# Lecture 21: 9 April, 2026

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Data Mining and Machine Learning
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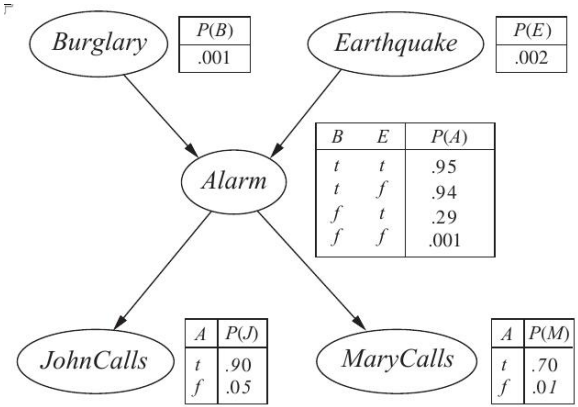
# Probabilistic graphical models

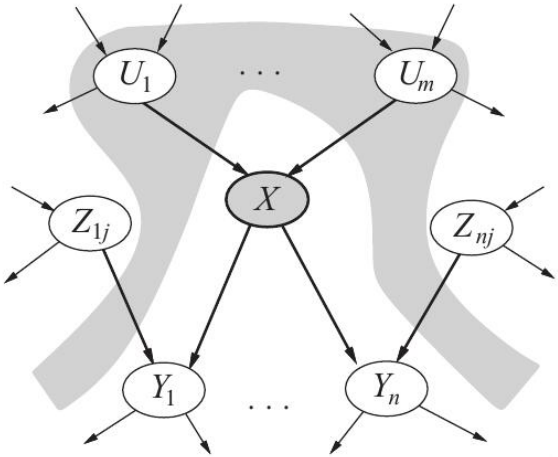
- Underlying DAG, no cyclic dependencies
- Each node has a local (conditional) probability table



# Probabilistic graphical models

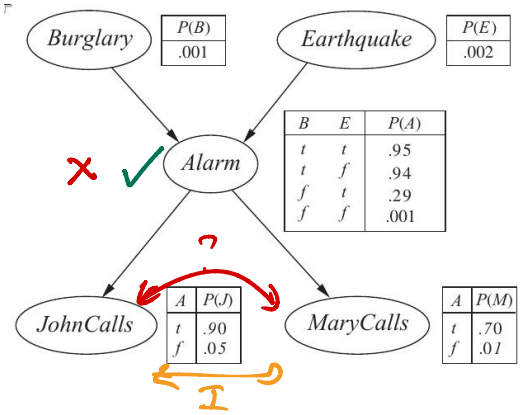
- Fundamental assumption

  A node is conditionally independent of non-descendants, given its parents



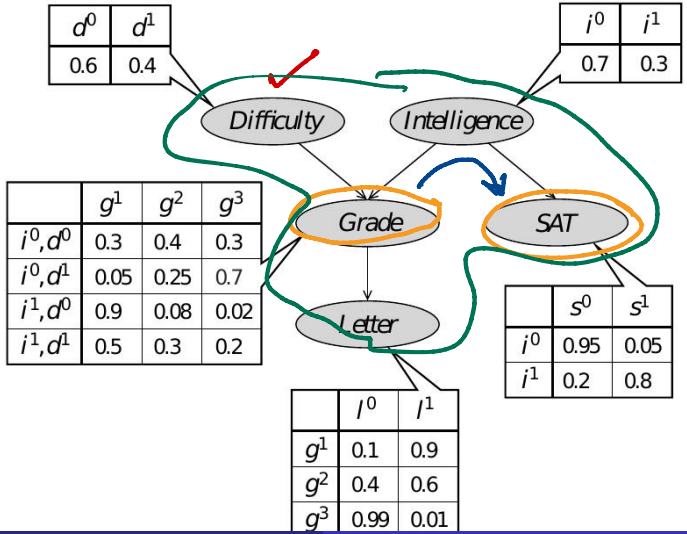
# Conditional independence

- $x \perp y$ — $x$ and $y$ are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - $x$ and $y$ are independent given $z$
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$
- Is *JohnCalls* independent of *MaryCalls* ($j \perp m$)?
  - No — value of $j$ tells us something about value of $m$ and vice versa
- Is *JohnCalls* independent of *MaryCalls* given *Alarm* ($j \perp m \mid a$)?
  - Yes — by semantics of network, local independence



# Student example

- $SAT \perp Grade \mid Difficulty$ ?
  - No
- Can we calculate conditional independence from the graph?
- In general, check if $X \perp Y \mid Z$ for sets of variables $X$, $Y$, $Z$

