Programming Language Concepts: Lecture 21

Madhavan Mukund

Chennai Mathematical Institute

madhavan@cmi.ac.in

http://www.cmi.ac.in/~madhavan/courses/pl2009

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Type inference with shallow types

Consider

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applypair f x y = (f x,f y)
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We have to unify the following set of constraints

```
id  :: a -> a
7   :: Int
'c' :: Char
a = Int              (from id 7)
a = Char             (from id 'c')
```

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We have to unify the following set of constraints

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7   :: Int
'c' :: Char
a = Int             (from id 7)
a = Char            (from id 'c')
```

Not possible! Haskell compiler says

```
applypair :: (a -> b) -> b -> b -> (b,b)}
```

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Now, we have

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What's going on?

Type inference with shallow types

Extend λ-calculus with "local" definitions, like `where`

$$\Lambda = C_i \mid x \mid \lambda x.M \mid MN \mid \text{let } f = e \text{ in } M$$

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Here is the λ-term for the second version of `applypair`

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Here is the λ-term for the second version of `applypair`

$$\text{let } f = \lambda z.z \text{ in } \lambda xy.pair\ (fx)(fy)$$

In fact, Haskell allows both

```
let f z = z in applypair x y = (f x,f y)
```

and

```
applypair x y = (f x,f y) where f z = z
```

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Type inference with shallow types

- $\text{let } f = e \text{ in } \lambda x.M$ and $(\lambda f x.M)e$ are equivalent with respect to β-reduction
- ... but type inference works differently for the two
- One may be typeable while the other is not
  - $(\lambda I.(II))(\lambda x.x)$
  - $\text{let } I = \lambda x.x \text{ in } (II)$

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First attempt

- Assume $f :: t$ where $\alpha, \beta, \ldots$ are type variables occurring in t
- Make a separate copy of type variables for each instance of f in M'

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Type inference for $M = \text{let } f = e \text{ in } M'$

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Example

- $\text{let } f = \lambda z.z \text{ in } \lambda xy.pair\ (fx)(fy)$
- First instance of f has type $\alpha_1 \rightarrow \beta_1$
- Second instance of f has type $\alpha_2 \rightarrow \beta_2$

Type inference with shallow types

A subtle problem

```
applypair2 w x y = ((tag x),(tag y))
  where
    tag      = pair w
    pair s t = (s,t)
```

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- `applypair2 w x y` $\rightsquigarrow$ `((w,x),(w,y))`
- Type should be
  `applypair2 ::  a -> b -> c -> ((a,b),(a,c))`

Type inference with shallow types

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applypair2 w x y = ((tag x),(tag y))
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Type inference

```
applypair2 :: a -> b -> c -> (d,e)
pair       :: f -> g -> (f,g)
tag        :: h -> (i,h)
```

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- Using `let` rule, two instances of `tag` get different types
  - `d = h1 -> (i1,h1)`
  - `e = h2 -> (i2,h2)`

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- `a = i` because `tag` uses input `w` from `applypair2`
- Using `let` rule, two instances of `tag` get different types
  - `d = h1 -> (i1,h1)`
  - `e = h2 -> (i2,h2)`
- End up with
  `applypair2 ::  a -> b -> c -> ((i1,b),(i2,c))`
  - The connection `a = i = i1 = i2` is lost!

Type inference with shallow types

- In `tag :: h -> (i,h)`
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  - Should not make copies of non-generic variables!

Correct type inference rule for $M = \text{let } f = e \text{ in } M'$

- Assume $f :: t$ where $\alpha, \beta, \ldots$ are generic type variables occurring in t
- Make a separate copy of these generic type variables for each instance of f in M'
- Non-generic variables retain their name across all copies of f

Logic programming

- Programming with relations . . .
  - . . . as opposed to programming with functions

Logic programming

- Programming with relations ...
  - ... as opposed to programming with functions
- Function f with n arguments defines a relation R_f with $n+1$ arguments

$$f(x_1, x_2, \ldots, x_n) = y \text{ iff } (x_1, x_2, \ldots, x_n, y) \in R_f$$

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- Functional programs compute y given $(x_1, x_2, \ldots, x_n)$
- Logic programming allows computation of more general relations
- Will follow `Prolog` syntax

Variables and constants

Two kinds of entities

- Variables
  - Names starting with a capital letter
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  - Uninterpreted — no types like `Char`, `Bool` etc!
  - Exception: natural numbers, some arithmetic

Defining relations

A Prolog program describes a relation

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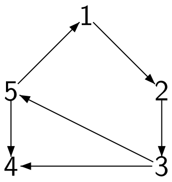
Example: A graph



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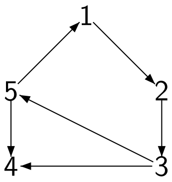
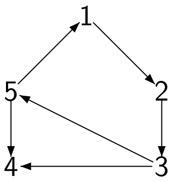


- Want to define a relation `path(X,Y)`
- `path(X,Y)` holds if there is a path from `X` to `Y`

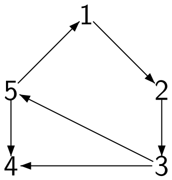
Facts and rules



Represent edge relation using the following facts.

