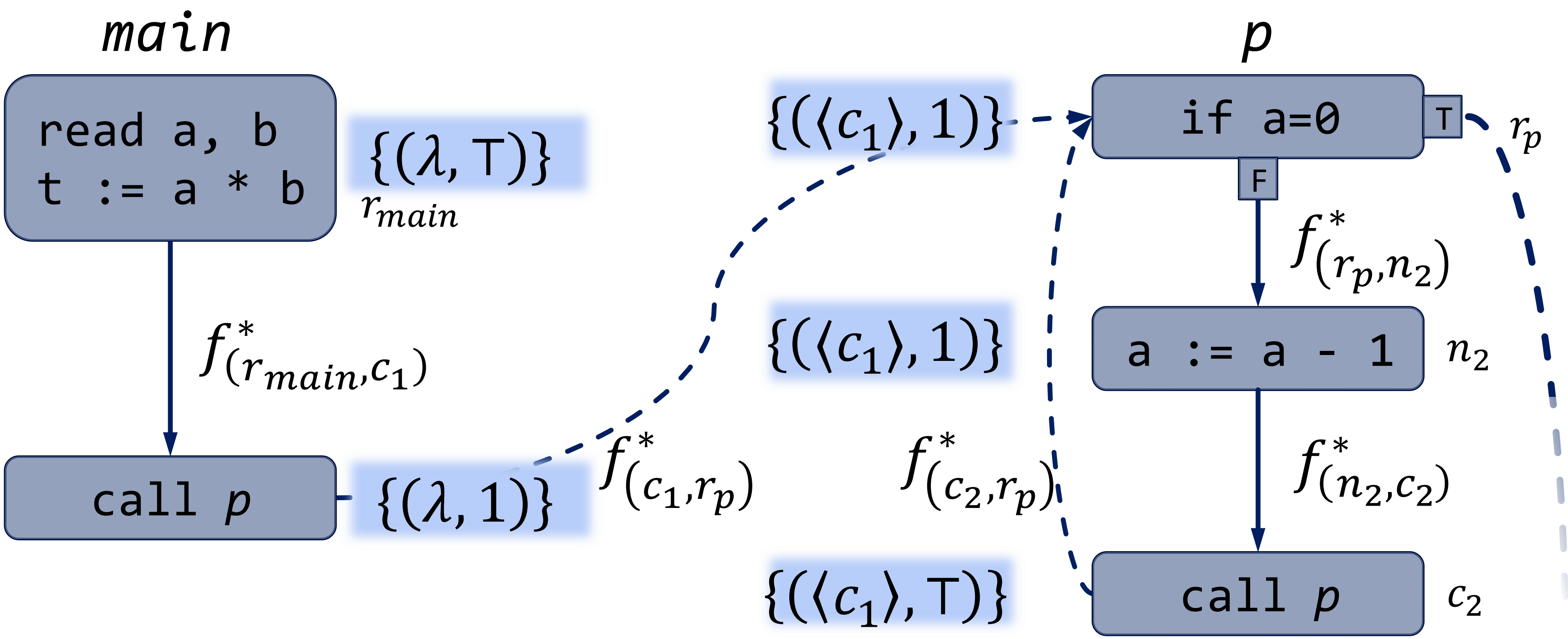
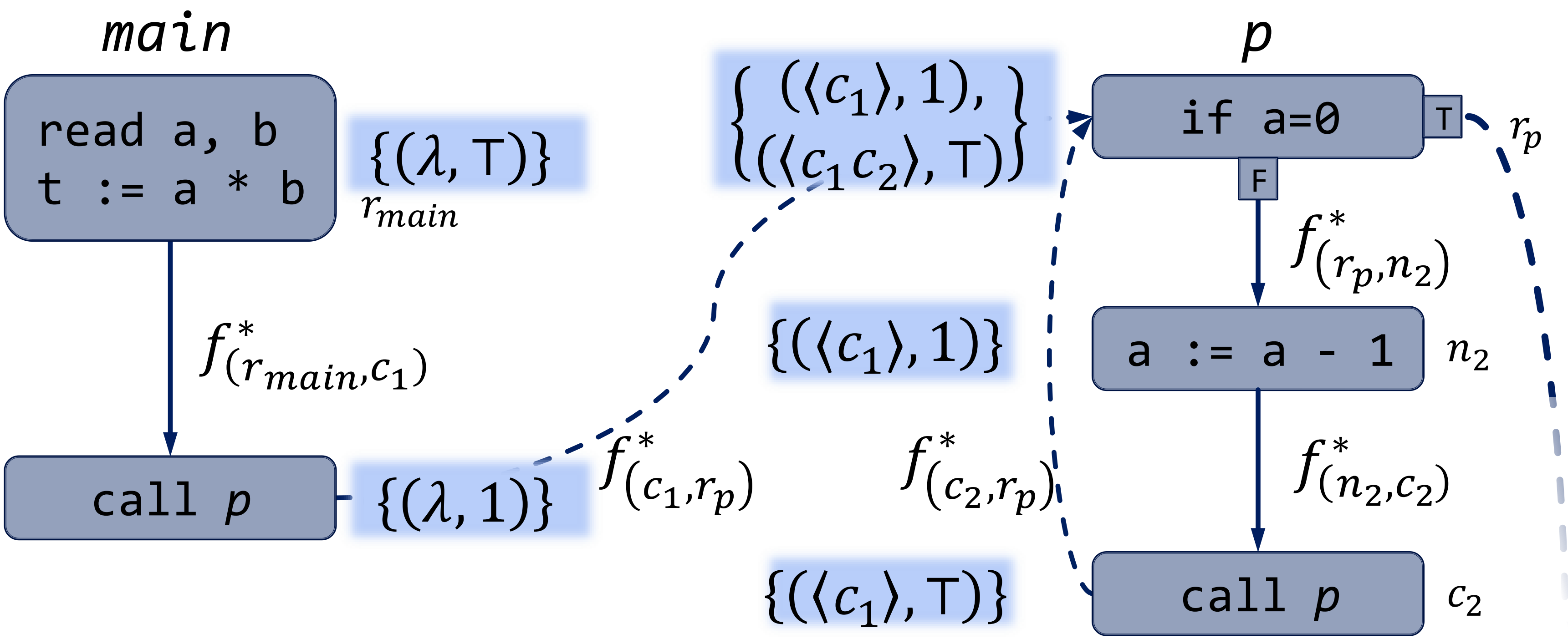


main
p
read a, b
t := a * b
{(λ, T)}
r_main
f*(r_main, c1)
call p
{(λ, 1)}
f*(c1, rp)
{((⟨c1⟩, 1)}
{((⟨c1⟩, 1)}
f*(c2, rp)
{((⟨c1⟩, T)}
if a=0
T
F
rp
f*(rp, n2)
a := a - 1
n2
f*(n2, c2)
call p
c2

Example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t := a * b
{(λ, ⊤)}
r_{main}
$f^*_{(r_{main},c_1)}$
call p
{(λ, 1)}
$f^*_{(c_1,r_p)}$
{ ((⟨c_1⟩, 1), ((⟨$c_1 c_2$⟩, ⊤)) }
{((⟨c_1⟩, 1)}
{((⟨c_1⟩, ⊤)}
$f^*_{(c_2,r_p)}$
p
if a=0
T
F
r_p
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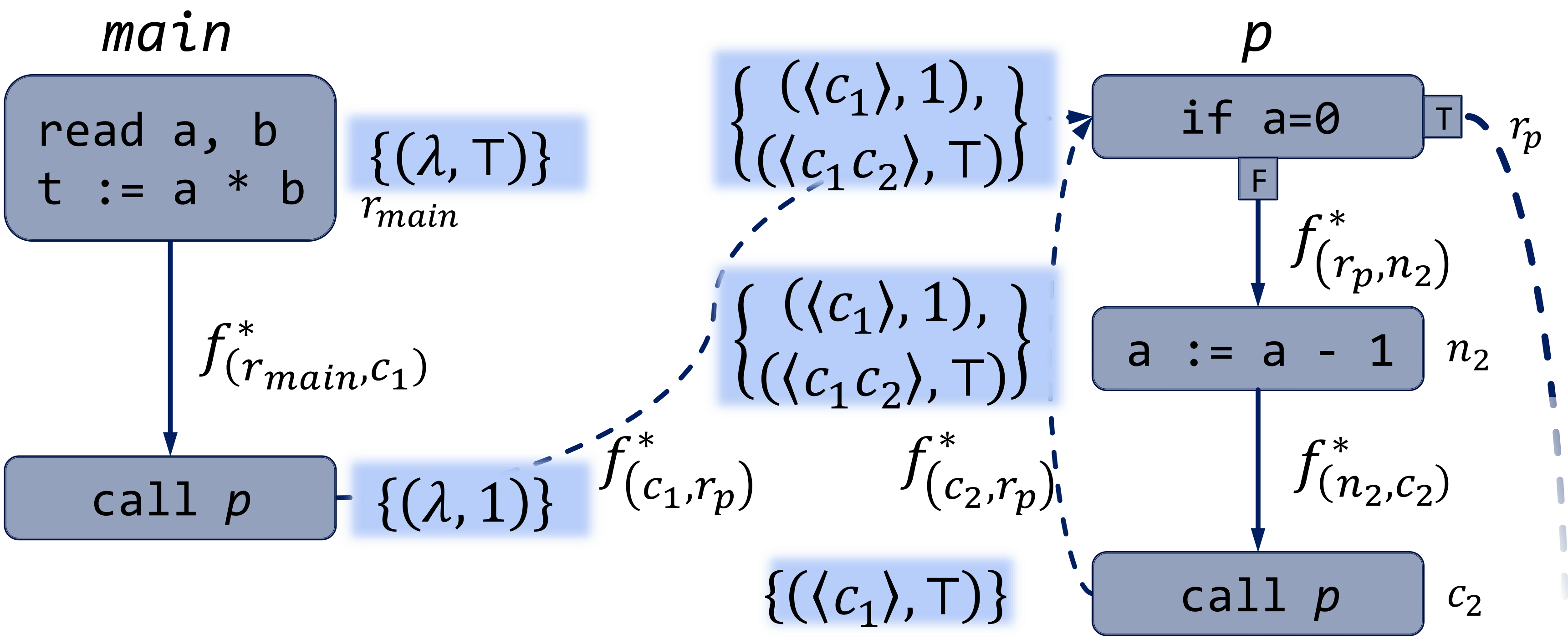
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$f^*_{(c_2, r_p)}$
$\{(\langle c_1 \rangle, \top)\}$
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if a=0
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F
r_p
$f^*_{(r_p, n_2)}$
a := a - 1
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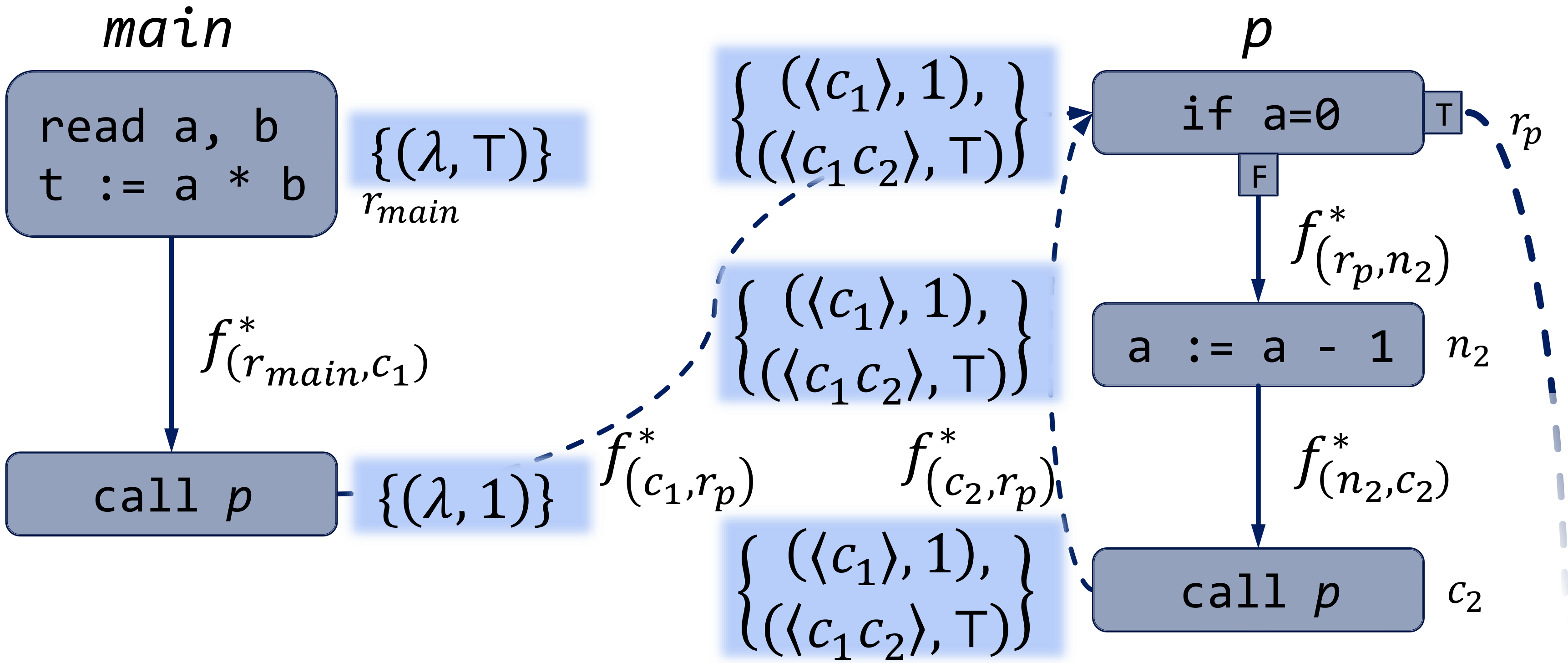
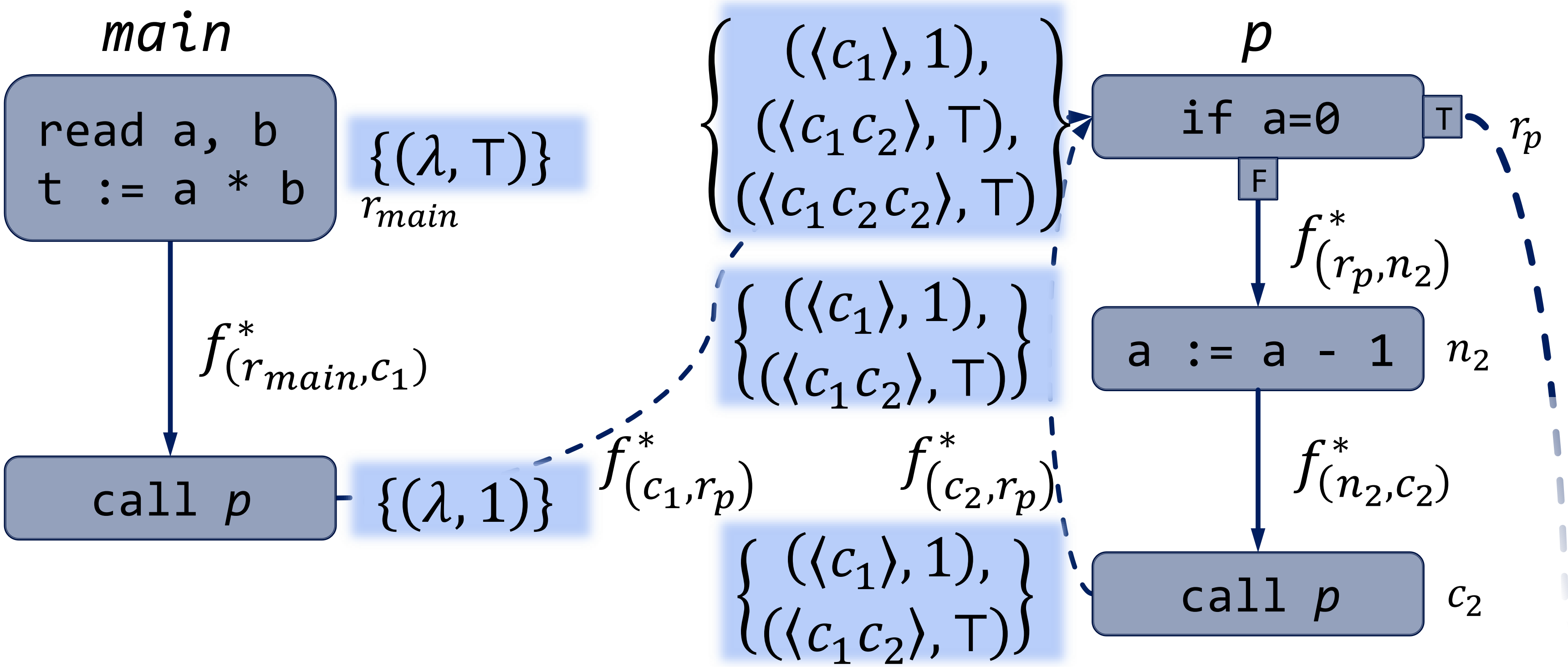
$f^*_{(n_2, c_2)}$
call p
c_2