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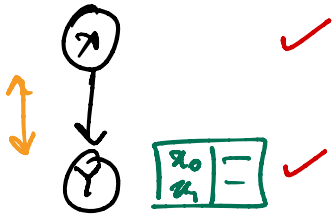

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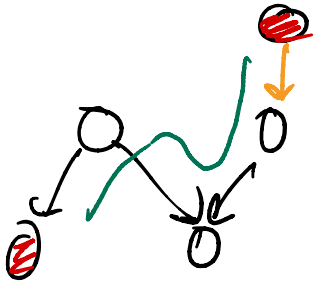
- Examine trails between nodes
  - Paths in the underlying undirected graph



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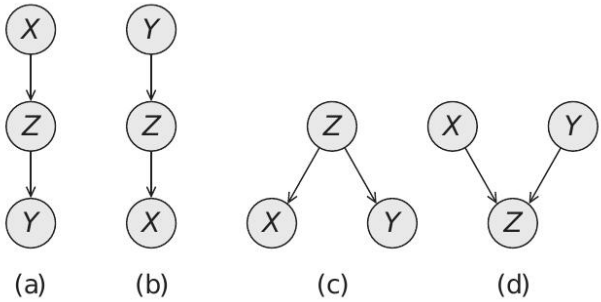
  - Paths in the underlying undirected graph
- Basic trails — (undirected) paths of length 2
  - Four basic trails

$P(X \perp Y \mid Z)$



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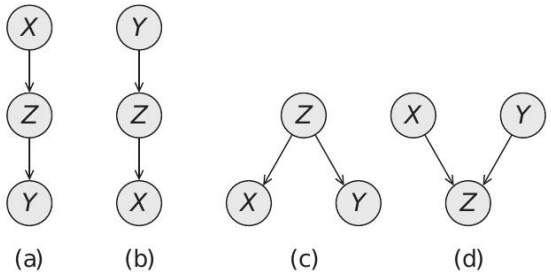

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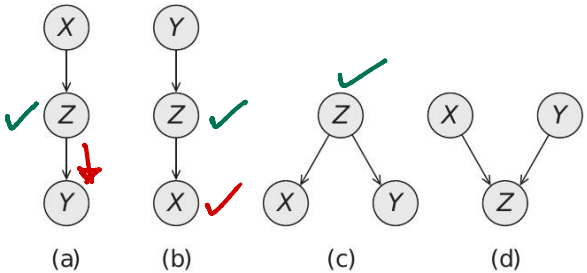

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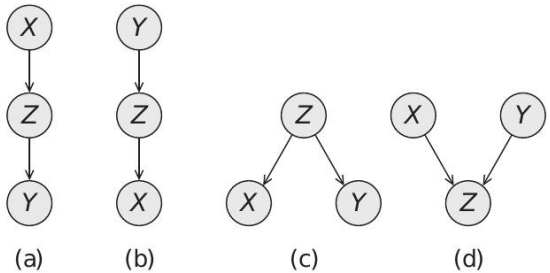
$P(X \perp Y \mid Z)$ ✗



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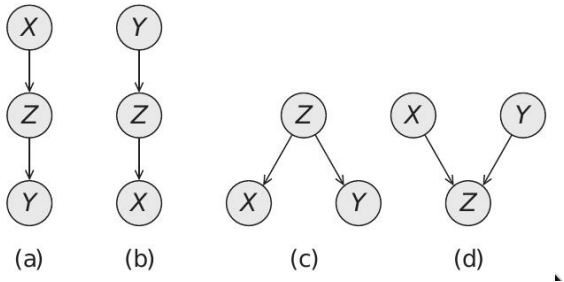
  - $Z$: Car does not start
    $X$: Low Battery, $Y$: No Fuel



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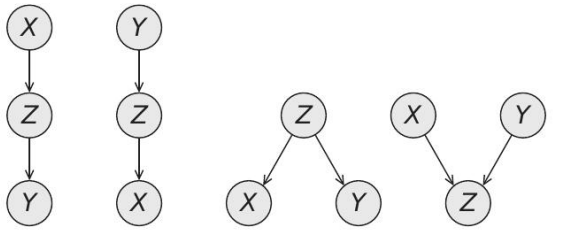


