Monotone Frameworks

- Monotone and Distributive Frameworks
- Instances of Frameworks
- Constant Propagation Analysis

The Overall Pattern

Each of the four classical analyses take the form

$$\mathit{Analysis}_\circ(\ell) = \begin{cases} \iota & \text{if } \ell \in E \\ \bigsqcup\{\mathit{Analysis}_\bullet(\ell') \mid (\ell',\ell) \in F\} & \text{otherwise} \end{cases}$$

$$\mathit{Analysis}_\bullet(\ell) = f_\ell(\mathit{Analysis}_\circ(\ell))$$

where

- $\bigsqcup$ is $\bigcap$ or $\bigcup$ (and $\sqcup$ is $\cup$ or $\cap$),
- F is either *flow*$(S_\star)$ or *flow*$^R(S_\star)$,
- E is $\{$*init*$(S_\star)\}$ or *final*$(S_\star)$,
- ι specifies the initial or final analysis information, and
- f_ℓ is the transfer function associated with $B^\ell \in$ *blocks*$(S_\star)$.

The Principle: forward versus backward

- The *forward analyses* have F to be *flow*$(S_\star)$ and then *Analysis*$_\circ$ concerns entry conditions and *Analysis*$_\bullet$ concerns exit conditions; the equation system presupposes that $S_\star$ has isolated entries.

- The *backward analyses* have F to be *flow*$^R(S_\star)$ and then *Analysis*$_\circ$ concerns exit conditions and *Analysis*$_\bullet$ concerns entry conditions; the equation system presupposes that $S_\star$ has isolated exits.

The Principle: union versus intersecton

- When $\bigsqcup$ is $\bigcap$ we require the greatest sets that solve the equations and we are able to detect properties satisfied by *all execution paths* reaching (or leaving) the entry (or exit) of a label; the analysis is called a must-analysis.

- When $\bigsqcup$ is $\bigcup$ we require the smallest sets that solve the equations and we are able to detect properties satisfied by *at least one execution path* to (or from) the entry (or exit) of a label; the analysis is called a may-analysis.

Property Spaces

The *property space*, L, is used to represent the data flow information, and the *combination operator*, $\bigsqcup$: $\mathcal{P}(L) \to L$, is used to combine information from different paths.

- L is a *complete lattice*, that is, a partially ordered set, $(L, \sqsubseteq)$, such that each subset, Y, has a least upper bound, $\bigsqcup Y$.

- L satisfies the *Ascending Chain Condition*; that is, each ascending chain eventually stabilises (meaning that if $(l_n)_n$ is such that $l_1 \sqsubseteq l_2 \sqsubseteq l_3 \sqsubseteq \cdots$,then there exists n such that $l_n = l_{n+1} = \cdots$).

Example: Reaching Definitions

- $L = \mathcal{P}(\mathbf{Var}_\star \times \mathbf{Lab}_\star)$ is partially ordered by subset inclusion so $\sqsubseteq$ is $\subseteq$

- the least upper bound operation $\bigsqcup$ is $\bigcup$ and the least element $\bot$ is $\emptyset$

- L satisfies the Ascending Chain Condition because $\mathbf{Var}_\star \times \mathbf{Lab}_\star$ is finite (unlike $\mathbf{Var} \times \mathbf{Lab}$)

Example: Available Expressions

- $L = \mathcal{P}(\mathbf{AExp}_\star)$ is partially ordered by superset inclusion so $\sqsubseteq$ is $\supseteq$

- the least upper bound operation $\bigsqcup$ is $\bigcap$ and the least element $\bot$ is $\mathbf{AExp}_\star$

- L satisfies the Ascending Chain Condition because $\mathbf{AExp}_\star$ is finite (unlike $\mathbf{AExp}$)

Transfer Functions

The set of transfer functions, $\mathcal{F}$, is a set of monotone functions over L, meaning that

$$l \sqsubseteq l' \text{ implies } f_\ell(l) \sqsubseteq f_\ell(l')$$

and furthermore they fulfil the following conditions:

- $\mathcal{F}$ contains *all* the transfer functions $f_\ell : L \to L$ in question (for $\ell \in \mathbf{Lab}_\star$)

- $\mathcal{F}$ contains the *identity function*

- $\mathcal{F}$ is *closed under composition* of functions

Frameworks

A *Monotone Framework* consists of:

- a complete lattice, L, that satisfies the Ascending Chain Condition; we write $\bigsqcup$ for the least upper bound operator
- a set $\mathcal{F}$ of monotone functions from L to L that contains the identity function and that is closed under function composition

A *Distributive Framework* is a Monotone Framework where additionally all functions f in $\mathcal{F}$ are required to be distributive:

$$f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

Instances

An *instance* of a Framework consists of:

– the complete lattice, L, of the framework

– the space of functions, $\mathcal{F}$, of the framework

– a finite flow, F (typically $\mathit{flow}(S_\star)$ or $\mathit{flow}^R(S_\star)$)

– a finite set of *extremal labels*, E (typically $\{\mathit{init}(S_\star)\}$ or $\mathit{final}(S_\star)$)

– an *extremal value*, $\iota \in L$, for the extremal labels

– a mapping, $f_\cdot$, from the labels $\mathbf{Lab}_\star$ to transfer functions in $\mathcal{F}$

Equations of the Instance:

$$\mathit{Analysis}_\circ(\ell) = \bigsqcup\{\mathit{Analysis}_\bullet(\ell') \mid (\ell',\ell) \in F\} \sqcup \iota_E^\ell$$

$$\text{where } \iota_E^\ell = \begin{cases} \iota & \text{if } \ell \in E \\ \bot & \text{if } \ell \notin E \end{cases}$$

$$\mathit{Analysis}_\bullet(\ell) = f_\ell(\mathit{Analysis}_\circ(\ell))$$

Constraints of the Instance:

$$\mathit{Analysis}_\circ(\ell) \sqsupseteq \bigsqcup\{\mathit{Analysis}_\bullet(\ell') \mid (\ell',\ell) \in F\} \sqcup \iota_E^\ell$$

$$\text{where } \iota_E^\ell = \begin{cases} \iota & \text{if } \ell \in E \\ \bot & \text{if } \ell \notin E \end{cases}$$

$$\mathit{Analysis}_\bullet(\ell) \sqsupseteq f_\ell(\mathit{Analysis}_\circ(\ell))$$

The Examples Revisited

	Available Expressions	Reaching Definitions	Very Busy Expressions	Live Variables
L	$\mathcal{P}(\mathbf{AExp}_\star)$	$\mathcal{P}(\mathbf{Var}_\star \times \mathbf{Lab}_\star)$	$\mathcal{P}(\mathbf{AExp}_\star)$	$\mathcal{P}(\mathbf{Var}_\star)$
$\sqsubseteq$	$\supseteq$	$\subseteq$	$\supseteq$	$\subseteq$
$\bigsqcup$	$\bigcap$	$\bigcup$	$\bigcap$	$\bigcup$
$\bot$	$\mathbf{AExp}_\star$	$\emptyset$	$\mathbf{AExp}_\star$	$\emptyset$
ι	$\emptyset$	$\{(x,?) \mid x \in \mathit{FV}(S_\star)\}$	$\emptyset$	$\emptyset$
E	$\{\mathit{init}(S_\star)\}$	$\{\mathit{init}(S_\star)\}$	$\mathit{final}(S_\star)$	$\mathit{final}(S_\star)$
F	$\mathit{flow}(S_\star)$	$\mathit{flow}(S_\star)$	$\mathit{flow}^R(S_\star)$	$\mathit{flow}^R(S_\star)$
$\mathcal{F}$	$\{f : L \to L \mid \exists l_k, l_g : f(l) = (l \setminus l_k) \cup l_g\}$			
f_ℓ	$f_\ell(l) = (l \setminus \mathit{kill}(B^\ell)) \cup \mathit{gen}(B^\ell)$ where $B^\ell \in \mathit{blocks}(S_\star)$			