Example



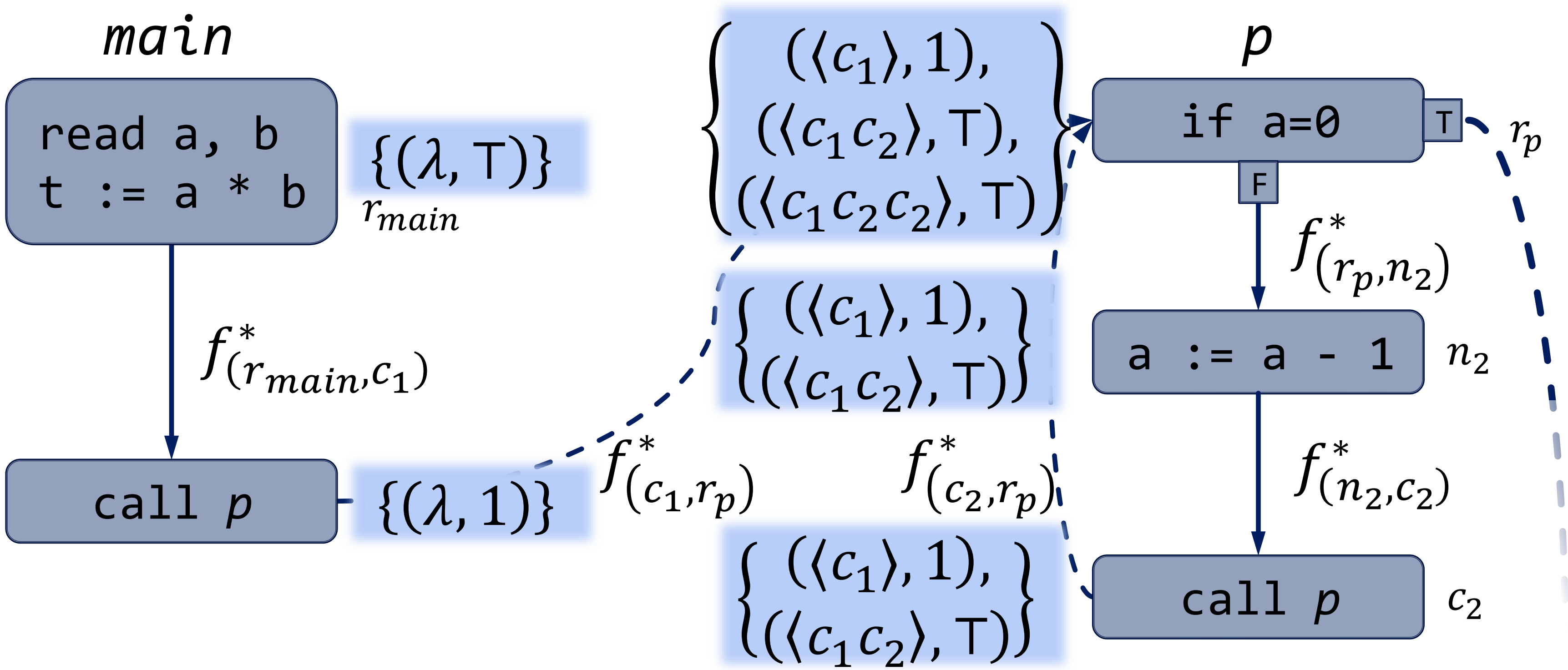
main
read a, b
t := a * b
{(λ, T)}
r_main
f*(r_main, c1)
call p
{(λ, 1)}
f*(c1, rp)
{ ((⟨c1⟩, 1), ((⟨c1 c2⟩, T)) }
{ ((⟨c1⟩, 1), ((⟨c1 c2⟩, T)) }
{ ((⟨c1⟩, 1), ((⟨c1 c2⟩, T)) }
f*(c2, rp)
p
if a=0
T
F
rp
f*(rp, n2)
a := a - 1
n2
f*(n2, c2)
call p
c2

Example



main
read a, b
t := a * b
$\{(\lambda, \mathsf{T})\}$
r_{main}
$f^*_{(r_{main}, c_1)}$
call p
$\{(\lambda, 1)\}$
$f^*_{(c_1, r_p)}$
$\left\{ \begin{array}{c} (\langle c_1 \rangle, 1), \\ (\langle c_1 c_2 \rangle, \mathsf{T}), \\ (\langle c_1 c_2 c_2 \rangle, \mathsf{T}) \end{array} \right\}$
$\left\{ \begin{array}{c} (\langle c_1 \rangle, 1), \\ (\langle c_1 c_2 \rangle, \mathsf{T}) \end{array} \right\}$
$f^*_{(c_2, r_p)}$
$\left\{ \begin{array}{c} (\langle c_1 \rangle, 1), \\ (\langle c_1 c_2 \rangle, \mathsf{T}) \end{array} \right\}$
p
if a=0
T
F
r_p
$f^*_{(r_p, n_2)}$
a := a - 1
n_2
$f^*_{(n_2, c_2)}$
call p
c_2

Example



main
read a, b
t := a * b
$\{(\lambda, \top)\}$
r_{main}
$f^*_{(r_{main},c_1)}$
call p
$\{(\lambda, 1)\}$
$f^*_{(c_1,r_p)}$
$\left\{ (\langle c_1\rangle, 1), (\langle c_1 c_2\rangle, \top), (\langle c_1 c_2 c_2\rangle, \top) \right\}$
$\left\{ (\langle c_1\rangle, 1), (\langle c_1 c_2\rangle, \top) \right\}$
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$f^*_{(r_p,n_2)}$
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n_2
$f^*_{(n_2,c_2)}$
call p
c_2

...

Chapter summary

Call string approach:

- New dataflow problem on top of (L, F)
  - Data "tagged" with propagation history
  - Interprocedural flow explicit
- Existence of $\mathrm{MFP}_{\mathrm{CS}}$
  - No functional compositions
  - Only iteratively computable if Γ finite

Structure

1. Definition of a new DF problem (L^*, F^*)
2. **Proof: Solution to $(\boldsymbol{L}^*, \boldsymbol{F}^*) \equiv$ MOP solution**
3. Feasibility and precise variants
4. Approximative solution

MOP solution and MFP_{CS}

- This chapter: Show that MOP solution

$$y_n = \bigwedge \{f_q : q \in IVP(r_{main}, n)\} \ (\top)$$

can be acquired from MFP_{CS}.

- Chapter outline:
  1. Define MOP_{CS}
  2. Show $\text{MOP}_{\text{CS}} = \text{MFP}_{\text{CS}}$
  3. Extract results $x'_n \in L$ from MFP_{CS}
  4. Using MOP_{CS} and MFP_{CS}, show $x'_n = \text{MOP}$

MOP solution for call strings (MOP_{CS})

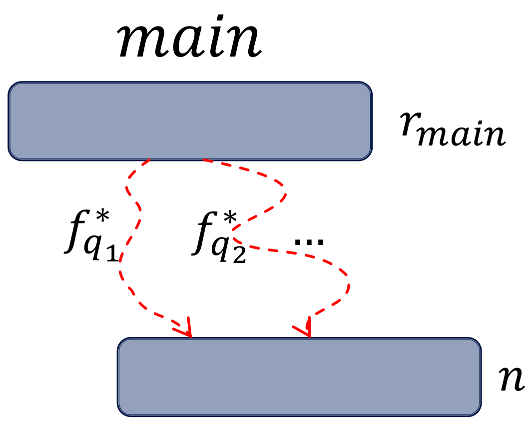
- Propagation of data along paths in G^*

  For $q = (r_{main}, s_1, \dots, s_k, n) \in path_{G^*}$:

  $$f_q^* \coloneqq f_{(s_k,n)}^* \circ f_{(s_{k-1},s_k)}^* \circ \cdots \circ f_{(s_1,s_2)}^* \circ f_{(r_{main},s_1)}^*$$

- Call string MOP solution (***MOP_{CS}***)

  $$\forall n \in N^*: \; y_n^* \coloneqq \bigwedge \left\{ f_q^*\left(x_{r_{main}}^*\right) : q \in path_{G^*}(r_{main}, n) \right\}$$



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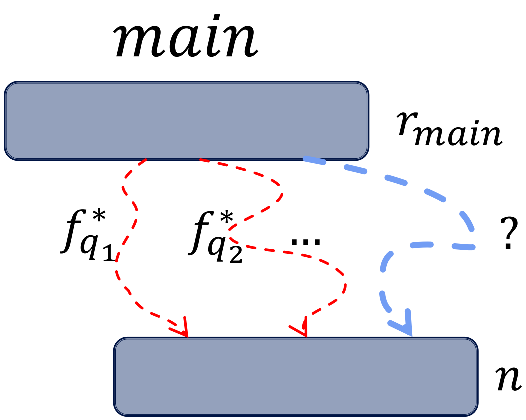


$path_{G^*} \supseteq IVP$

Valid-path-only propagation

Lemma:

Let $n \in N^*, q \in path_{G^*}(r_{main}, n), \gamma \in \Gamma$. Then

1. $q \in IVP(r_{main}, n)$ and $CM(q) = \gamma$
   $\Leftrightarrow\ f_q^*\left(x_{r_{main}}^*\right)(\gamma)$ defined

2. $f_q^*\left(x_{r_{main}}^*\right)(\gamma) = f_q(\top)$

$\Rightarrow$ "MOP_{CS} only contains paths in IVP and for each such p yields the same result as (L, F)"

Valid-path-only propagation (Proof)

By induction on $l(q)$ for $q = (r_{main}, s_1, \dots, s_k, n)$:

- $\boldsymbol{l(q) = 0}$: Let $n \in N^*$.

  $q = (r_{main}) \in IVP(r_{main}, r_{main}) \Rightarrow CM(q) = \lambda$.

  $f_q^*(r_{main}) = \{(\lambda, \top)\}$ only def. for λ with $\top = f_q(\top)$.

- IH:

  Iff $q \in IVP(r_{main}, n), l(q) < k$ and $\gamma = CM(q)$

  then $f_q^*\left(x_{r_{main}}^*\right)(\lambda) = f_p(\top)$.

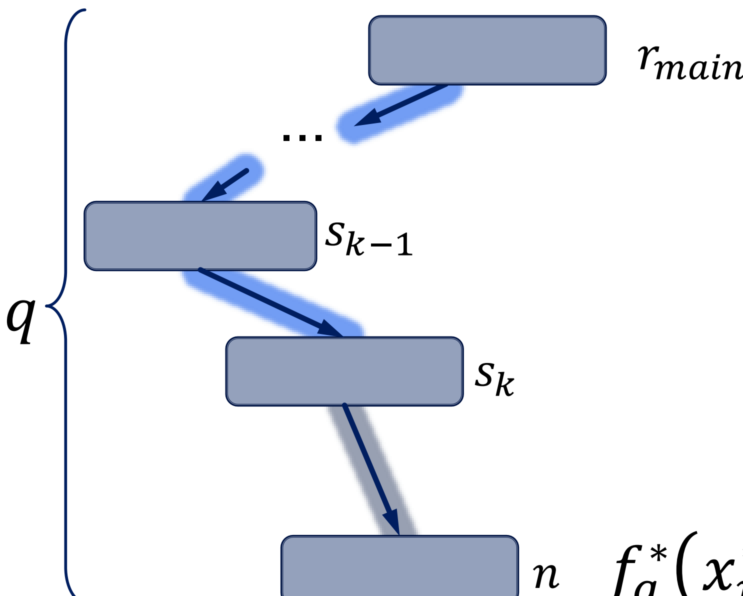
Valid-path-only propagation (Proof)

- **IH:**

  Iff $q \in IVP(r_{main}, n), l(q) < k$ and $\gamma = CM(q)$

  then $f_q^*\left(x_{r_{main}}^*\right)(\lambda) = f_p(\top)$

- **IS:** $l(q) = k$.



$$f_q^*\left(x_{r_{main}}^*\right)(\gamma) \stackrel{\text{def}}{=} f_{(s_k,n)}\left(x_{s_k}^*\right)$$

Valid-path-only propagation (Proof)

- **IH:**

  Iff $q \in IVP(r_{main}, n), l(q) < k$ and $\gamma = CM(q)$

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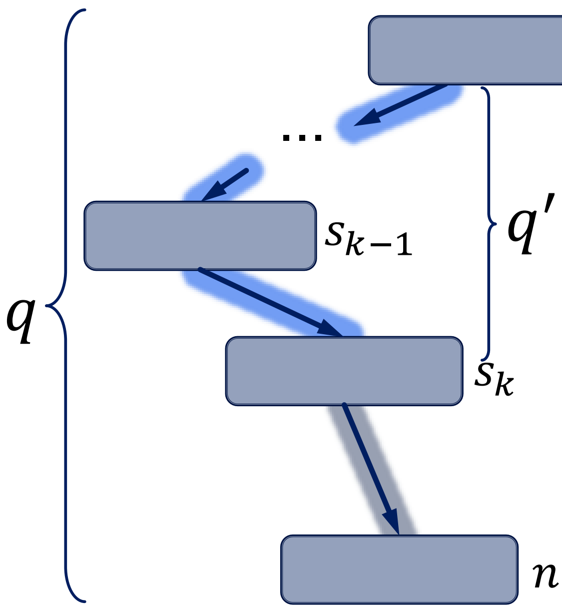

$\gamma = \gamma' \circ (m, n), \qquad \gamma' = CM(q')$

$f_{q'}^*\left(x_{r_{main}}^*\right)(\gamma')$

$f_q^*\left(x_{r_{main}}^*\right)(\gamma) \stackrel{\text{def}}{=} f_{(s_k, n)}\left(x_{s_k}^*\right)$

Valid-path-only propagation (Proof)

- **IH:**

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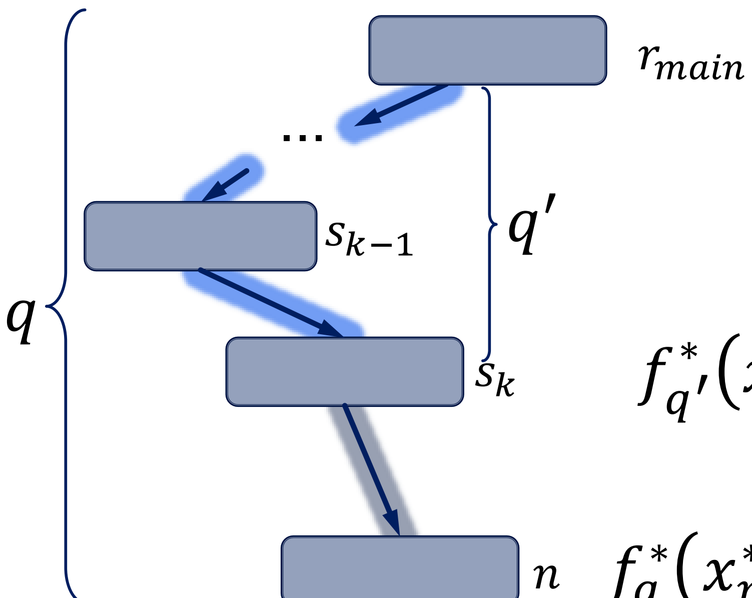

$$\gamma = \gamma' \circ (m, n), \qquad \gamma' = CM(q')$$

$$f_{q'}^*(x_{r_{main}}^*)(\gamma') =^{IH} f_{q'}(\top)$$

$$f_q^*(x_{r_{main}}^*)(\gamma) \stackrel{\text{def}}{=} f_{(s_k, n)}(x_{s_k}^*)$$

