Introduction to Lattice Theory

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Approximations and correctness

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The 'may be' version

- Solution $\{a, b, c, d\}$ is approximate and correct.
- Solution $\{a, b\}$ is *incorrect*.
- So, the 'fastest but most useless' analysis for this problem returns the universal set.

Sets and relations

A (binary) relation $\mathcal{R}$ between sets S_1 and S_2 is just a subset of $S_1 \times S_2$. Similarly, any subset $\mathcal{R}$ of $S_1 \times S_2$ is a relation between sets S_1 and S_2. That is, for any sets S_1, S_2, $\mathcal{R}$ is a relation between S_1 and S_2 *iff*

$$\mathcal{R} \subseteq S_1 \times S_2$$

If $(s_1, s_2) \in \mathcal{R}$, we also write $s_1 \mathcal{R} s_2$
Let $S_1 = \{a, b, c, d\}$, $S_2 = \{1, 2, 3\}$
$\mathcal{R}_1 = \{(a, 1), (b, 2), (c, 3), (d, 1)\}$?
$\mathcal{R}_2 = \{(a, 1), (b, 1), (c, 1), (c, 2), (c, 3)\}$?
$\mathcal{R}_3 = \{\}$?
$\mathcal{R}_4 = \{(a, a), (b, b), (c, c)\}$?
$\mathcal{R}_5 = \{(a, x), (b, y)\}$?

Some ‘real’ relations

$S_1 = \{$Rajeev, Sanjay, Bhim, Mohandas, Duryodhan$\}$
$S_2 = \{$Pandu, Gandhari, Indira, Feroz, Bhishma$\}$
$\texttt{Child} = \{(\text{Rajeev}, \text{Indira}), (\text{Rajeev}, \text{Feroz}),$
$(\text{Sanjay}, \text{Indira}), (\text{Sanjay}, \text{Feroz}),$
$(\text{Bhim}, \text{Pandu}), (\text{Duryodhan}, \text{Gandhari})\}$

$Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
$<= \{\cdots, (-2, -1), (-2, 0), (-2, 1), (-2, 2), (-1, 0), \cdots\}$
$\texttt{double} = \{\cdots, (-1, -2), (0, 0), (1, 2), (2, 4) \cdots\}$

- Relations can be $N - N$, $1 - N$, $N - 1$ or $1 - 1$.
- Relations can be ‘total’ (all of S_1) or ‘onto’ (all of S_2)
- Functions are just $N - 1$ relations!

Kinds of relations

A relation $\mathcal{R}$ from S to S is

- *reflexive iff* $\forall a \in S.\ (a, a) \in \mathcal{R}$. Example: $\leq$ is reflexive, but $<$ is not.
- *symmetric iff* $(a, b) \in \mathcal{R} \Rightarrow (b, a) \in \mathcal{R}$. Example: $\neq$ and $=$ are symmetric, but $\leq$ is not.
- *anti-symmetric iff* $a \neq b \wedge (a, b) \in \mathcal{R} \Rightarrow (b, a) \notin \mathcal{R}$. Example: $\leq$ is anti-symmetric, while $\neq$, $=$ are not.
- *transitive iff* $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \Rightarrow (a, c) \in \mathcal{R}$. Example: $<$, $\leq$, `Ancestor` are transitive, but $\neq$, `Parent` are not.

Note: $\mathcal{R}$ is symmetric and transitive $\not\Rightarrow$ $\mathcal{R}$ is reflexive! Example: the empty relation!!

Equivalence relations

Relation $\equiv$ from S to S is an equivalence relation *iff* it is reflexive, symmetric and transitive. $\equiv$ *partitions* S into disjoint subsets (equivalence classes) where all elements of each sub-set are $\equiv$-related to each other, and no two elements across the subsets are $\equiv$-related. Examples:

- $=$ partitions Z into an infinite number of singleton equivalence classes: $\{\cdots, \{-1\}, \{0\}, \{1\}, \cdots\}$.
- $=_{mod\ n}$ partitions N into n infinitely large equivalence classes: $\{\{0, n, 2n, \cdots\}, \{1, n+1, 2n+1, \cdots\} \cdots \{n-1, 2n-1, 3n-1 \cdots\}\}$.
- `sibling` partitions the entire human population into equivalence classes, where a `sibling` b *iff* a and b have *both* parents in common.