```
edge(3,4).
edge(5,4).
edge(5,1).
edge(1,2).
edge(3,5).
edge(2,3).
```

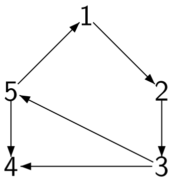
Facts and rules . . .



Define path using the following rules.

```
path(X,Y) :- X = Y.
path(X,Y) :- edge(X,Z), path(Z,Y).
```

Facts and rules . . .



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Read the rules read as follows:

Rule 1 For all X,Y, (X,Y) $\in$ path if X is same as (i.e., unifies with) Y.

Rule 2 For all X,Y, (X,Y) $\in$ path if there exists Z such that (X,Z) $\in$ edge and (Z,Y) $\in$ path.

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- Each rule is of the form

  Conclusion if Premise_1 and Premise_2 . . . and Premise_n

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  - Variables in goal are universally quantified
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- Quantification of variables
  - Variables in goal are universally quantified
    - `X`, `Y` above
  - Variables in premise are existentially quantified
    - `Z` above

Computing in Prolog

- Ask a question (a query)

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    - (Z,1) $\in$ path.

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    - `(3,Z)` ∈ `edge` and
    - `(Z,1)` ∈ `path`.
- Sub goals are tried depth-first
  - `(3,Z)` ∈ `edge`?
    - `(3,4)` ∈ `edge`, set `Z = 4`

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    - `(3,4)` $\in$ `edge`, set `Z = 4`
  - `(4,1)` $\in$ `path`? 4 cannot be unifed with 1, two subgoals, new `Z'`
    - `(4,Z')` $\in$ `edge`
    - `(Z',1)` $\in$ `path`

Computing in Prolog

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  ?- path(3,1).
  ```

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  - `3` cannot be unified with `1`, skip Rule 1.
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    - `(4,Z')` $\in$ `edge`
    - `(Z',1)` $\in$ `path`
  - Cannot find `Z'` such that `(4,Z')` $\in$ `edge`!

Backtracking

- `(3,Z)` $\in$ `edge`?
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- `(4,1)` $\in$ `path`? `4` cannot be unified with `1`, two subgoals, new `Z'`
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- No `Z'` such that `(4,Z')` $\in$ `edge`

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- No `Z'` such that `(4,Z')` $\in$ `edge`
- Backtrack and try another value for `Z`
  - `edge(3,5)` $\in$ `edge`, set `Z = 5`

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Backtracking is sensitive to order of facts

- We had put `edge(3,4)` before `edge(3,5)`

Reversing the question

- Consider the question

```
?- edge(3,X).
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- Find all `X` such that `(3,X)` $\in$ `edge`

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- Find all `X` such that `(3,X)` $\in$ `edge`
- Prolog lists out all satisfying values, one by one

```
X=4;
X=5;
X=2;
No.
```