Bit Vector Frameworks

A *Bit Vector Framework* has

- $L = \mathcal{P}(D)$ for D finite
- $\mathcal{F} = \{f \mid \exists l_k, l_g : f(l) = (l \setminus l_k) \cup l_g\}$

Examples:

- Available Expressions
- Live Variables
- Reaching Definitions
- Very Busy Expressions

Lemma: Bit Vector Frameworks are always Distributive Frameworks

Proof

$$f(l_1 \sqcup l_2) = \left\{\begin{array}{l} f(l_1 \cup l_2) \\ f(l_1 \cap l_2) \end{array}\right. = \left\{\begin{array}{l} ((l_1 \cup l_2) \setminus l_k) \cup l_g \\ ((l_1 \cap l_2) \setminus l_k) \cup l_g \end{array}\right.$$

$$= \left\{\begin{array}{l} ((l_1 \setminus l_k) \cup (l_2 \setminus l_k)) \cup l_g \\ ((l_1 \setminus l_k) \cap (l_2 \setminus l_k)) \cup l_g \end{array}\right. = \left\{\begin{array}{l} ((l_1 \setminus l_k) \cup l_g) \cup ((l_2 \setminus l_k) \cup l_g) \\ ((l_1 \setminus l_k) \cup l_g) \cap ((l_2 \setminus l_k) \cup l_g) \end{array}\right.$$

$$= \left\{\begin{array}{l} f(l_1) \cup f(l_2) \\ f(l_1) \cap f(l_2) \end{array}\right. = f(l_1) \sqcup f(l_2)$$

- $id(l) = (l \setminus \emptyset) \cup \emptyset$
- $f_2(f_1(l)) = (((l \setminus l_k^1) \cup l_g^1) \setminus l_k^2) \cup l_g^2 = (l \setminus (l_k^1 \cup l_k^2)) \cup ((l_g^1 \setminus l_k^2) \cup l_g^2)$
- monotonicity follows from distributivity
- $\mathcal{P}(D)$ satisfies the Ascending Chain Condition because D is finite

The Constant Propagation Framework

An example of a Monotone Framework that is not a Distributive Framework

The aim of the *Constant Propagation Analysis* is to determine

> For each program point, whether or not a variable has a constant value whenever execution reaches that point.

Example:

$[\texttt{x:=6}]^1; [\texttt{y:=3}]^2; \texttt{while } [\texttt{x} > \texttt{y}]^3 \texttt{ do } ([\texttt{x:=x} - 1]^4; [\texttt{z:=y} * \texttt{y}]^6)$

The analysis enables a transformation into

$[\texttt{x:=6}]^1; [\texttt{y:=3}]^2; \texttt{while } [\texttt{x} > 3]^3 \texttt{ do } ([\texttt{x:=x} - 1]^4; [\texttt{z:=9}]^6)$

Elements of L

$$\widehat{\mathbf{State}}_{\mathsf{CP}} = ((\mathbf{Var}_\star \to \mathbf{Z}^\top)_\perp, \sqsubseteq)$$

Idea:

- $\perp$ is the least element: no information is available
- $\widehat{\sigma} \in \mathbf{Var}_\star \to \mathbf{Z}^\top$ specifies for each variable whether it is constant:
  - $\widehat{\sigma}(x) \in \mathbf{Z}$: x is constant and the value is $\widehat{\sigma}(x)$
  - $\widehat{\sigma}(x) = \top$: x might not be constant

Partial Ordering on L

The partial ordering $\sqsubseteq$ on $(\mathbf{Var}_\star \to \mathbf{Z}^\top)_\perp$ is defined by

$$\forall \widehat{\sigma} \in (\mathbf{Var}_\star \to \mathbf{Z}^\top)_\perp : \quad \perp \sqsubseteq \widehat{\sigma}$$

$$\forall \widehat{\sigma}_1, \widehat{\sigma}_2 \in \mathbf{Var}_\star \to \mathbf{Z}^\top : \quad \widehat{\sigma}_1 \sqsubseteq \widehat{\sigma}_2 \quad \underline{\text{iff}} \quad \forall x : \widehat{\sigma}_1(x) \sqsubseteq \widehat{\sigma}_2(x)$$

where $\mathbf{Z}^\top = \mathbf{Z} \cup \{\top\}$ is partially ordered as follows:

$$\forall z \in \mathbf{Z}^\top : z \sqsubseteq \top$$

$$\forall z_1, z_2 \in \mathbf{Z} : (z_1 \sqsubseteq z_2) \Leftrightarrow (z_1 = z_2)$$