Partial orders

Relation $\sqsubseteq$ from S to S is a *partial order iff* it is reflexive, anti-symmetric and transitive. Note that there may be elements a, b in S such that neither $a \sqsubseteq b$ nor $b \sqsubseteq a$. If either $a \sqsubseteq b$ or $b \sqsubseteq a$ for all a, b, then $\sqsubseteq$ is called a *total* order. Examples:

- $\leq$ is a total order over Z.
- Relation $\subseteq$ is a partial order over any set S of sets. For all sets A, B, C: $A \subseteq A$; $A \subseteq B \wedge A \neq B \Rightarrow B \not\subseteq A$; and $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$. And of course, there exist sets S_1, S_2 such that neither $S_1 \subseteq S_2$ nor $S_2 \subseteq S_1$.
- Relation $|$ (divides) is a partial order over N. For any natural numbers a, b, c, $a|a$; $a|b \wedge a \neq b \Rightarrow b \not| a$; and $a|b \wedge b|c \Rightarrow a|c$.

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- Everything approximates itself (perfectly!).
- a approximates b and $a \neq b \Rightarrow b$ definitely does not approximate a.
- a approximates b and b approximates $c \Rightarrow a$ approximates c. In such cases, b is a ‘more precise’ approximation of c than a.
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Example: $3 \sqsupset 3.1 \sqsupset 3.14 \sqsupset 3.141 \sqsupset \cdots \sqsupset \pi$.

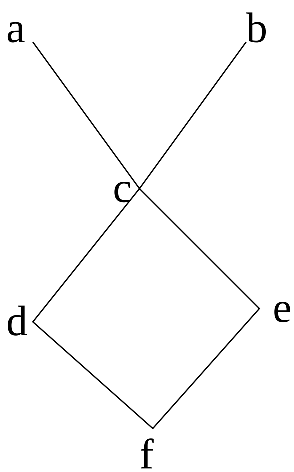
Exercises

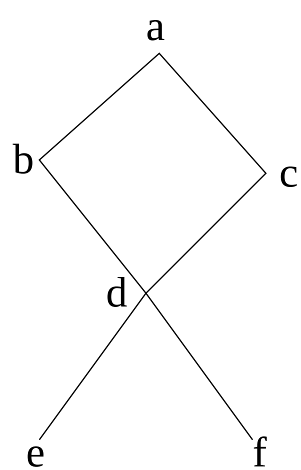
- Which of the following are reflexive, symmetric, anti-symmetric, equivalence relations, partial orders?
  - $\{(a,a),(b,b),(c,c)\}$ over set $S = \{a,b,c\}$
  - $\{(a,a),(b,b),(c,c),(a,b),(b,a)\}$ over set $S = \{a,b,c\}$
  - $\mathcal{R} = \{\}$ over any set S.
  - $\mathcal{R} = S \times S$ over any set S.
  - $f(x) = x^2$
  - $f(x) = -x$
  - $\mathcal{R}$ is defined over S, the set of functions from N to N, as $f\mathcal{R}g$ *iff* $\forall x. f(x)|g(x)$.
- Give examples of other 'real' equivalence classes and partial orders.

