Mathematical Logic, August–December 2015

Assignment 1: Propositional Logic

August 28, 2015

Due: September 7, 2015

Note: *Only electronic submissions accepted, via Moodle.*

1. Let $|\alpha|$ denote the size of the formula α. Define $|\cdot| : \Phi \to \mathbb{N}$ as follows:

   - $|p| = 0$ where p is an atomic proposition.
   - $|\neg\alpha| = 1 + |\alpha|$
   - $|\alpha \vee \beta| = 1 + |\alpha| + |\beta|$

   Let $sf(\alpha)$ denote the set of subformulas of the formula α. Define $sf : \Phi \to 2^\Phi$ as follows:

   - $sf(p) = \{p\}$ for $p \in \mathcal{P}$
   - $sf(\neg\alpha) = \{(\neg\alpha)\} \cup sf(\alpha)$
   - $sf((\alpha \vee \beta)) = \{(\alpha \vee \beta)\} \cup sf(\alpha) \cup sf(\beta)$

   Let $\#(sf(\alpha))$ denote the number of subformulas of α.

   Show that:

   - $\#(sf(\alpha)) \leq 2|\alpha| + 1$
   - For every $i > 0$, there is at least one formula α_i with $|\alpha_i| = i$ and $\#(sf(\alpha_i)) = 2|\alpha_i| + 1$ and at least one β_i with $|\beta_i| = i$ and $\#(sf(\beta_i)) < 2|\beta_i| + 1$

2. A finite/infinite k-sequence is a finite/infinite string over the alphabet $\{0, 1, \ldots, k-1\}$. For example, this is a finite 3-sequence: 0102101.

   More formally, an infinite k-sequence is a function $s : \mathbb{N} \to \{0, 1, \ldots, k-1\}$ and a k-sequence of length ℓ is a function $s : \{0, 1, \ldots, \ell-1\} \to \{0, 1, \ldots, k-1\}$.

   We say that a k-sequence s (finite or infinite) is n-free if there *does not exist* a finite k-sequence x such that x^n (x repeated n times) is a *substring* of s.

   (a) Show that there is no infinite 2-sequence that is 2-free.

   (b) Using König's lemma, show that there is an infinite 3-free 2-sequence.

   (c) Prove a similar result for 2-free 3-sequences.

3. Determine if each of the following sets of formulas is satisfiable or not. If it is satisfiable, provide the satisfying assignment. If not, provide a proof why it is not satisfiable.

   (a) $\{(p_0 \to p_1), (p_1 \to p_2), ((p_2 \vee p_3) \leftrightarrow \neg p_1)\}$
   (b) $\{\neg(\neg p_1 \vee p_0), (p_0 \vee \neg p_2), (p_1 \to \neg p_2)\}$.
   (c) $\{(p_3 \to p_1), (p_0 \vee \neg p_1), \neg(p_3 \wedge p_0), p_3\}$.

4. **Alternate proof of Completeness** [Kalmár, 1935]:

   Consider the system with three axiom schemes and one inference rule:

   - **Axiom 1**: $\alpha \to (\beta \to \alpha)$
   - **Axiom 2**: $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
   - **Axiom 3**: $(\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$
   - **Modus Ponens**: $\dfrac{\alpha, \alpha \to \beta}{\beta}$.

   Let α be a formula from Φ such that $\neg$ and $\to$ are the only connectives appearing in α.

   For completeness, we need to show that if α is valid then α is a theorem.

   Let $Voc(\alpha) = \{p_0, p_1, \ldots, p_k\}$

   Let v be a valuation over $Voc(\alpha)$.

   For $0 \leq i \leq n$ define

$$p_i' = \begin{cases} p_i & \text{if } v(p_i) = \top \\ \neg p_i & \text{if } v(p_i) = \bot \end{cases}$$

   Similarly define

$$\alpha' = \begin{cases} \alpha & \text{if } v(\alpha) = \top \\ \neg\alpha & \text{if } v(\alpha) = \bot \end{cases}$$

   Let n be the number of occurences of $\neg$ and $\to$ in α.

   Subtask a: When $n = 0$ show that $p_0', p_1', \ldots, p_k' \vdash \alpha'$.

   For the next two subtasks, assume that for all $n < j$, we have $p_0', p_1', \ldots, p_k' \vdash \alpha'$.

   Subtask b: When $n = j$ and α is of the form $\neg\beta$ show that $p_0', p_1', \ldots, p_k' \vdash \alpha'$.
   Hint: Observe that β has fewer than j occurences of $\neg$ and $\to$. Argue seperately for the cases when $v(\beta) = \top$ and $v(\beta) = \bot$.

Subtask c: For $n = j$ and α of the form $\beta \to \gamma$, show that $p'_0, p'_1, \ldots, p'_k \vdash \alpha'$.

Hint: As before, use the fact that β and γ have fewer than j occurences of $\neg$ and $\to$. Argue separately for the different valuations of β and γ.

Subtask d: Conclude that for any α and a valuation v, $p'_0, p'_1, \ldots, p'_k \vdash \alpha'$.

Now, suppose α is a valid formula.

Subtask e: Show that for any valuation v, $p'_0, p'_1, \ldots, p'_k \vdash \alpha$.

Suppose $p_k = \top$. Then $p'_k = p_k$ and $p'_0, p'_1, \ldots, p'_{k-1}$, $p_k \vdash \alpha$. Similarly if $p_k = \bot$, then $p'_k = \neg p_k$ and $p'_0, p'_1, \ldots, p'_{k-1}$, $\neg p_k \vdash \alpha$.

Subtask f: From this conclude that $p'_0, p'_1, \ldots, p'_{k-1} \vdash \alpha$.

Subtask g: Show that $\vdash \alpha$ and conclude that if α is valid then α is a theorem.