Valid–path–only propagation (Proof)

- **IH:**

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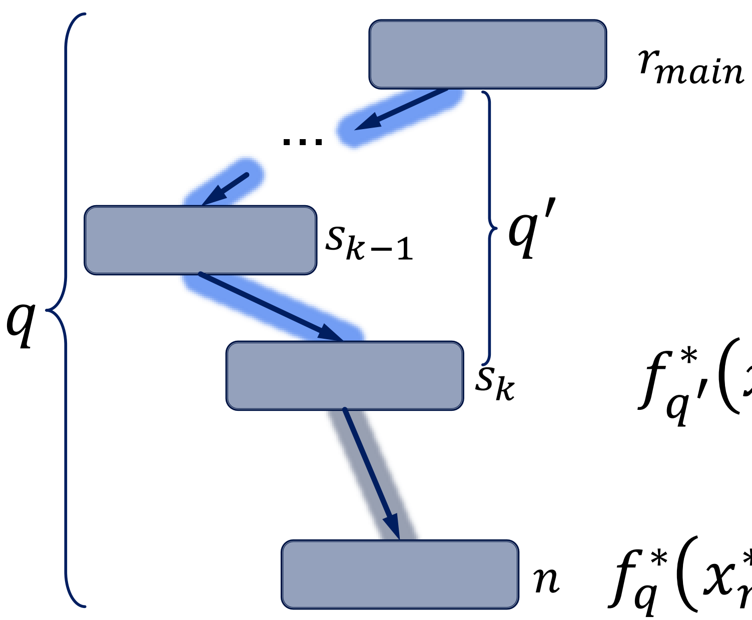


$$\gamma = \gamma' \circ (m, n), \qquad \gamma' = CM(q')$$

$$f_{q'}^*\left(x_{r_{main}}^*\right)(\gamma') =^{IH} f_{q'}(\top)$$

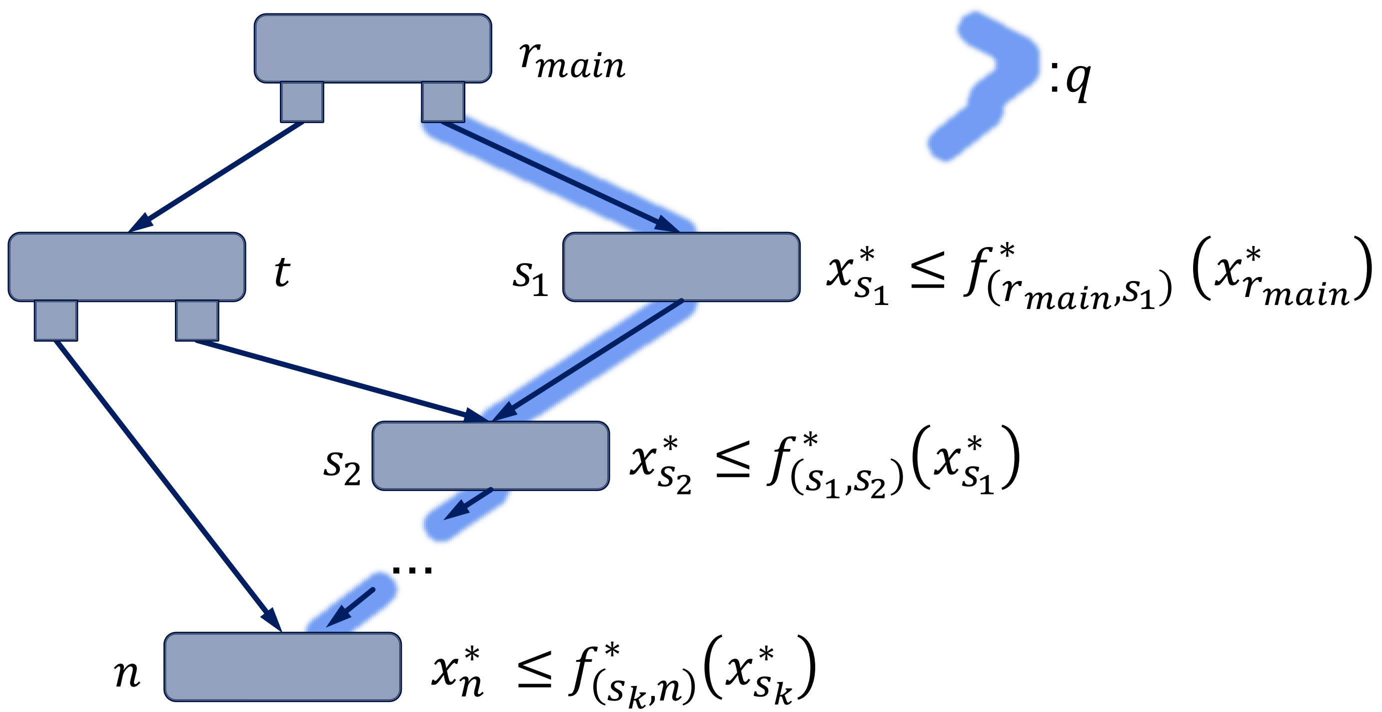
$$f_q^*\left(x_{r_{main}}^*\right)(\gamma) = f_{(s_k, n)}\left(f_{q'}(\top)\right) \stackrel{\text{def}}{=} f_q(\top) \quad \blacksquare$$

MFP$_{CS}$ = MOP$_{CS}$: $x_n^* = y_n^*$

Theorem: (L,F) distributive $\Rightarrow \forall n \in N^*$: $x_n^* = y_n^*$

Proof:

- $\boldsymbol{x_n^* \le y_n^*}$: Let $q \in path_{G^*} = (r_{main}, s_1, \dots, s_k, n)$.



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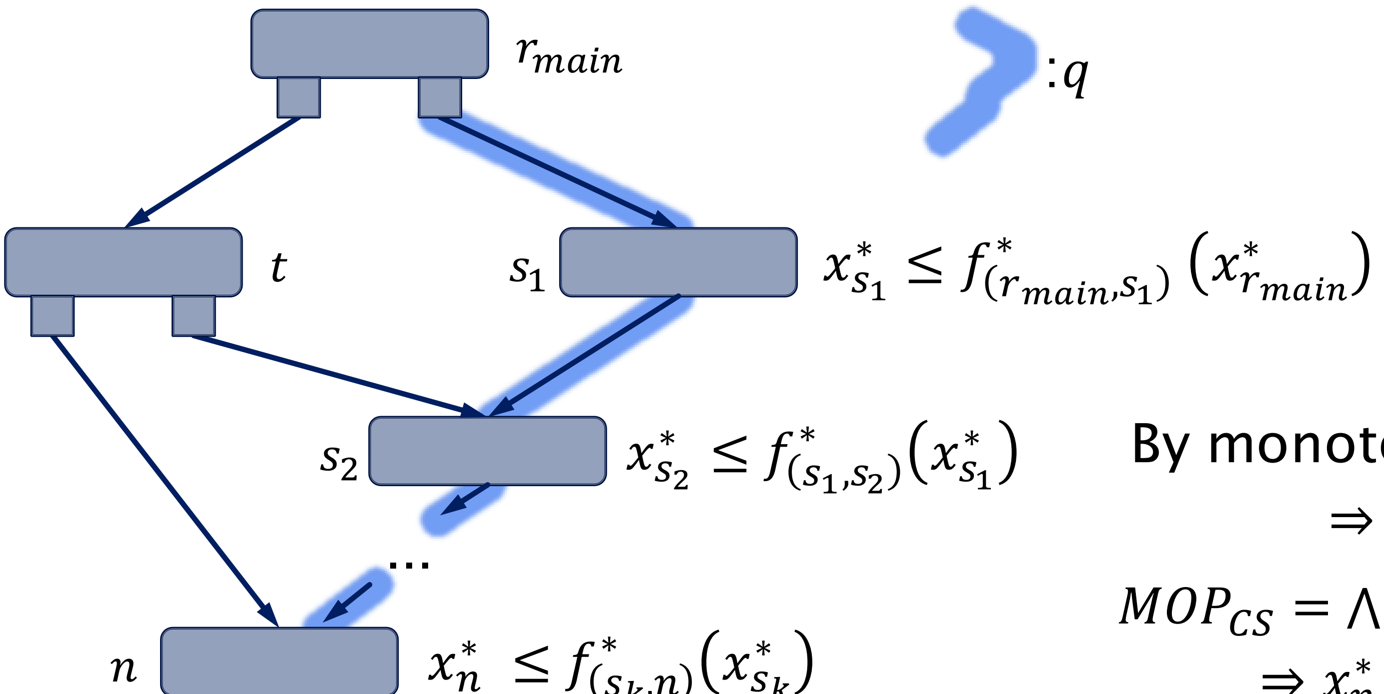


By monotonicity

$$\Rightarrow x_n^* \leq f_q^*\left(x_{r_{main}}^*\right)$$

$$MOP_{CS} = \bigwedge_q f_q^*$$

$$\Rightarrow x_n^* \leq \bigwedge_q f_q^*\left(x_{r_{main}}^*\right)$$

$\text{MFP}_{\text{CS}} = \text{MOP}_{\text{CS}}$: $x_n^* = y_n^*$

- $\boldsymbol{x_n^* \geq y_n^*}$:
  By induction on i: $\forall i \geq 0, n \in N^*$: $x_n^{*(i)} \geq y_n^*$
  - $\boldsymbol{i = 0}$:

    $$n = r_{main} : x_{r_{main}}^{*(0)} \overset{\text{def}}{\equiv} \{(\lambda, \top)\} = f_{q_0}^*\left(x_{r_{main}}^*\right) \geq y_{r_{main}}^*$$

    $$n \neq r_{main} : x_n^{*(0)} \overset{\text{def}}{\equiv} \bot^* \geq y_n^*$$
  - IH:

    $$\forall i \leq k, n \in N^*: x_n^{*(i)} \geq y_n^*$$
  - IS:

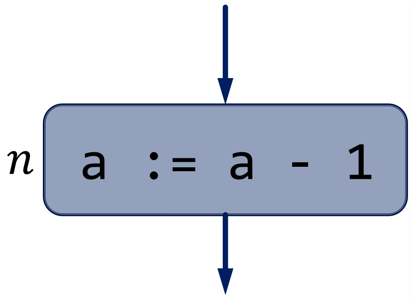
$$x_n^{*(k+1)} \overset{\text{def}}{\equiv} \bigwedge_{(m,n) \in E^*} f_{(m,n)}^*\left(x_m^{*(k)}\right) \overset{\text{IH}}{\geq} \bigwedge_{(m,n) \in E^*} f_{(m,n)}^*\left(y_m^*\right) \overset{\substack{\text{Cont.+Dist.} \\ \text{of } f_{(m,n)}^*}}{\geq} y_n^*$$

■

Result for (L, F) from MFP_{CS}

Define $x'_n \in L$ as follows:

$$x'_n := \bigwedge_{\gamma \in \Gamma} x^*_n(\gamma)$$



$$\left\{\begin{matrix} (\langle c_1 \rangle, 1), \\ (\langle c_1 c_2 \rangle, \top), \\ (\langle c_1 c_2 c_2 \rangle, \top), \\ \ldots \end{matrix}\right\} = x^*_n$$

$$\begin{aligned} x'_n &= x^*_n(\lambda) \wedge x^*_n(\langle c_1 \rangle) \wedge \\ &\quad x^*_n(\langle c_1 c_2 \rangle) \wedge \cdots \\ &= \bot \wedge 1 \wedge \top \wedge \cdots \\ &= \quad 1 \wedge \top \wedge \cdots \\ &= \top \end{aligned}$$

Claim: x'_n is the MOP solution in (L, F)

$MFP_{CS} \equiv MOP$

Theorem: For each $n \in N^*$: $x'_n = y_n$

Proof:

$$x'_n = \bigwedge_{\gamma \in \Gamma} x^*_n(\gamma)$$ Def x'_n

$$= \bigwedge_{\gamma \in \Gamma} \{ f^*_q(x^*_{r_{main}})(\gamma) \mid q \in path_{G^*}(r_{main}, n) \}$$ MFP_{CS}= MOP_{CS}

$$= \bigwedge_{\gamma \in \Gamma} \{ f_q(\top) \mid q \in IVP(r_{main}, n) \text{ sth. } CM(q) = \gamma \}$$ Only-valid-path propag. Lemma

$$= \bigwedge \{ f_q(\top) \mid q \in IVP(r_{main}, n) \}$$ $CM^{-1}(\Gamma) \cap IVP = IVP$

$$= y_n$$ ■

Chapter summary

- Shown:
  - Get MOP solution from MFP_{CS}
  - Conversion $x_n^* \rightarrow x_n'$ straightforward and without functional composition

$\Rightarrow$ Call string approach is actually useful

- But: Computing MFP_{CS} still problematic

Structure

1. Definition of a new DF problem (L^*, F^*)
2. Proof: Solution to $(L^*, F^*) \equiv$ MOP solution
3. **Feasibility and precise variants**
4. Approximative solution

Motivation

- Observation: L finite $\Rightarrow$ functional approach converges, CS approach not necessarily
- Idea: Ensure termination of CS approach by limiting to finite CS subset $\Gamma_0 \subseteq \Gamma$
- Show: Can be done for all (L, F) with finite L without losing precision

Redefinitions using Γ_o

- Γ_0 is finite subset of Γ fulfilling:
  If $\gamma \in \Gamma_0$ and γ' initial subtuple of $\gamma \Rightarrow \gamma' \in \Gamma_0$
- $IVP' := \{q \in IVP \mid \forall \text{ prefix } q' \text{of } q\text{: } CM(q') \in \Gamma_0\}$
- Let $\circ_0$ only act in Γ_0
  - discards γ' entirely iff $\gamma' \circ_0 (m,n) \notin \Gamma_0$
  - $\circ_0$ is consistent with IVP'

Definitions: DF analysis with Γ_0

Now consider (L_0^*, F_0^*) with $L_0^* \coloneqq \Gamma_0 \to L$ finite

- Dataflow equations now iteratively solvable

$$x'_{n_0} \coloneqq \bigwedge_{\gamma \in \Gamma_0} x_n^*(\gamma) \qquad \textbf{(}\boldsymbol{MFP_{CS0}}\textbf{)}$$

- MOP solution using only paths in IVP'

$$y'_{n_0} \coloneqq \bigwedge \{ f_q(0) \mid q \in IVP'(r_{main}, n) \} \qquad \textbf{(}\boldsymbol{MOP_{CS0}}\textbf{)}$$