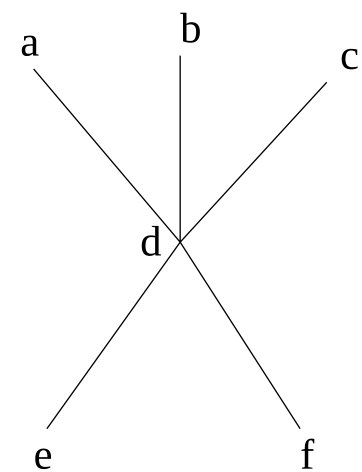
Posets

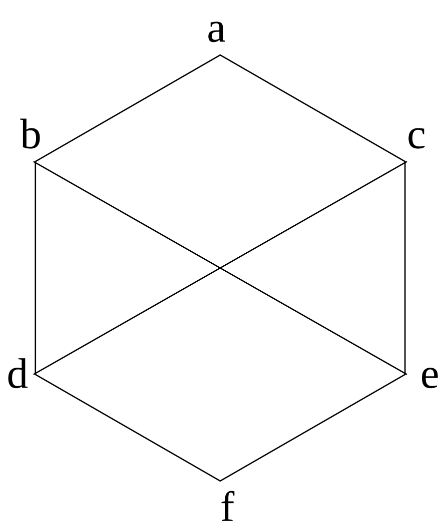
- If partial ordering $\sqsubseteq$ is defined over set S, then $(S, \sqsubseteq)$ is called a *partially-ordered set* or a *poset*.
- $b \sqsupseteq a$ is the same as $a \sqsubseteq b$. Note that if $\sqsubseteq$ is a partial order, then so is $\sqsupseteq$.
- $a \sqsubset b$ is the same as $a \sqsubseteq b$ and $a \neq b$. $\sqsubset$ is *not* a partial order. Similarly $\sqsupset$.
- Element b is a *minimal* element or *lower bound* of poset $(S, \sqsubseteq)$ *iff* $\forall x \in S . x \not\sqsubset b$.
- Similarly t is a *maximal* element or *upper bound* of the poset *iff* $\forall x \in S . x \not\sqsupset t$. Note: Minimal and maximal elements may not be unique for a poset.

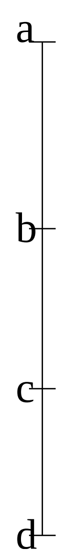
Maximal and minimal elements

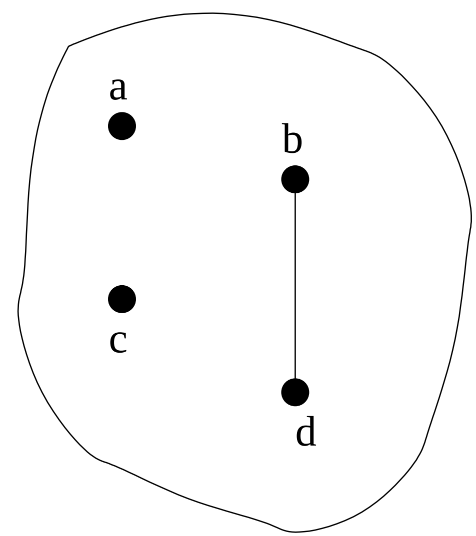












Lower and upper bounds

Consider a poset $(L, \sqsubseteq)$ and a subset Y of L. Element $l \in L$ is an upper bound of Y if $\forall l' \in Y . l' \sqsubseteq l$. Similarly, l is a lower bound if $\forall l' \in Y . l' \sqsupseteq l$. Note: l may not belong to the subset Y. l may not be unique, i.e. Y may have many (or no) lower and upper bounds.

