Mathematical Logic, August–December 2012

Assignment 2: First order Logic

September 21, 2012

Due: October 7, 2012

1. Let L be a finite first-order language and let $\mathcal{M}$ be a finite L-structure. Show that there is an L-sentence $\varphi_{\mathcal{M}}$ whose models are precisely the L-structures isomorphic to $\mathcal{M}$.

2. (a) Let $L = (\emptyset, \{+, \times\}, \{0\})$ where $+$ and $\times$ are binary function symbols and 0 is a constant symbol. Consider an L-structure $(\mathbb{R}, +, \times, 0)$, where $\mathbb{R}$ is the set of real numbers with the conventional interpretation of $+$, $\times$ and 0 as addition, multiplication and zero.

   Show that the relation $<$ ("less-than") is *elementary definable* in $(\mathbb{R}, +, \times, 0)$—that is, there is a formula $\varphi(x, y)$ over L such that for all $a, b \in \mathbb{R}$,

   $$((\mathbb{R}, +, \times, 0), [x \mapsto a, y \mapsto b]) \models \varphi(x, y) \text{ iff } a < b.$$

   (b) Let $L = (\emptyset, \{+\}, \{0\})$. Show that $<$ is not elementary definable in $(\mathbb{R}, +, 0)$. [Hint: Work with a suitable automorphism of $(\mathbb{R}, +, 0)$ – that is, an isomorphism of $(\mathbb{R}, +, 0)$ onto itself.]

   (c) (Bonus Question): Is $<$ elementary definable in $(\mathbb{Z}, +, \times, 0)$, where $\mathbb{Z}$ is the set of integers ?

3. A set of natural numbers M is called a *spectrum* if there is a language L and a sentence φ over L such that

   $$M = \{n \mid \varphi \text{ has a model of size exactly } n\}.$$

   Show that:

   (a) Every finite subset of $\{1, 2, 3, \dots\}$ is a spectrum.

   (b) For every $m \geq 1$, the set of numbers greater than 0 that are divisible by m is a spectrum.

   (c) The set of squares greater than 0 is a spectrum.

4. Let L be the first-order language consisting of a single binary relation $<$. The following axioms capture the fact that $<$ is a dense linear order. The resulting theory is called **DLO**.

   - $\forall x \forall y \forall z\ (x < y \wedge y < z \supset x < z)$
   - $\forall x\ \neg(x < x)$
   - $\forall x \forall y\ (x < y \vee x = y \vee y < x)$
   - $\forall x \exists y \exists z\ (x < y \wedge z < x)$
   - $\forall x \forall y\ (x < y \supset \exists z\ (x < z \wedge z < y))$

   For example, the structures $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ both satisfy **DLO**.

   Show that any two countable models of **DLO** are isomorphic.

   Hint: Prove that a countable model $(A, <)$ such that $(A, <) \models$ **DLO** is isomorphic to $(\mathbb{Q}, <)$ by building an isomorphism $A \to \mathbb{Q}$ inductively. At any stage in the construction we have n distinct elements $a_1, a_2, \ldots, a_n$ of A and n distinct elements $q_1, q_2, \ldots, q_n$ of $\mathbb{Q}$. Inductively assume $a_i < a_j$ if and only if $q_i < q_j$ for all i, j. Show that the DLO axioms allow the following: given a new element a_{n+1} from A, we can add a new element q_{n+1} from $\mathbb{Q}$ while preserving the induction hypothesis, and, symmetrically, given a new element q_{n+1} from $\mathbb{Q}$, we can add a new element a_{n+1} from A while preserving the induction hypothesis. (These two steps are traditionally called 'back-and-forth'.) Explain also how the countability of A and $\mathbb{Q}$ allows this inductive back-and-forth construction to build an isomorphism between $(A, <)$ and $(\mathbb{Q}, <)$.

5. Using Ehrenfeucht Fraïssé games show that *acyclicity* of finite graphs is not first-order definable in the language $L = (\{E\}, \emptyset, \emptyset)$ where E is interpreted as the edge relation