Transfer Functions in $\mathcal{F}$

$$\mathcal{F}_{\mathsf{CP}} = \{f \mid f \text{ is a monotone function on } \widehat{\mathbf{State}}_{\mathsf{CP}}\}$$

Lemma

Constant Propagation as defined by $\widehat{\mathbf{State}}_{\mathsf{CP}}$ and $\mathcal{F}_{\mathsf{CP}}$ is a Monotone Framework

Instances

Constant Propagation is a forward analysis, so for the program $S_\star$:

- the flow, F, is *flow*$(S_\star)$,
- the extremal labels, E, is $\{$*init*$(S_\star)\}$,
- the extremal value, ι_{CP}, is $\lambda x.\top$, and
- the mapping, $f_\cdot^{\mathsf{CP}}$, of labels to transfer functions is as shown next

Constant Propagation Analysis

$$\mathcal{A}_{\mathsf{CP}} : \mathbf{AExp} \to (\widehat{\mathbf{State}}_{\mathsf{CP}} \to \mathbf{Z}_\perp^\top)$$

$$\mathcal{A}_{\mathsf{CP}}[\![x]\!]\widehat{\sigma} = \begin{cases} \perp & \text{if } \widehat{\sigma} = \perp \\ \widehat{\sigma}(x) & \text{otherwise} \end{cases}$$

$$\mathcal{A}_{\mathsf{CP}}[\![n]\!]\widehat{\sigma} = \begin{cases} \perp & \text{if } \widehat{\sigma} = \perp \\ n & \text{otherwise} \end{cases}$$

$$\mathcal{A}_{\mathsf{CP}}[\![a_1 \; op_a \; a_2]\!]\widehat{\sigma} = \mathcal{A}_{\mathsf{CP}}[\![a_1]\!]\widehat{\sigma} \; \widehat{\mathsf{op}}_a \; \mathcal{A}_{\mathsf{CP}}[\![a_2]\!]\widehat{\sigma}$$

transfer functions: f_ℓ^{CP}

$$[x := a]^\ell : \quad f_\ell^{\mathsf{CP}}(\widehat{\sigma}) = \begin{cases} \perp & \text{if } \widehat{\sigma} = \perp \\ \widehat{\sigma}[x \mapsto \mathcal{A}_{\mathsf{CP}}[\![a]\!]\widehat{\sigma}] & \text{otherwise} \end{cases}$$

$$[\mathtt{skip}]^\ell : \quad f_\ell^{\mathsf{CP}}(\widehat{\sigma}) = \widehat{\sigma}$$

$$[b]^\ell : \quad f_\ell^{\mathsf{CP}}(\widehat{\sigma}) = \widehat{\sigma}$$

Lemma

Constant Propagation is not a Distributive Framework

Proof

Consider the transfer function f_ℓ^{CP} for $[\mathtt{y:=x*x}]^\ell$

Let $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ be such that $\widehat{\sigma}_1(\mathtt{x}) = 1$ and $\widehat{\sigma}_2(\mathtt{x}) = -1$

Then $\widehat{\sigma}_1 \sqcup \widehat{\sigma}_2$ maps $\mathtt{x}$ to $\top$ — $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_1 \sqcup \widehat{\sigma}_2)$ maps $\mathtt{y}$ to $\top$

Both $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_1)$ and $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_2)$ map $\mathtt{y}$ to 1 — $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_1) \sqcup f_\ell^{\mathsf{CP}}(\widehat{\sigma}_2)$ maps $\mathtt{y}$ to 1