$MOP_{CS0} = MFP_{CS0}$

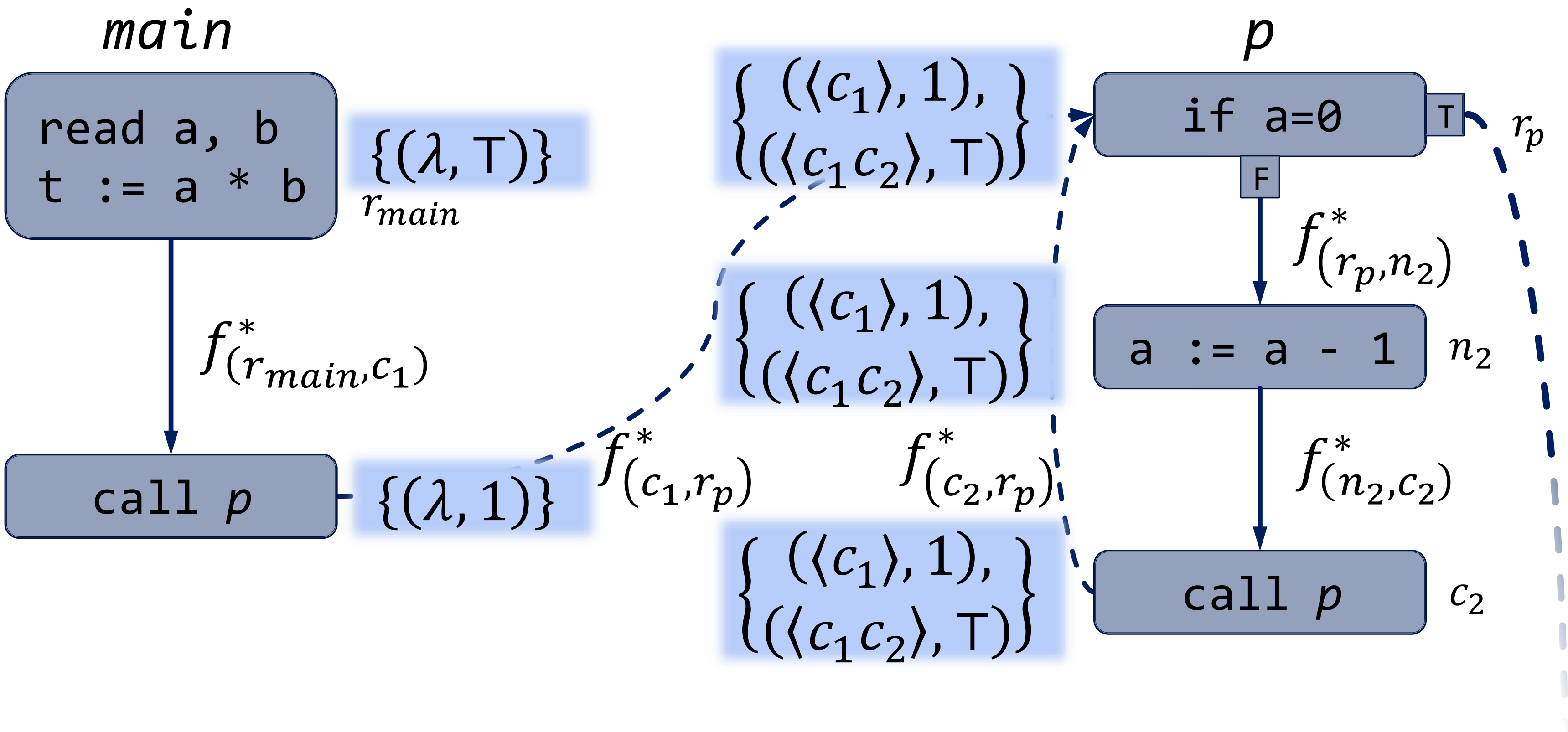
Theorem: $\forall n \in N^*: x'_{n_0} = y'_{n_0}$

Proof:

- Completely analogous to MOP_{CS}=MFP_{CS} proof by replacing $\Gamma, IVP, \circ$ by $\Gamma_0, IVP', \circ_0$
- No reasoning about infinite meets or continuity of F_0^* required

■

Motivation: CS of limited length



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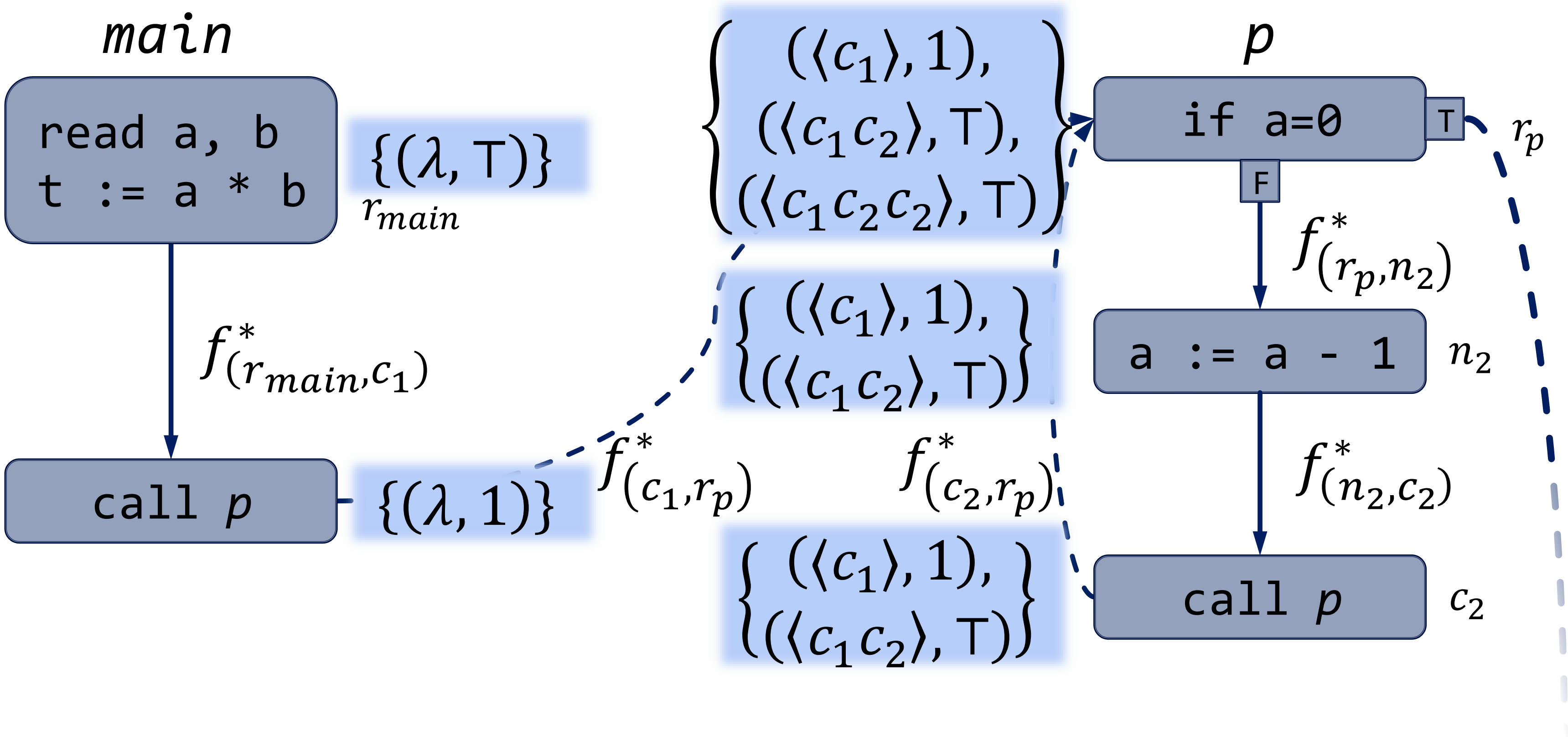

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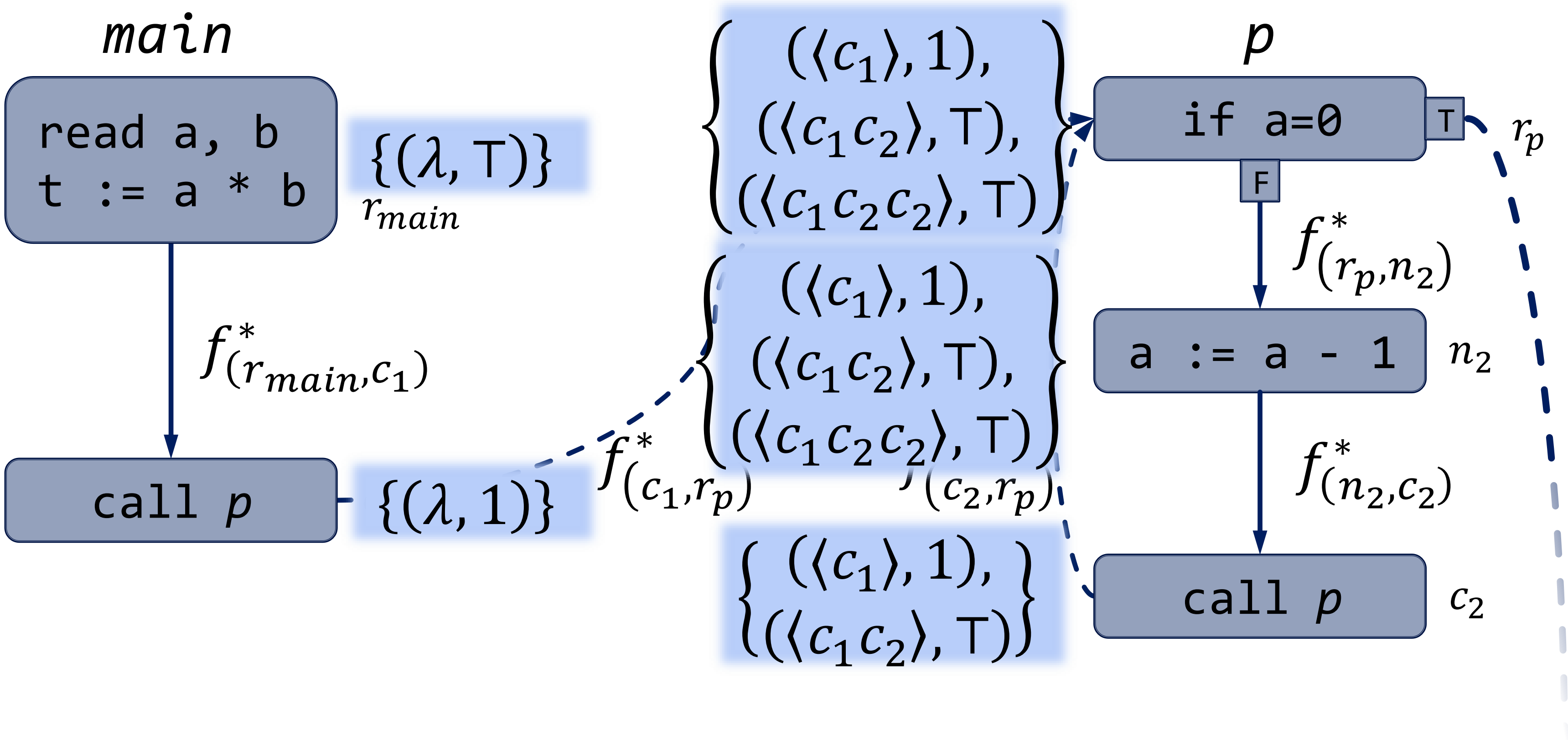

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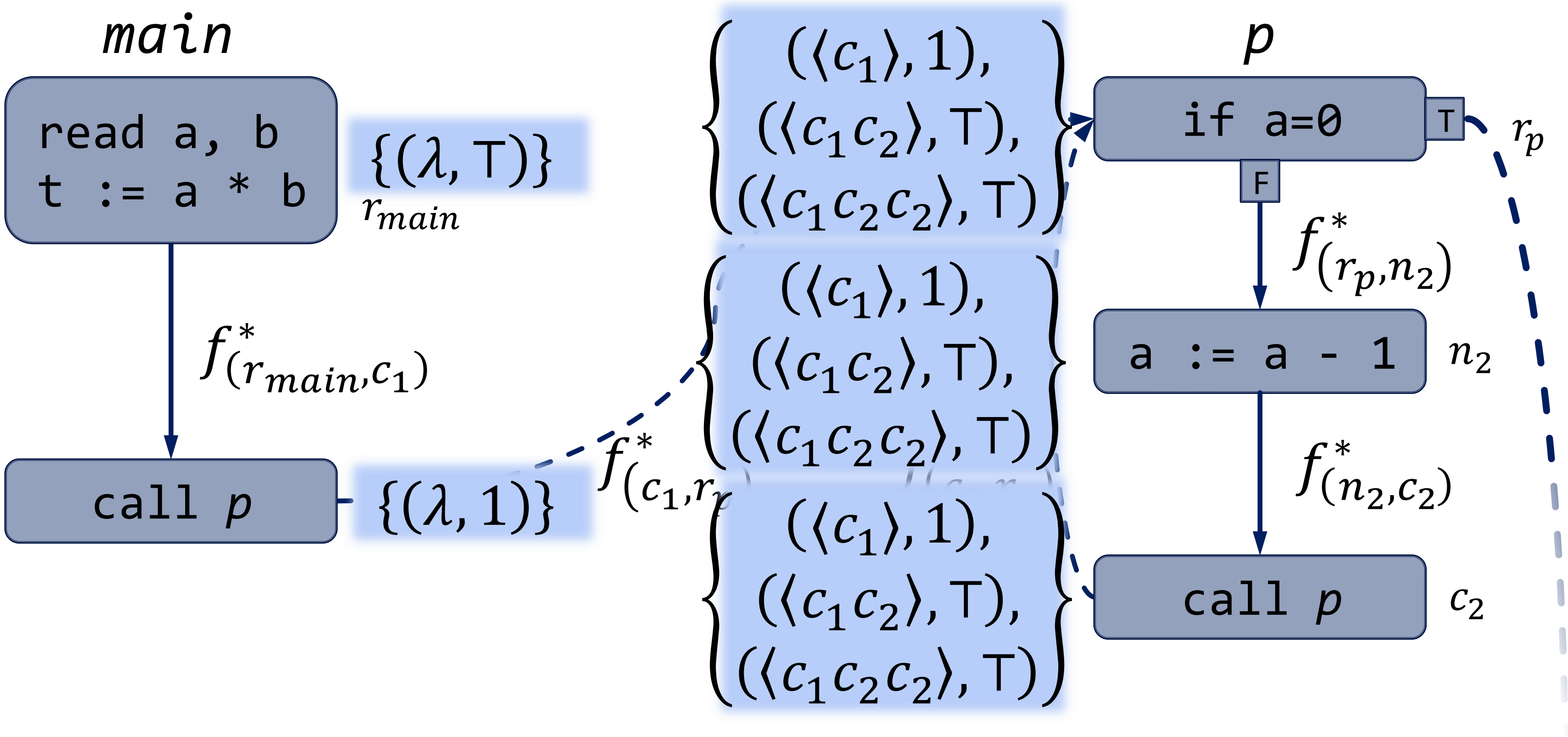

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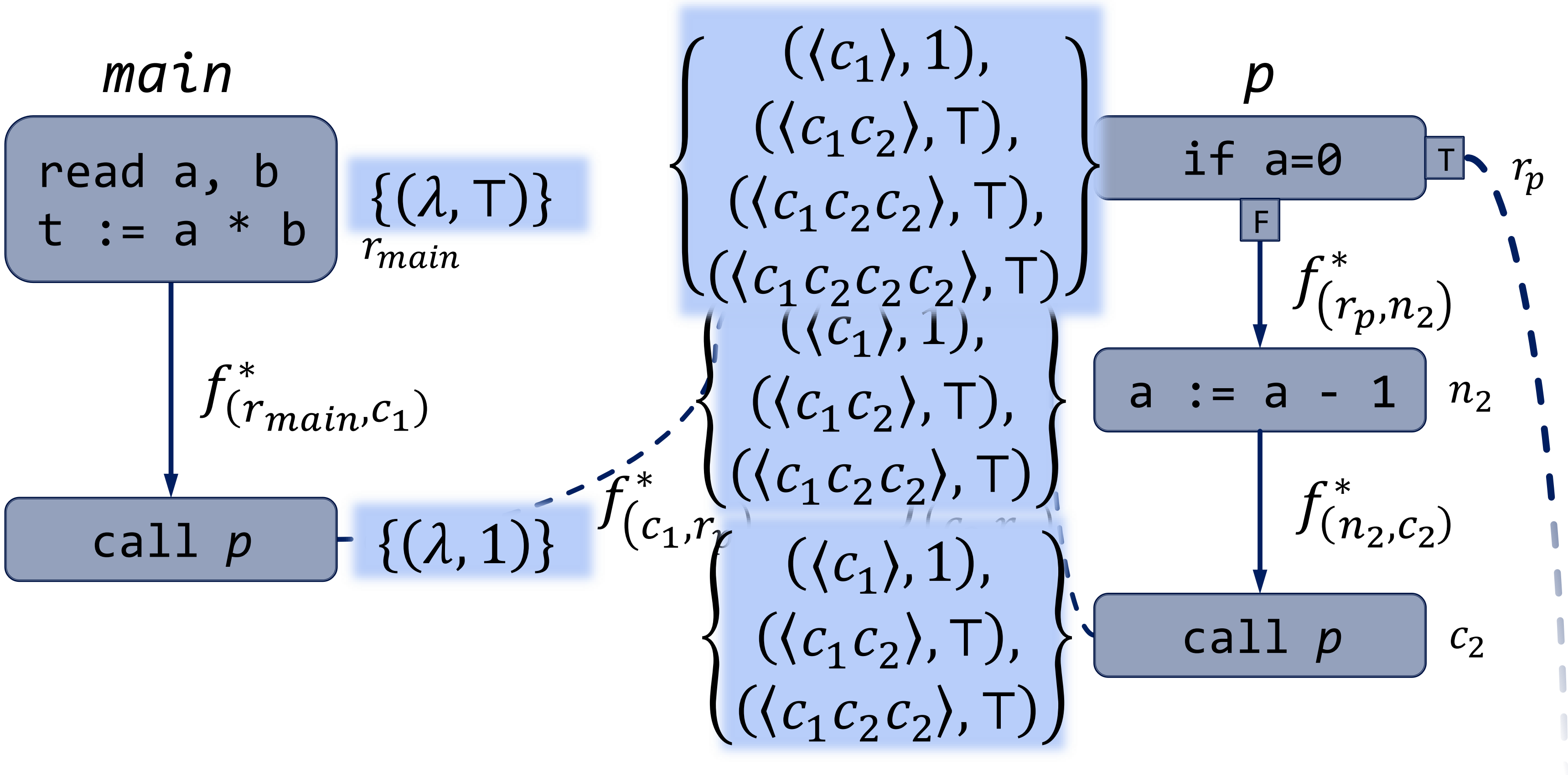