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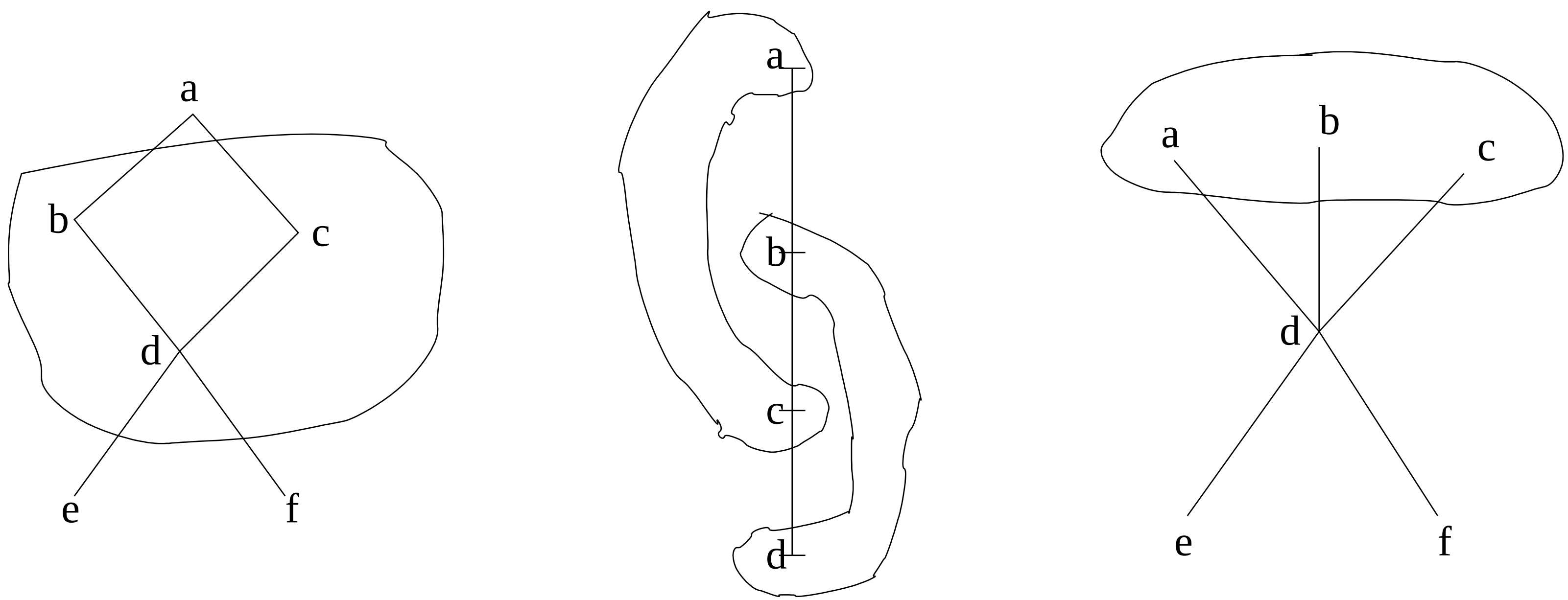

LUBs and GLBs

- $l \in L$ is a least upper bound (LUB) of a subset Y *iff* l is an upper bound of Y and $l \sqsubseteq l'$ for all other upper bounds l' of Y.
- Similarly, greatest lower bound (GLB) of a subset Y is a lower bound that is $\sqsupseteq$ all other lower bounds.
- LUB of a set Y is denoted as $\bigsqcup Y$ and is also called the *join* operator.
- GLB of a set Y is denoted as $\bigsqcap Y$ and is also called the *meet* operator.
- A subset Y may be such that $\bigsqcup Y$ or $\bigsqcap Y$ do not exist. But if they exist, they are *unique*.
- $\bigsqcup\{y_1, y_2\}$ is also written $y_1 \sqcup y_2$. Similarly $\bigsqcap\{y_1, y_2\} = y_1 \sqcap y_2$.

Lattices

- A poset $(L, \sqsubseteq)$ is a complete lattice $(L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ *iff* all subsets Y of L have greatest lower bounds as well as least upper bounds.
- $\bot = \bigsqcup \phi = \bigsqcap L$ is the *least element* of L.
- $\top = \bigsqcap \phi = \bigsqcup L$ is the *greatest element* of L.
- $x \sqcup \top = \top$, $x \sqcup \bot = x$
- $x \sqcap \bot = \bot$, $x \sqcap \top = x$

More on lattices

- For any set S, $(2^S, \subseteq, \bigcup, \bigcap, \phi, S)$ is a complete lattice.
- Is $(Z, \leq)$ a complete lattice? And what about $(N, \leq)$?

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Properties of $\sqcup, \sqcap$:

Idempotence $x \sqcup x = x \sqcap x = x$

Commutativity $x \sqcup y = y \sqcup x$ and $x \sqcap y = y \sqcap x$

Associativity $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$. Similarly for $\sqcap$.

