Mathematical Logic, August–December 2012

Assignment 3: First order theory of $\mathbb{N}$

October 19, 2012

Due: October 26, 2012

Let $\mathfrak{N}_S$, $\mathfrak{N}_<$ and $\mathfrak{N}_+$ denote the structures $(\mathbb{N}, 0, S)$, $(\mathbb{N}, 0, S, <)$ and $(\mathbb{N}, 0, S, +, <)$, respectively.

1. A relation $R \subseteq \mathbb{N} \times \mathbb{N}$ is said to be *linear* if all pairs in R, seen as points in the xy-plane, can be covered by a finite number lines. Show that if a binary relation R over $\mathbb{N}$ is definable in $\mathfrak{N}_S$, either R or its complement is linear.

2. Show that a subset $D \subseteq \mathbb{N}$ is definable in $\mathfrak{N}_<$ iff D or its complement is finite.

3. A subset D of natural numbers is said to be *eventually periodic* iff there exist positive numbers M and p such that for all n greater than M, $n \in D$ iff $n + p \in D$. Show that any eventually periodic set of natural numbers is definable in $\mathfrak{N}_+$.

4. Consider the structure $(\mathbb{N}, +)$. Show that the following relations are definable.

   (a) Ordering, $\{(m, n) \mid m < n\}$.

   (b) Zero, $\{0\}$.

   (c) Successor, $\{(m, n) \mid n = S(m)\}$.