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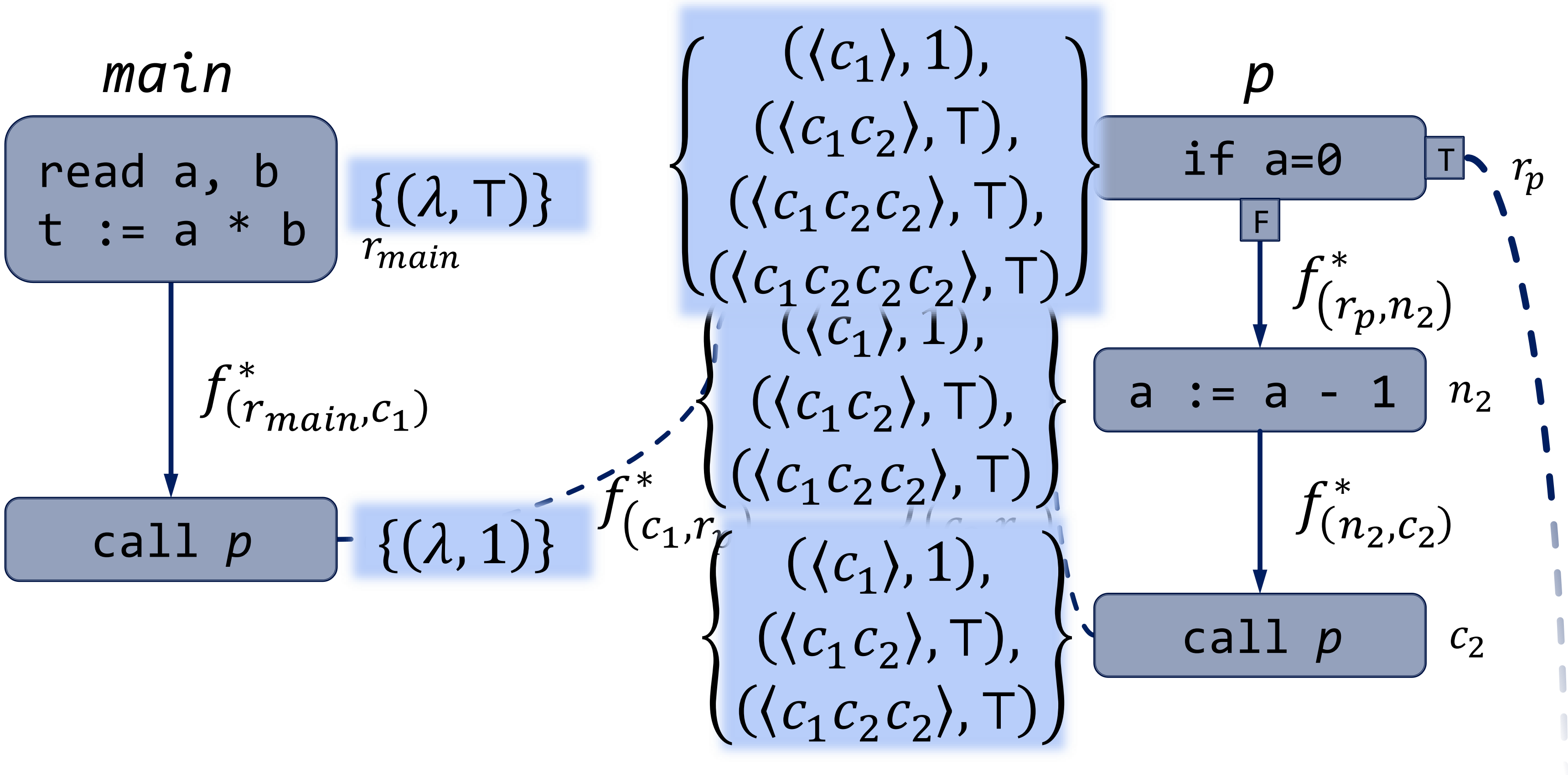

- No information gain on further iteration
- Stop if data tagged by longer CS „irrelevant“

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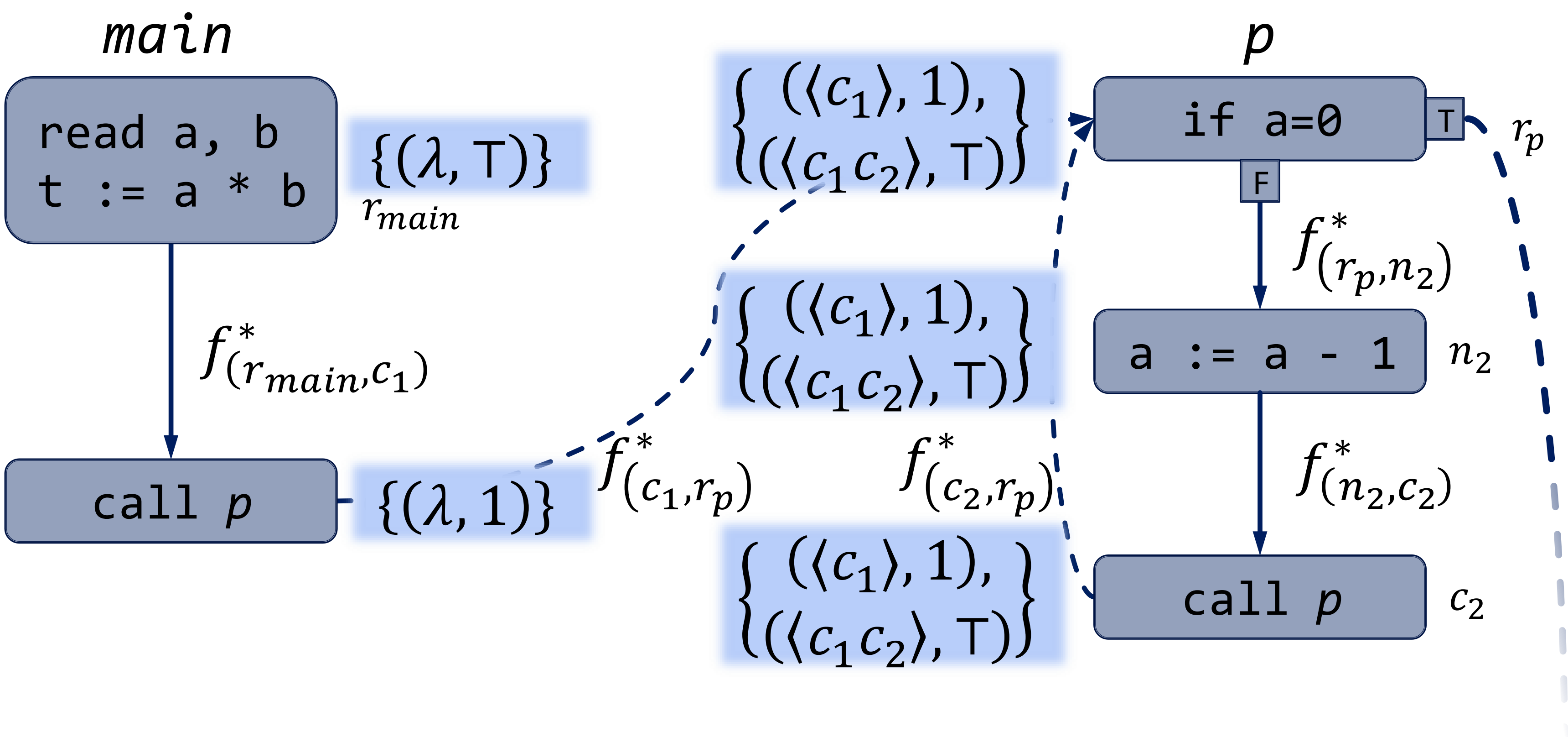

- No information gain on further iteration
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Redundant information in analysis

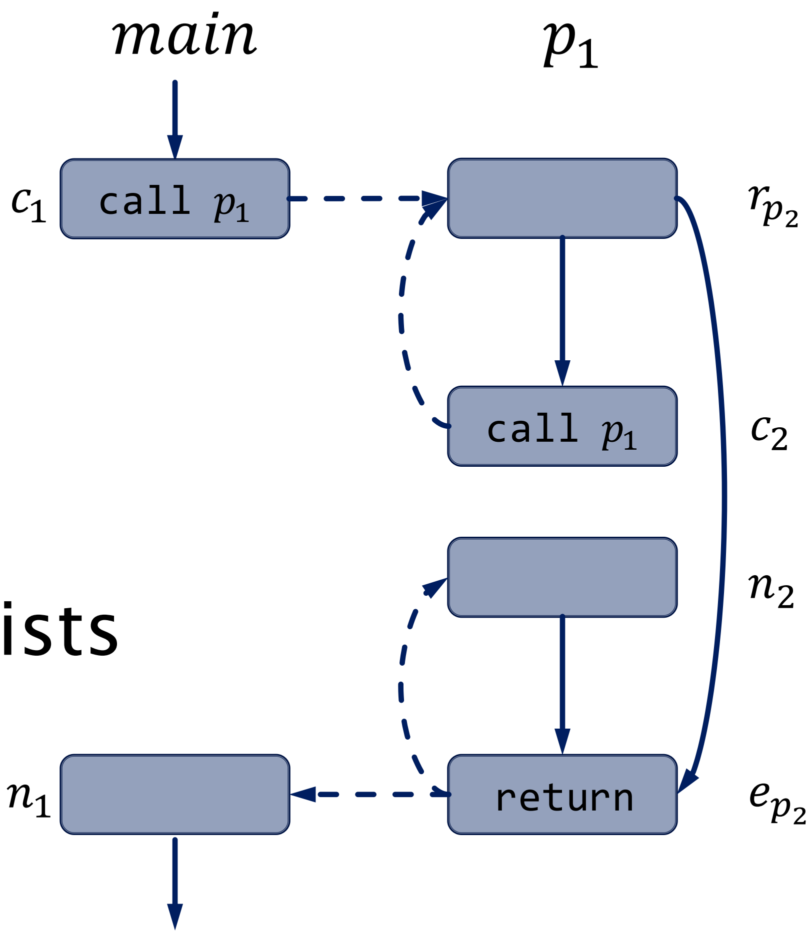
- Why paths of arbitrary length?
  - K := number of call sites in program $\in \mathbb{N}$
    $\Rightarrow$ only possible by recurring on call site(s)
  - At each call, analysis arrives with value $x \in L$ and returns with value $y \in L$.
    L is finite $\Rightarrow |L|^2$ many distinct begin/end value combinations per call
- **Idea:** Recurring on any of the K calls more than $|L|^2$ times is redundant.

Lemma: Path shortening

Let

- (L, F) with L finite
- $M := K * |L|^2$
  - $K :=$ #(call blocks in G^*)
- $\Gamma_0 = \Gamma_M$.

Then $\forall\, n \in N^*$:
If $q \in IVP(r_{main}, n)$ then exists $q' \in IVP'(r_{main}, n)$ with $f_q(0) = f_{q'}(0)$.



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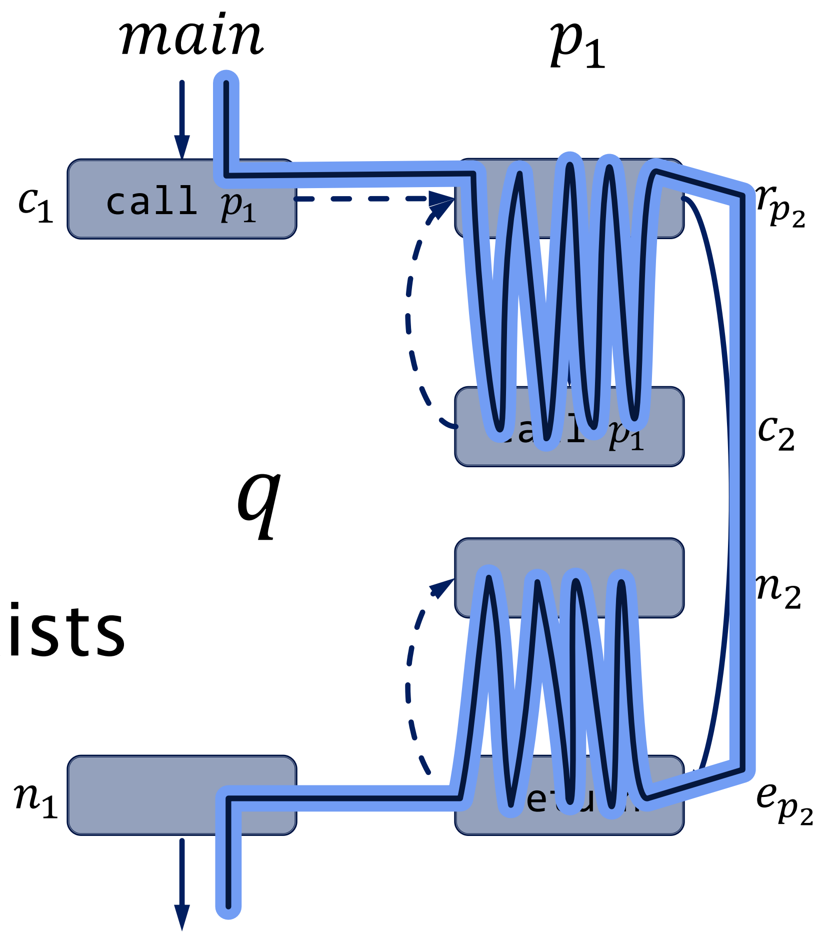

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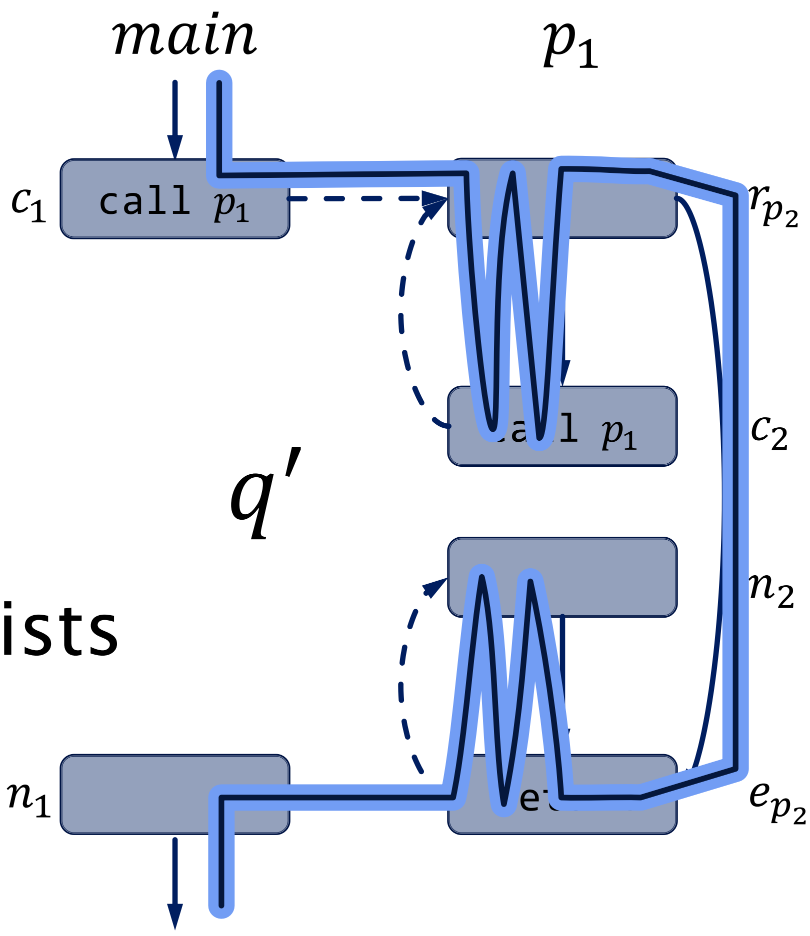


Lemma: Path shortening (proof)

- **Proof:** By induction on $l(q) \coloneqq$ length of q.
  - $\boldsymbol{l(q) = 0}$: $\lambda \in \Gamma$ and $\lambda \in \Gamma_0$.
  - **IH**: Lemma holds for all paths with $l(q) < k, k \in \mathbb{N}^+$.
  - **IS**: Let $n \in N^*, q \in IVP(r_{main}, n), l(q) = k$.