Absorption $x \sqcup (x \sqcap y) = x$

Exercises

Prove the following:

- If $\sqsubseteq$ is a partial order, then so is $\sqsupseteq$.
- When they exist, $\bigsqcup Y$ and $\bigsqcap Y$ are unique for any subset Y of a poset $(L, \sqsubseteq)$.
- $(x \sqcup y = y) \Leftrightarrow (x \sqsubseteq y)$
- The idempotence, commutativity, associativity and absorption properties of $\sqcup$.

Chains

Any totally ordered subset S of poset $(L, \sqsubseteq)$ is called a *chain*.

That is, $\forall l_1, l_2 \in S.\ (l_1 \sqsubseteq l_2) \vee (l_2 \sqsubseteq l_1)$.

Note: An empty subset of L is also a chain!!

A *sequence* of elements $l_1, l_2, \cdots$ is an *ascending chain iff* $i < j \Rightarrow l_i \sqsubseteq l_j$. Similarly, a sequence is a *descending chain iff* $i < j \Rightarrow l_i \sqsupseteq l_j$.

Example ...

Chains $\cdots$

The *height* of a poset $(L, \sqsubseteq)$ is h if the largest chain in the lattice contains $h+1$ elements.

Poset $(L, \sqsubseteq)$ has a finite height *iff* all chains are finite, i.e. all ascending and descending chains are of the form $l_1, l_2, \cdots l_k, l_{k+1}, l_{k+2}, \cdots$ where $l_j = l_k \; \forall j \geq k$.

Obviously finite posets have finite heights!

Examples of infinite posets with finite and infinite heights?

Product lattices

- Given two posets $(L_1, \sqsubseteq_1)$ and $(L_2, \sqsubseteq_2)$, $(L, \sqsubseteq)$ is also a partial order where

$$L = \{(l_1, l_2) | l_1 \in L_1 \wedge l_2 \in L_2\}$$

and

$$(l_{11}, l_{12}) \sqsubseteq (l_{21}, l_{22}) \text{ iff } l_{11} \sqsubseteq_1 l_{21} \wedge l_{12} \sqsubseteq_2 l_{22}$$

- Prove the above!

Product lattices $\cdots$

- If each L_i is a complete lattice, then so is $(L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ as follows:

  $$\bigsqcup Y = (\bigsqcup_1 \{l_1 | \exists l_2 : (l_1, l_2) \in Y\}, \bigsqcup_2 \{l_2 | \exists l_1 : (l_1, l_2) \in Y\}$$

  Similarly for $\bigsqcap$
  $\bot = (\bot_1, \bot_2)$
  $\top = (\top_1, \top_2)$
- L often referred to as $L_1 \times L_2$, the cartesian product of L_1 and L_2.
- Cartesian products can be extended to any number of posets or lattices, i.e. $L_1 \times L_2 \times L_3 \times \cdots \times L_k$

Functions

Consider posets $(L_1, \sqsubseteq_1)$, $(L_2, \sqsubseteq_2)$, and a function $f : L_1 \rightarrow L_2$.

- f is a *monotonic* (or *monotone*) function *iff* $\forall x, y . x \sqsubseteq y \ \Rightarrow f(x) \sqsubseteq f(y)$
- f is a completely *additive* (or *distributive*) function if $\forall Y \subseteq L_1 . f(\bigsqcup_1 Y) = \bigsqcup_2 \{f(l') | l' \in Y\}$ whenever $\bigsqcup_1 Y$ exists.
- Similarly, it is completely *multiplicative* if $\forall Y \subseteq L_1 . f(\bigsqcap_1 Y) = \bigsqcap_2 \{f(l') | l' \in Y\}$ whenever $\bigsqcap_1 Y$ exists.
- A function is *strict* if $f(\bot_1) = \bot_2$. f is completely additive $\Rightarrow f$ is strict.