    Assume $q \notin IVP'(r_{main}, n)$.
    Let q' the shortest prefix with $q' \notin IVP'(r_{main}, n)$.
    Then q' contains $M + 1$ unreturned calls.

    By decomposition Lemma:

$$q' = q_0 || \left(c_1, r_{p_1}\right) || q_1 || \left(c_2, r_{p_2}\right) || \dots || \left(c_{M+1}, r_{p_{M+1}}\right) || q_{M+1}$$

Lemma: Path shortening (proof)

$$q' = q_0 || \left(c_1, r_{p_1}\right) || q_1 || \left(c_2, r_{p_2}\right) || \dots || \left(c_{M+1}, r_{p_{M+1}}\right) || q_{M+1}$$

- $M + 1$ calls, but max. M distinct elements. Let $\left(\hat{c}, r_{\hat{p}}\right)$ a duplicate call. If $\hat{c}$ returns, let $\left(e_{\hat{p}}, n\right)$ the return edge.
- Rewrite q as $q_0' \, || \left(\hat{c}, r_{\hat{p}}\right) || \, q_1' \, || \left(\hat{c}, r_{\hat{p}}\right) || \, q_2' \, || \left(e_{\hat{p}}, n\right) || \, q_3' \, || \left(e_{\hat{p}}, n\right) || \, q_4'$
- Shorter $\hat{q} \in IVP(r_{main}, n)$ with $f_{\hat{q}} = f_q$ by dropping redundant parts
- By **IH**, $\exists q' \in IVP'(r_{main}, n)$ for $\hat{q}$ with $f_{q'} = f_{\hat{q}}$ and thus $f_{q'} = f_q$.

$\blacksquare$

Lemma: Path shortening (proof)

$$q' = q_0 || \left(c_1, r_{p_1}\right) || q_1 || \left(c_2, r_{p_2}\right) || \ldots || \left(c_{M+1}, r_{p_{M+1}}\right) || q_{M+1}$$

- $M + 1$ calls, but max. M distinct elements. Let $\left(\hat{c}, r_{\hat{p}}\right)$ a duplicate call. If $\hat{c}$ returns, let $\left(e_{\hat{p}}, n\right)$ the return edge.
- Rewrite q as $q'_0 || \left(\hat{c}, r_{\hat{p}}\right) || q'_1 || \left(\hat{c}, r_{\hat{p}}\right) || q'_2 || \left(e_{\hat{p}}, n\right) || q'_3 || \left(e_{\hat{p}}, n\right) || q'_4$
- Shorter $\hat{q} \in IVP(r_{main}, n)$ with $f_{\hat{q}} = f_q$ by dropping redundant parts
- By **IH**, $\exists q' \in IVP'(r_{main}, n)$ for $\hat{q}$ with $f_{q'} = f_{\hat{q}}$ and thus $f_{q'} = f_q$.

■

MOP_{CS0}=MOP

- For $M \in \mathbb{N} \geq 0$, define Γ_M as set of all $\gamma \in \Gamma$ with length $\leq M$.

- **Theorem**:
  Let (L, F) a distributive DF framework with L finite, and $\Gamma_0 = \Gamma_M$ with $M = K * |L|^2$. Then

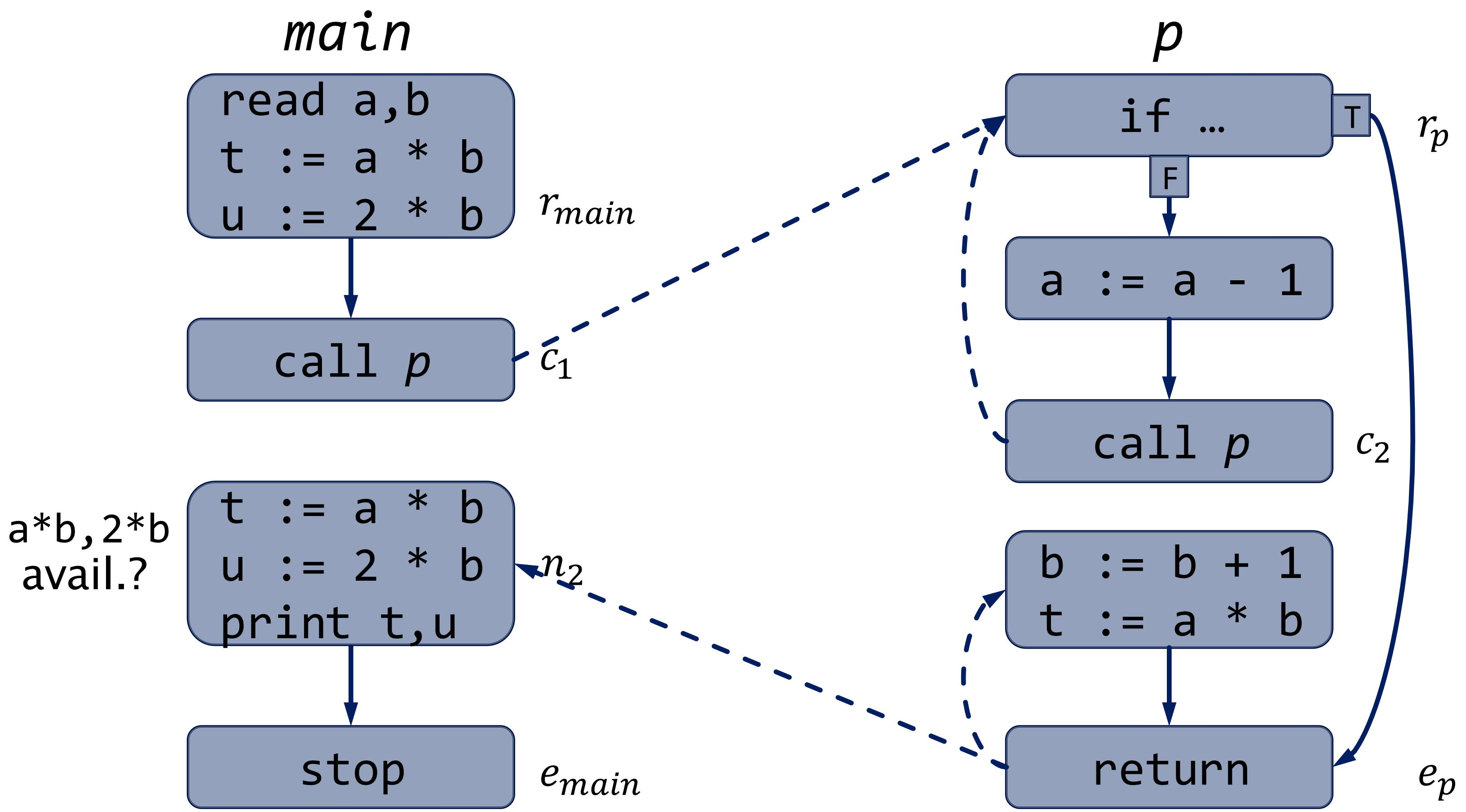
$$\forall n \in N^*: x'_{n_0} = y_n$$

⇒ "If γ is too long and gets discarded, a shorter one in the set represents the same data."

Better bounds for M

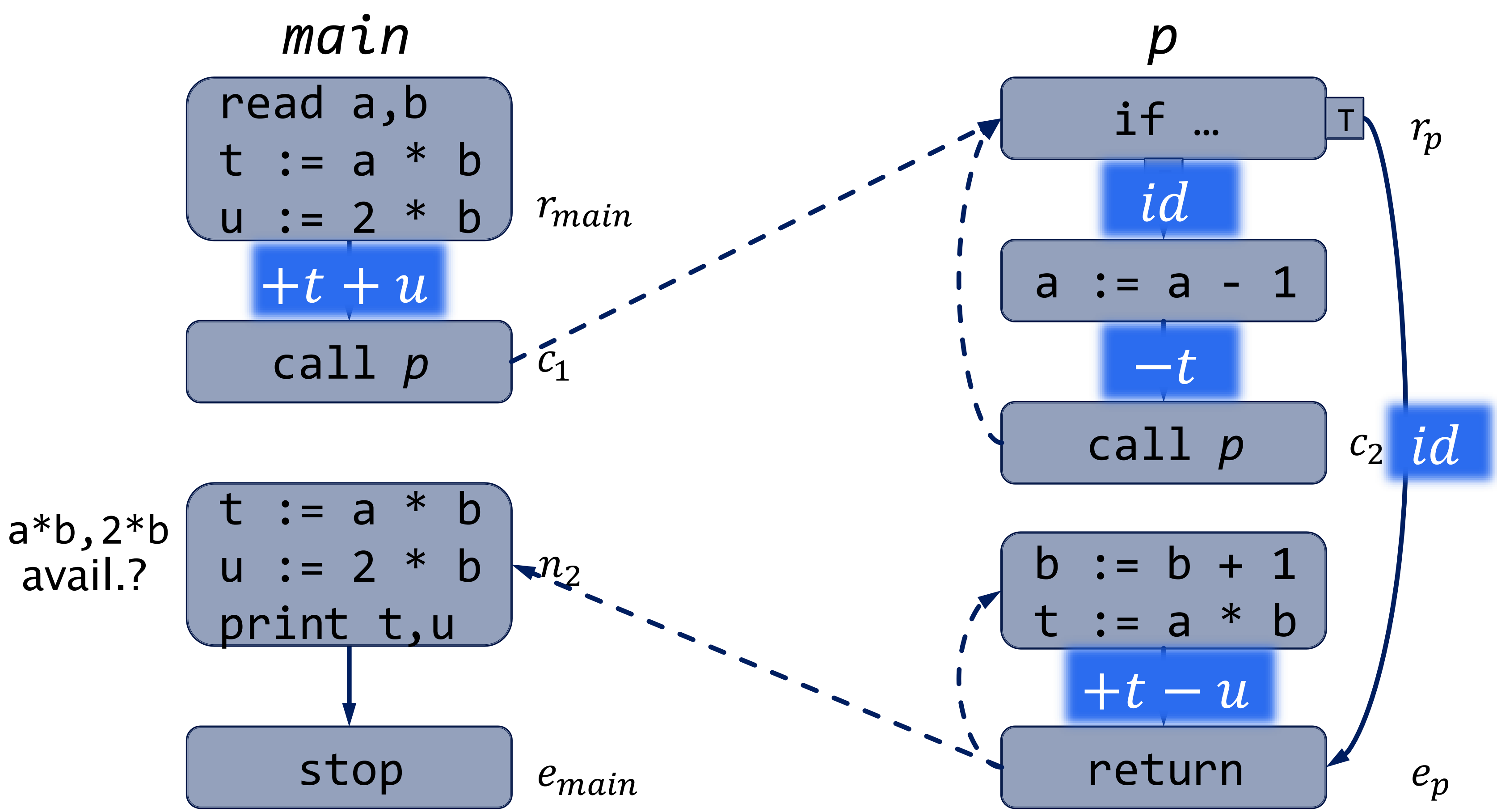
- M impractically large for most problems
- Lower bounds exists for certain problem classes.
  Example: Decomposable frameworks