Functions, fixed points $\cdots$

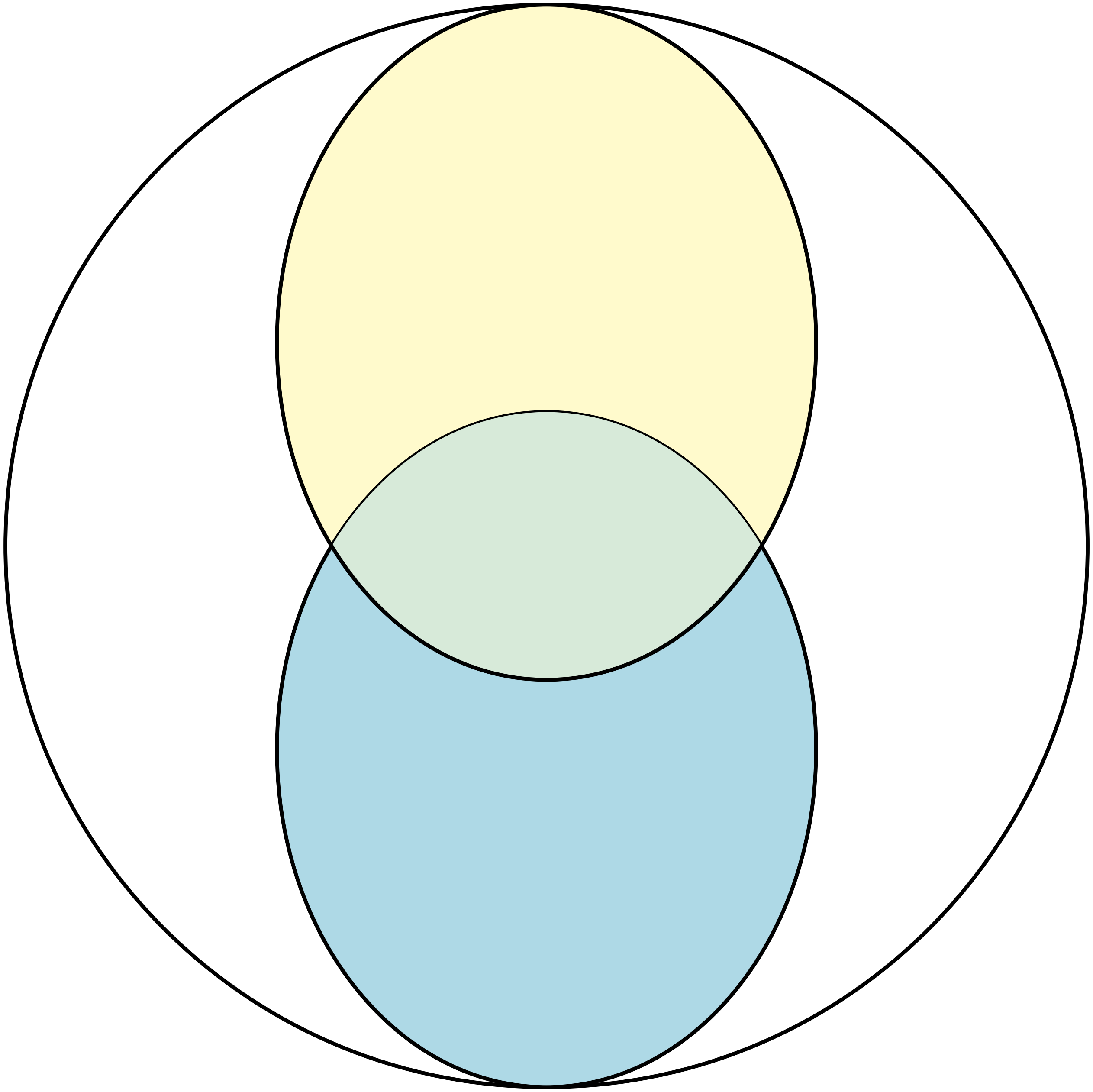
Consider a monotone function $f : L \rightarrow L$ on a complete lattice $(L, \sqsubseteq, \bigsqcup, \bigsqcap, \perp, \top)$.

- A *fixed point* of f is some $l \in L$ such that $f(l) = l$.
- What, if any, are the fixed point(s) of the following functions over Z?
  - $f(x) = x + 3$
  - $f(x) = x^2$
  - $f(x) = x$
  - $f(x) = x!$ (factorial, over N)
- $Fix(f) = \{l \in L | f(l) = l\}$, the set of fixed points of $f : L \rightarrow L$.