Decomposable frameworks



$$L = \{\{\,\}, \{t\}, \{u\}, \{t,u\}, \bot\} \qquad M = 50$$

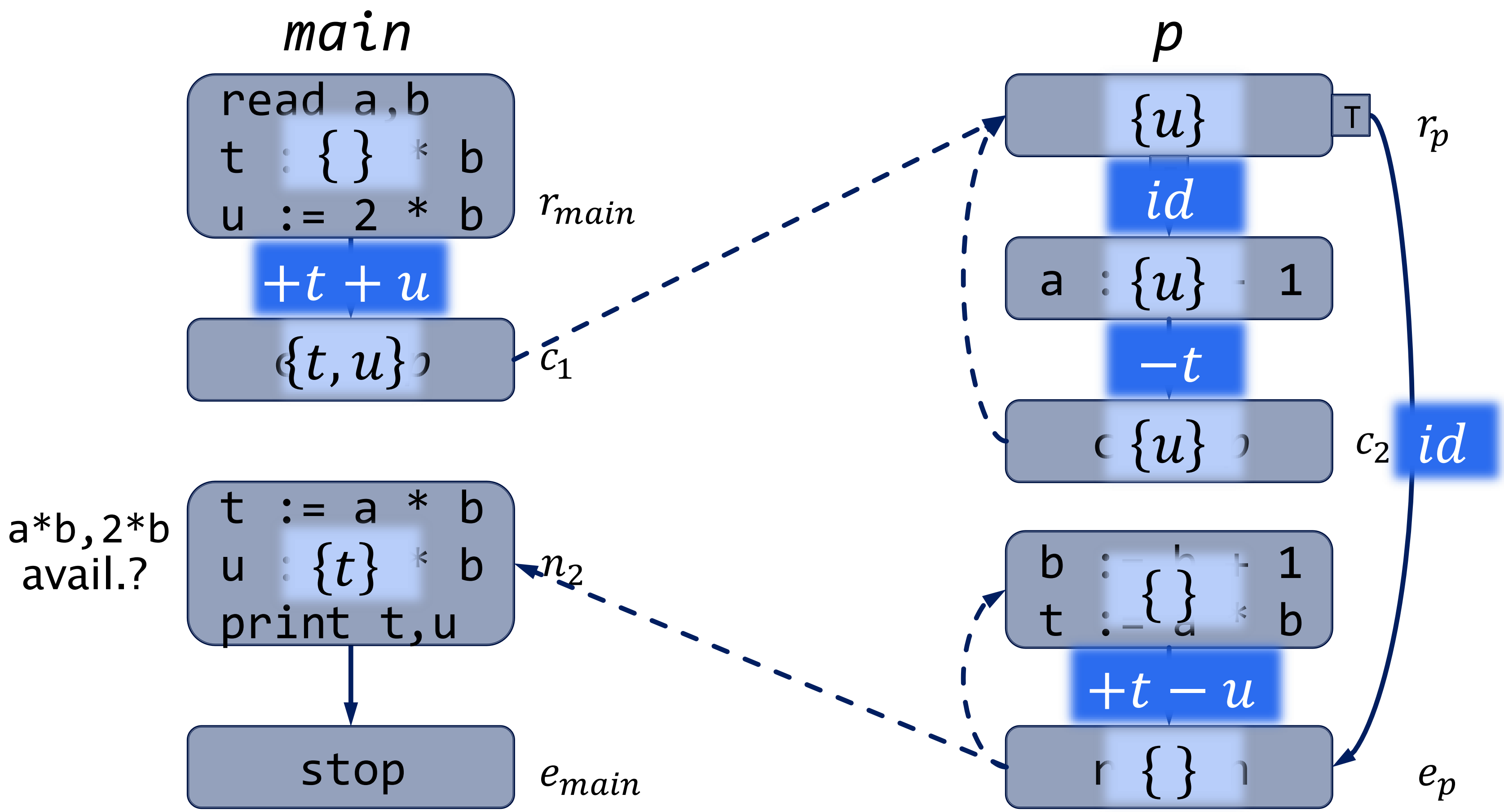
Decomposable frameworks



$$L = \{\{\,\}, \{t\}, \{u\}, \{t, u\}, \bot\}$$

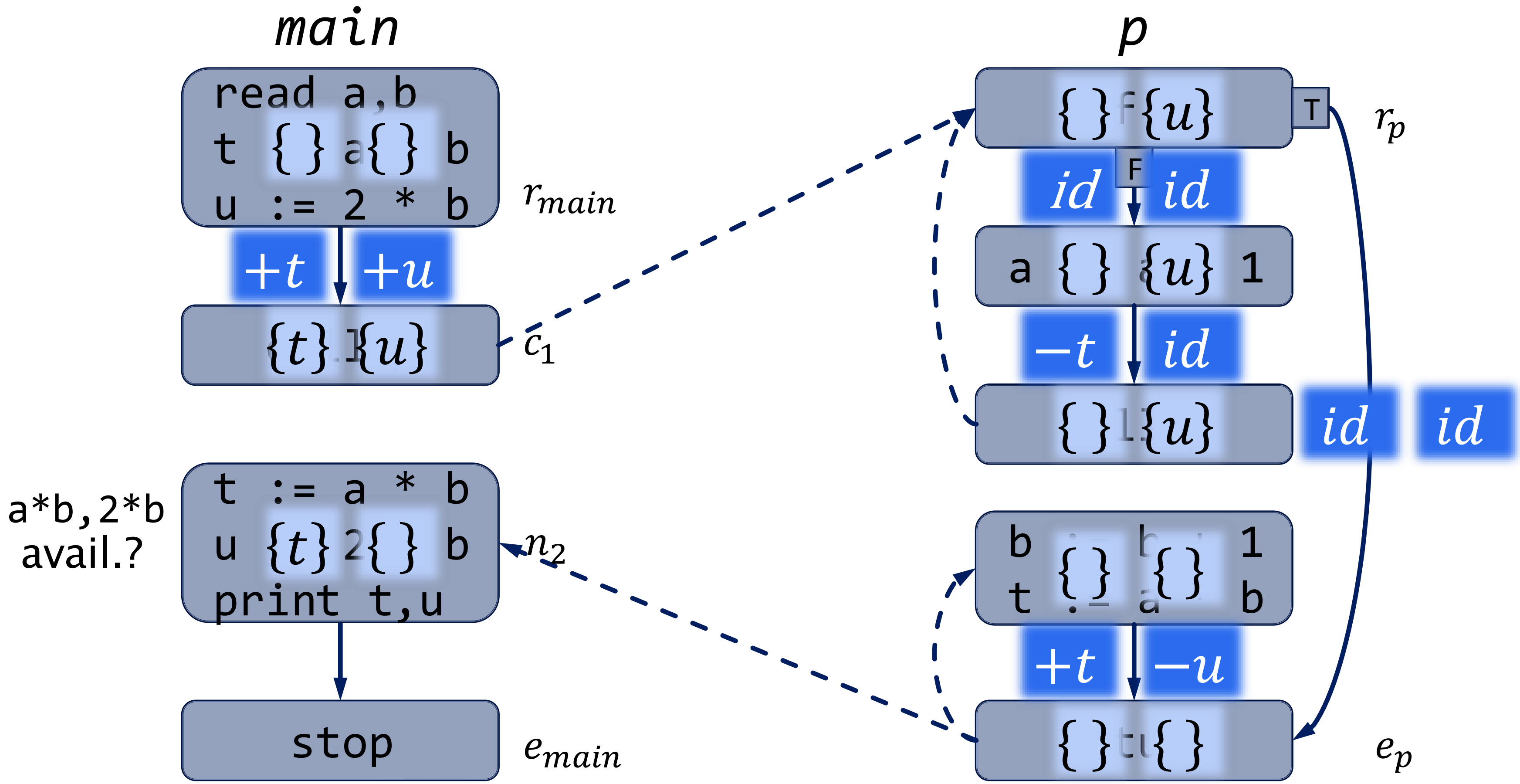
$$M = 50$$

Decomposable frameworks



$L = \{\{\,\},\{t\},\{u\},\{t,u\},\bot\}$ $M = 50$

Decomposable frameworks



$$L_t = \{\{\,\}, \{t\}, \bot\},\ L_u = \{\{\,\}, \{u\}, \bot\},\ \Rightarrow L = L_t \times L_u \qquad M = 18$$

Bound for decomposable frameworks

- **Theorem:** Let (L, F) decomposable into k frameworks $(L_i, F_i)_{i=1}^{k}$.
  Setting the maximum callstring length to

  $$M = K * \max_{i \in \{1..k\}} (|L_i|)^2$$

  yields $y'_{n_0} = y_n \; \forall n \in N^*$.

1−related frameworks

- **Definition:** A decomposable DF framework (L,F) is **1−related** if each F_i only consists of constant and identity functions.
- **Theorem:** In this case, using $\Gamma_0 = \Gamma_{3K}$ yields $y'_{n\,0} = y_n \; \forall n \in N^*$.
- **Example:** Available expressions is 1−related. Decomp. into subproblems (L_e, F_e) for each expression e with $L_e = \{\top, 1, \bot\}$, $F_e = \{id, f_\top, f_1, f_\bot\}$

Chapter summary

- If L finite, MFP_{CS} iteratively computable on Γ_0
  - Precision/safeness depending on choice for Γ_0
- Enforce termination by limiting max. CS length
  - Precision preserving bounds exist, but only of theoretical value
  - Better bounds exist for special problem classes

Structure

1. Definition of a new DF problem (L^*, F^*)
2. Proof: Solution to $(L^*, F^*) \equiv$ MOP solution
3. Feasibility and precise variants
4. **Approximative solutions**

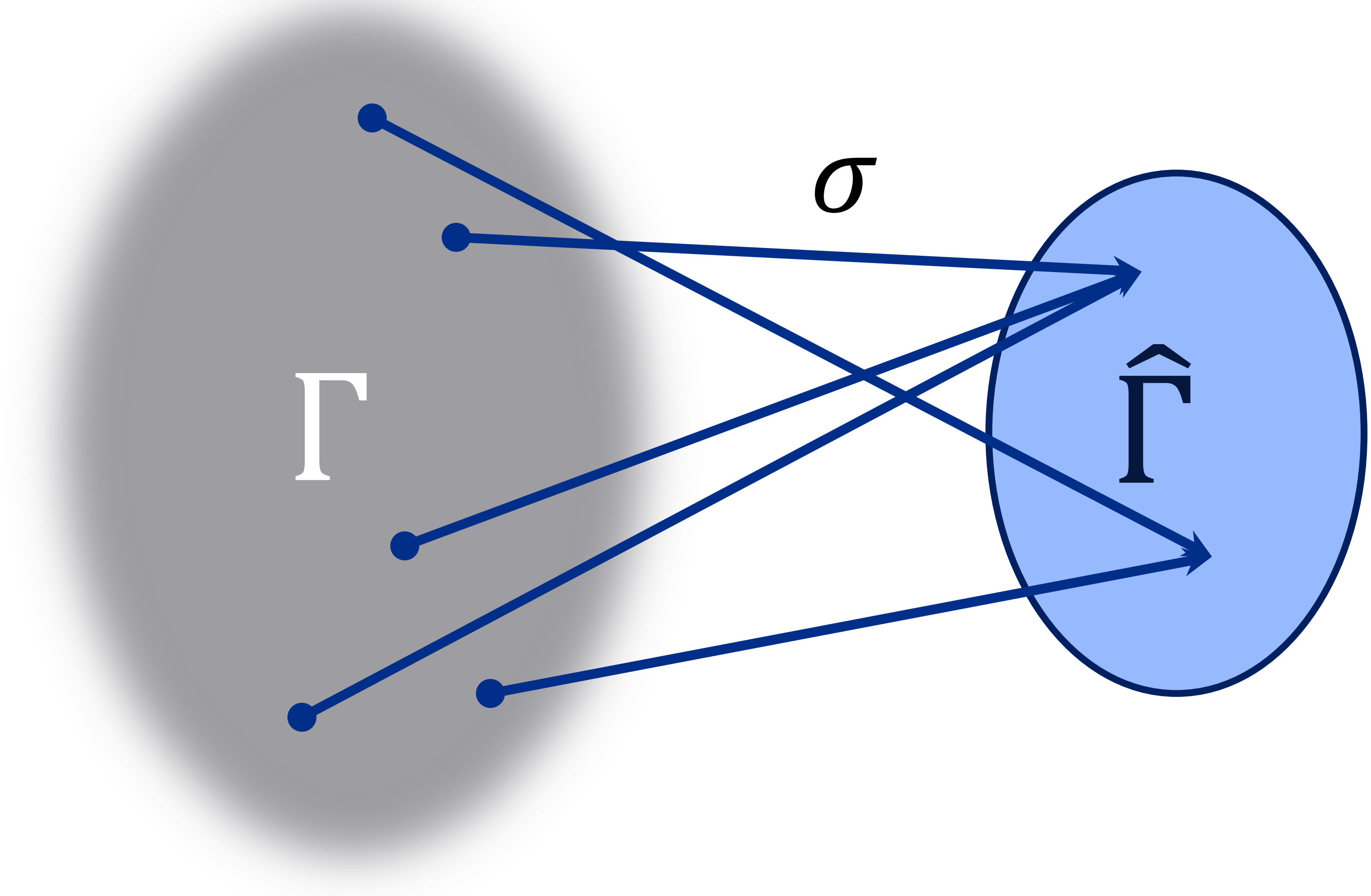
Motivation

- Prefer a safe approximative solution $\hat{x}_n \leq y_n$ over MFP_{CS} if
  - MFP_{CS} not (iteratively) computable
  - computation not feasible MFP_{CS} by time/space constraints

  and computation of $\hat{x}_n \leq y_n$ feasible.

⇒Accept precision loss for less complexity.

Reducing complexity of CS approach



Embedding Γ into $\hat{\Gamma}$

- Let $*$ an operation in $\hat{\Gamma}$ with left identity $w \in \hat{\Gamma}$.
- Encoding function $\sigma: \Gamma \to \hat{\Gamma}$
  - Define $\sigma(n)$ for each **call node** n in G^*.
  - For $\gamma = (c_1 c_2 \dots c_n) \in \Gamma$:

$$\sigma(\gamma) = \sigma(c_1) * \cdots * \sigma(c_n)$$

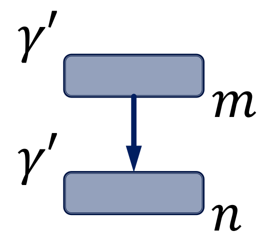
and

$$\sigma(\gamma) = w.$$

Transfer relation $R_{(m,n)}$

- For each $(m, n) \in E^*$, define transfer relation $R_{(m,n)} \in \hat{\Gamma} \times \hat{\Gamma}$:

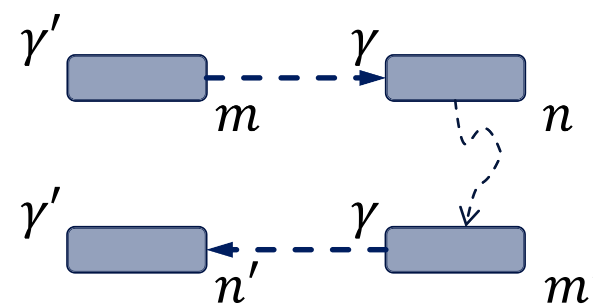
  - If (m, n) intraproc.:
    $\forall a \in \hat{\Gamma}$: $a \; R_{(m,n)} \; a$