Reductive and extensive regions

- $f : L \rightarrow L$ is *reductive* at $l \in L$ if $f(l) \sqsubseteq l$.
  $Red(f) = \{l \in L | f(l) \sqsubseteq l\}$
- $f : L \rightarrow L$ is *extensive* at $l \in L$ if $f(l) \sqsupseteq l$.
  $Ext(f) = \{l \in L | f(l) \sqsupseteq l\}$
- The function f itself is reductive (extensive) if $Red(f) = L$ ($Ext(f) = L$).
- Example: $f(x) = x + 3$ over $(Z, \leq)$ is extensive, while $f(x) = x - 3$ is reductive.

Reductive, extensive regions



Computation of fixed points

For a monotone function f over a complete lattice L:

- Least fixed point $lfp(f) = \bigsqcap Fix(f)$
- Greatest fixed point $gfp(f) = \bigsqcup Fix(f)$

Tarski's theorem: For a complete lattice $(L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and monotone $f : L \rightarrow L$
$lfp(x) = \bigsqcap Red(f) \in Fix(f)$
$gfp(x) = \bigsqcup Ext(f) \in Fix(f)$

- $lfp(f)$ is a fixed point of f that is $\sqsubseteq$ all other fixed points. Similarly, $gfp(f)$.
- If all chains in L are finite, $lfp(f)$ can be computed as the limit of the chain $f^n(\bot)$, i.e. $\bot, f(\bot), f^2(\bot) \cdots$. In other words, $lfp(f) = f^k(\bot)$ such that $f^k(\bot) = f^{k+1}(\bot)$

Analysis and lattices

- Lattices can be used to model approximations
- $a \sqsubseteq b$ means a is *more* precise than b in the semantics and some analysis literature.
- But in data flow analysis literature, it means a is *less* precise than b. This means $\cdots$
- $\sqsubseteq$ and $\sqsupseteq$ are interchanged;
- $\perp$ and $\top$ are interchanged;
- $\sqcup$ and $\sqcap$ are interchanged;
- lfp and gfp are interchanged.
- Basically, the lattice is hung ‘upside down’.

Analysis and fixed points

- Analysis equations of the form:
  $I_o(n) = f_n(I_i(n))$
  $I_i(n) = {f'}_n(I_o(n_1), I_o(n_2) \cdots I_o(n_k))$

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- If there are k nodes, there are $2k$ pieces of information, namely $I_i(1), I_o(1), I_i(2), I_o(2) \cdots I_i(k), I_o(k)$ defined mutually recursively by the above equations.

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- So, the above equations can be together written as:
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- So, the above equations can be together written as: $I = F(I)$ where $F(I) = \langle f'_1(I), f_1(I), \cdots f'_k(I), f_k(I) \rangle$
- So, the answer we seek, namely I is nothing but a fixed point of F! And the most precise answer is the least fixed point of F!!

Home work!

1. Let L be a complete lattice $(L, \sqsubseteq_L, \bigsqcup_L, \bigsqcap_L, \bot_L, \top_L)$. Consider the space of total functions over L, say F. That is, F consists of *all* total functions from L to L. Prove that $(F, \sqsubseteq_F, \bigsqcup_F, \bigsqcap_F, \bot_F, \top_F)$ is also a complete lattice, where:
   $f \sqsubseteq_F g$ *iff* $\forall x \in L. f(x) \sqsubseteq_L g(x)$
   $\forall Y \subseteq F. \bigsqcup_F Y = \lambda x. \bigsqcup_L \{f(x) | f \in Y\}$
   $\forall Y \subseteq F. \bigsqcap_F Y = \lambda x. \bigsqcap_L \{f(x) | f \in Y\}$
   $\bot_F = \lambda x. \bot_L$
   $\top_F = \lambda x. \top_L$

2. Prove that the limit of the chain $\bot, f(\bot), f^2(\bot) \cdots$ for monotone f over a complete lattice with only finite chains is indeed $lfp(f)$.

$f = \lambda x.e$ is the same as $f(x) = e$.