$\sigma(\gamma') = a$

  - If (m, n) interproc.:
    $\sigma(\gamma') \; R_{(m,n)} \; \sigma(\gamma)$

    $\Rightarrow R_{call} = R_{return}^{-1}$

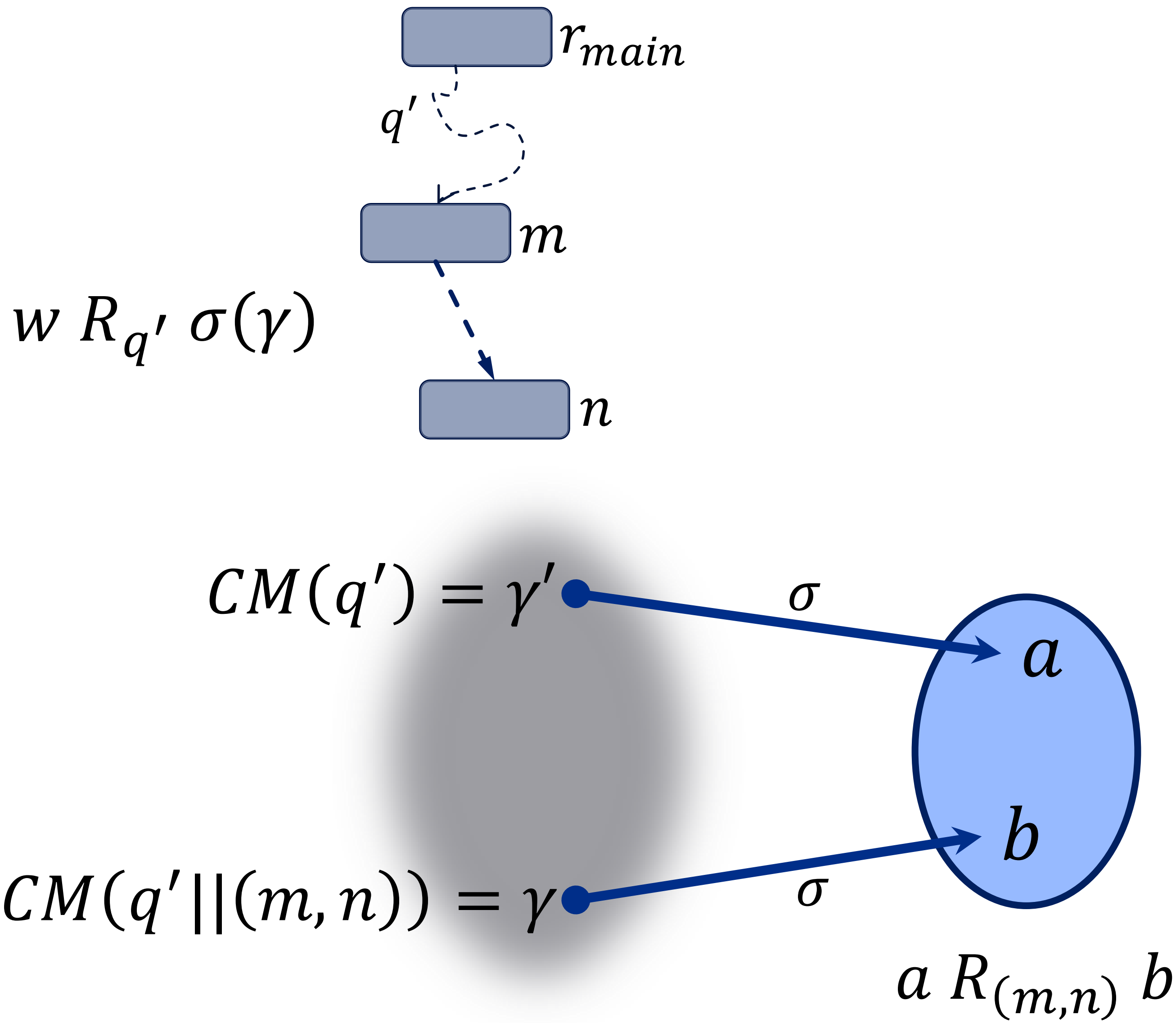


Acceptable paths

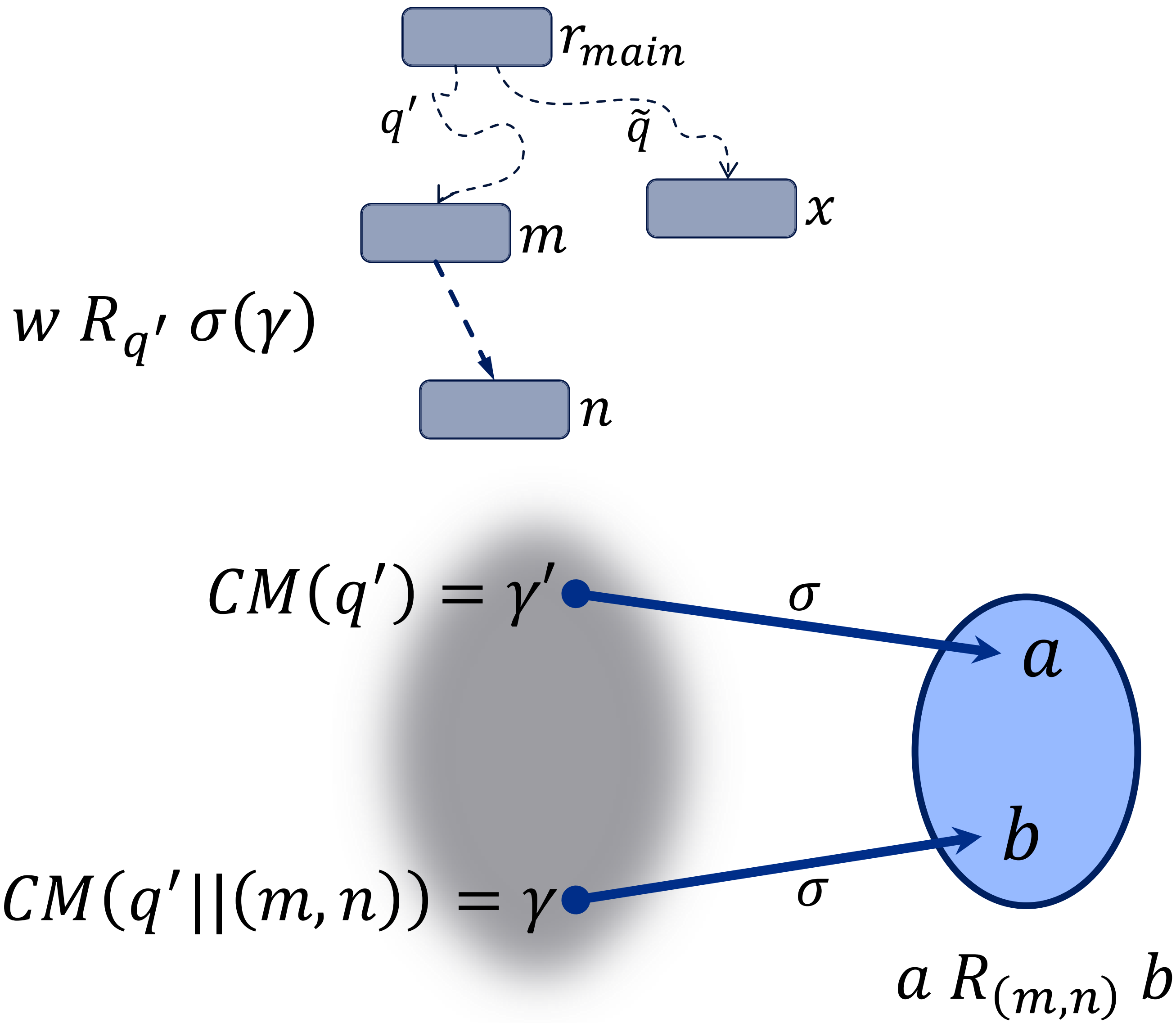
Let $q = (n_1, n_2, \dots, n_k) \in path_{G^*}$.

- Define $R_q := R_{(n_1,n_2)} \circ R_{(n_2,n_3)} \circ \cdots \circ R_{(n_{k-1},n_k)}$
- q **acceptable** $:\Leftrightarrow R_q \neq \emptyset$
- Intraprocedurally acceptable paths:
  $IAP(r_{main}, n) := \{q \in path_{G^*}(r_{main}, n) | R_q\{w\} \neq \emptyset\}$

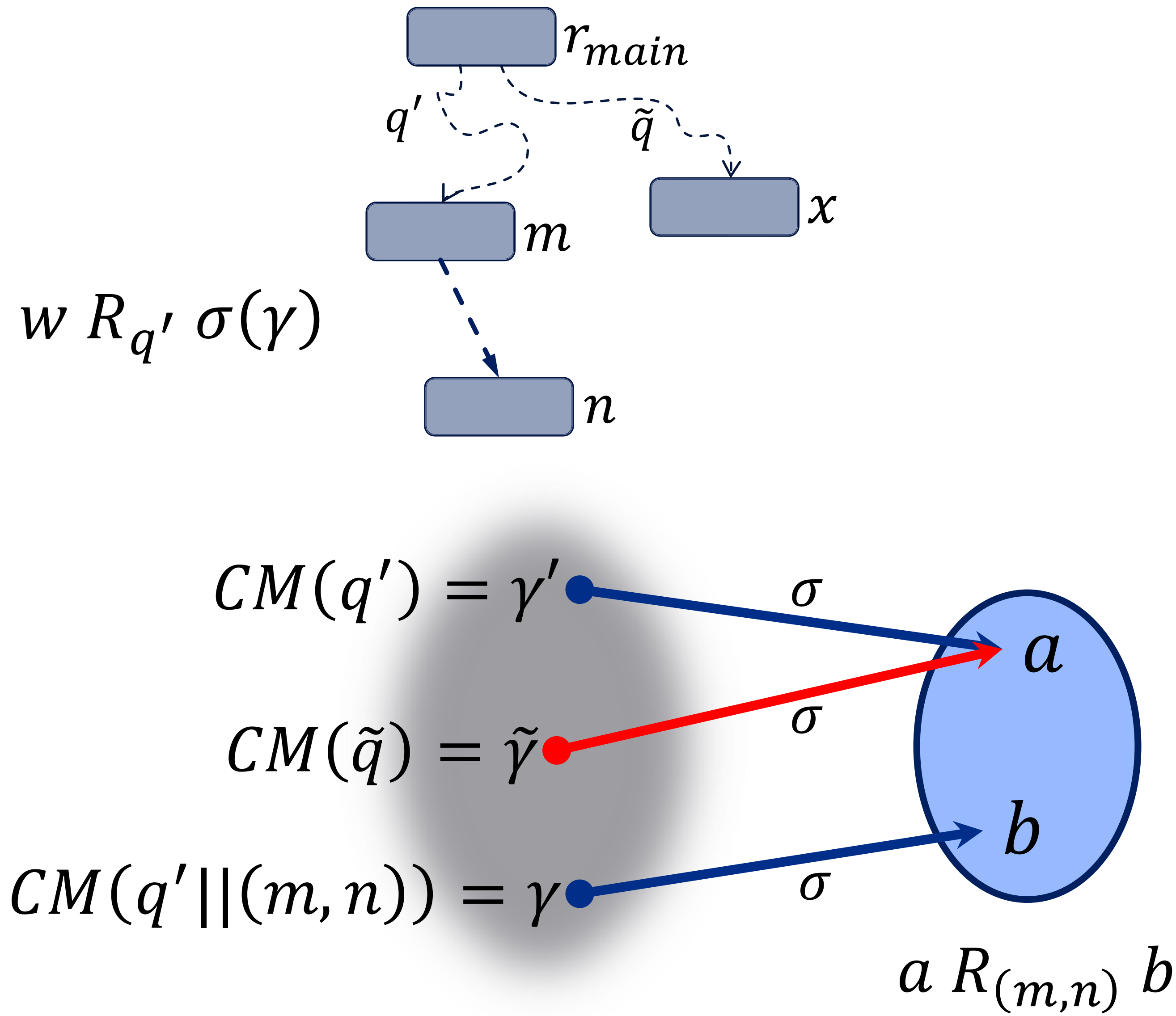
Invalid paths in IAP



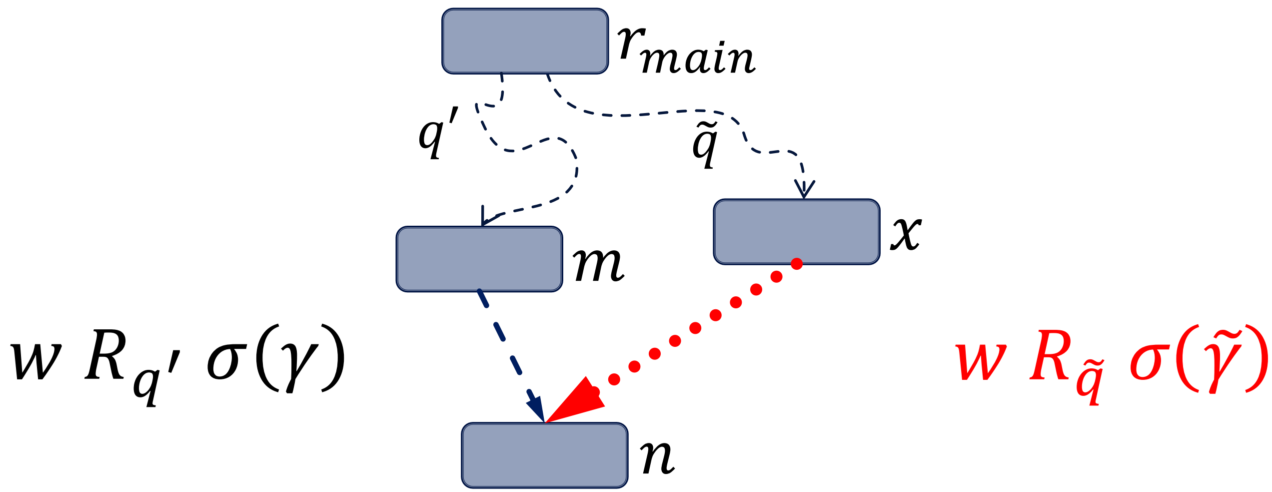
Invalid paths in IAP

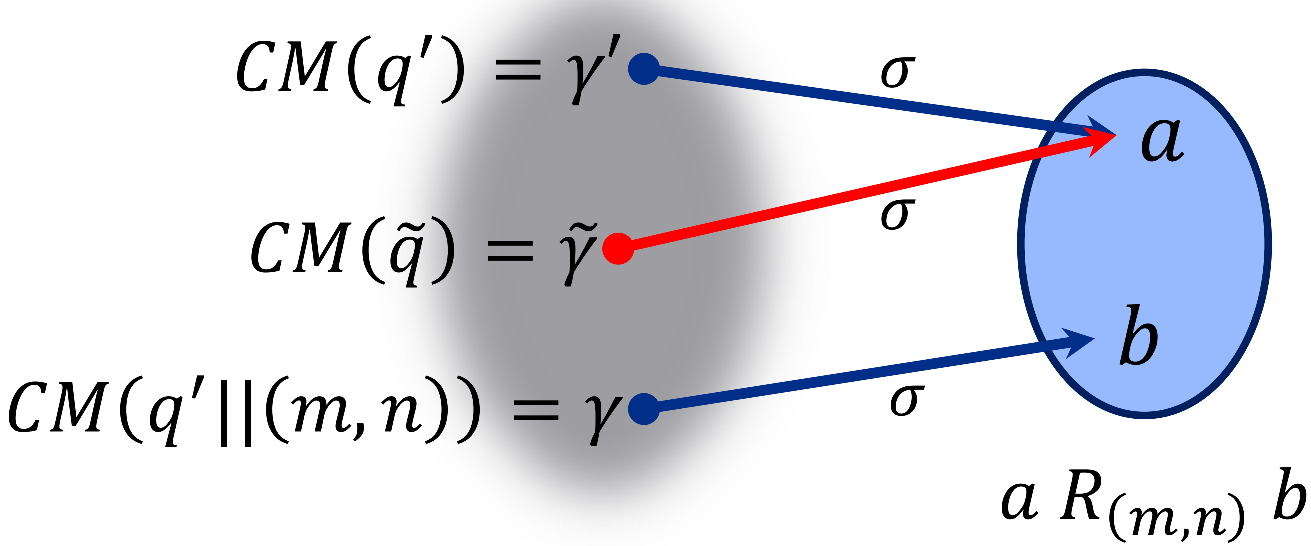


Invalid paths in IAP



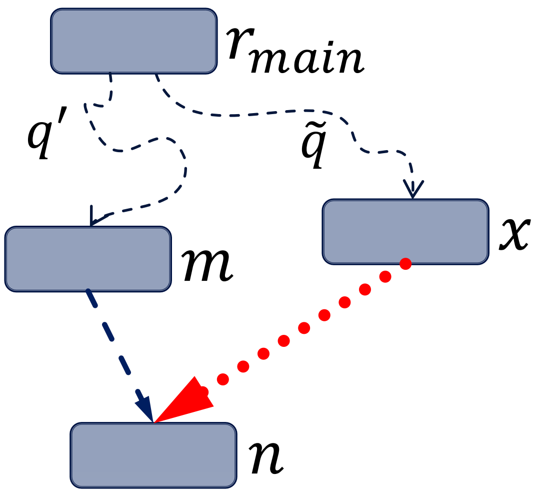
Invalid paths in *IAP*





Invalid paths in IAP

$w\ R_{q'}\ \sigma(\gamma)$



$w\ R_{\tilde{q}}\ \sigma(\tilde{\gamma})$

$CM(q') = \gamma'$

$$\Rightarrow IAP(r_{main}, n) \supseteq IVP(r_{main}, n)$$

$CM(q' \,||\, (m,n)) = \gamma$

$a\ R_{(m,n)}\ b$

Dataflow analysis using $\hat{\Gamma}$

Now define framework $(\hat{L}, \hat{F})$

- $\hat{L} := \hat{\Gamma} \to L$
- For edge (m, n), data in ξ_m is propagated to ξ_n from all $a \in \hat{\Gamma}$ to $b \in \hat{\Gamma}$ for which $a\, R_{(m,n)}\, b$:

$$f^*_{(m,n)}(\xi_m)(b) := \bigwedge\left\{f_{(m,n)}(\xi_m(a)) \,|\, a\, R_{(m,n)}\, b\right\}$$

- As before, define DF equations, the MFP solution $\{\hat{x}^*_n\}_{n \in N^*}$ (which exists as $\hat{\Gamma}$ finite), and the MOP solution $\hat{y}_n$ using IAP.
  $\Rightarrow$ Obviously $\hat{y}_n \leq y_n$, so a safe approximation.

Dataflow analysis using $\hat{\Gamma}$ (contd.)

- Without proof:
  - L distributive: $\hat{x}_n = \hat{y}_n \leq y_n$
  - L only monotone: $\hat{x}_n \leq \hat{y}_n \leq y_n$

- Examples:
  - $\hat{\Gamma} = \{w\}$
    $\sigma: \gamma \mapsto w$
    $*: \{w, w\} \mapsto w$
  - $\hat{\Gamma} = \mathbb{Z}/k\mathbb{Z},\ k \in \mathbb{N}$
    $\forall$ call $c_i \in N^*: \sigma(c_i) = i$
    $a * b \ := (a + b)\ mod\ k$

$\Rightarrow$ intraproc. analysis

$\Rightarrow$ only k call strings

Summary

Summary: CS Approach structure

- CS approach makes interprocedural flow explicit
  - Avoids functional composition of propagation functions
  - Keeps track of origin of data by tagging with CS
  - Complexity of approach “controlled” by Γ

Summary: Results of CS approach (1/2)

Assume (L, F) distributive.

- L and Γ finite:
  MFP_{CS} iteratively computable and yields MOP solution.
- Only L finite:
  Iterative calculation of MFP_{CS} will converge on MOP if Γ replaced by Γ_M, else likely to diverge
  - Depending on problem, better bounds exist.

Summary: Results of CS approach (2/2)

All other cases combinations of:

- (L, F) not distributive
- L infinite
- F not bounded

- Convergence then only guaranteed if Γ embedded into finite subset $\hat{\Gamma}$
  - Also likely to be the only feasible approach to the other situations

- Solution then is safe approximation of MOP

Thanks!