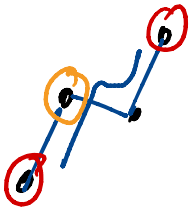

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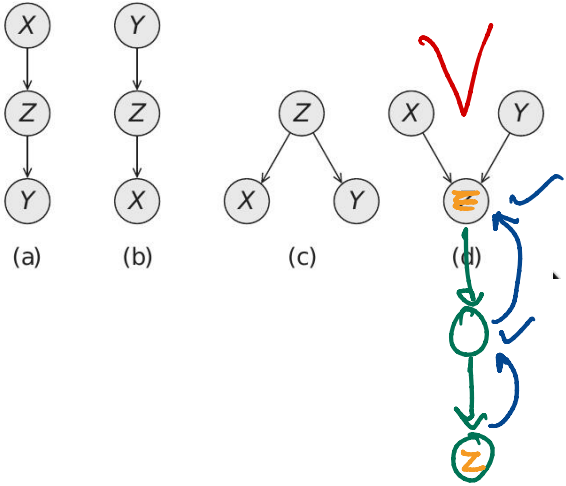
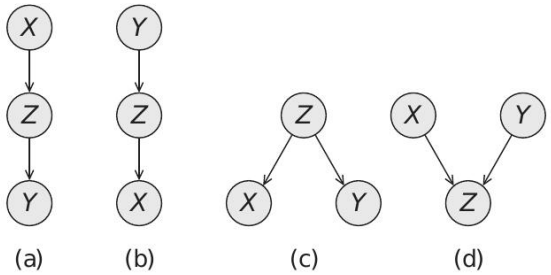
  - Simplest form of V-structure



# D-Separation

- Check if $X \perp Y \mid Z$



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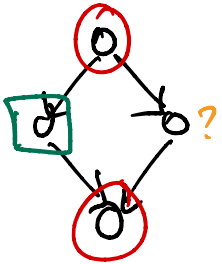


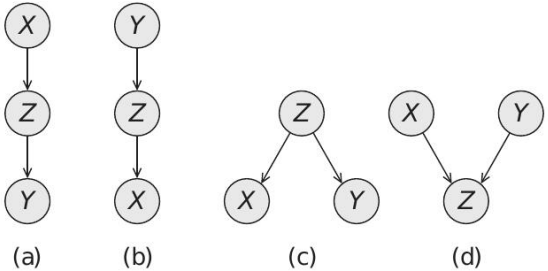

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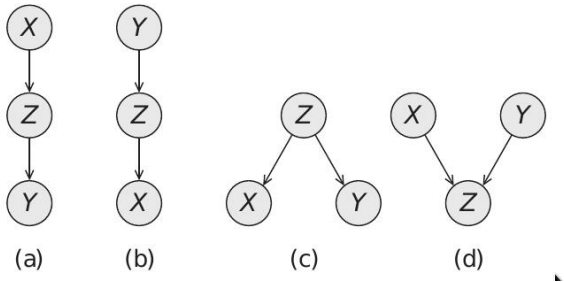

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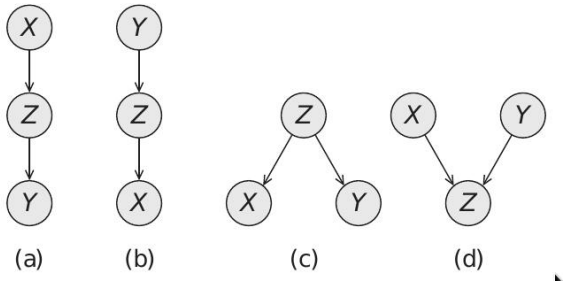

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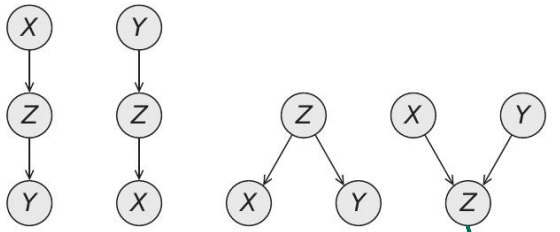

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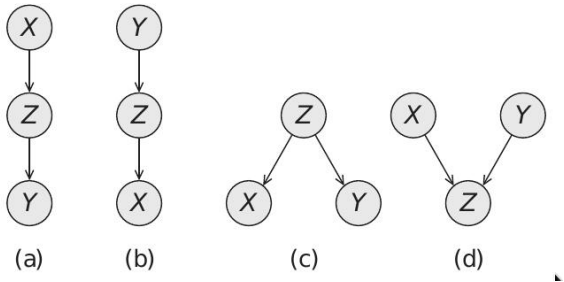
  - In general, V-structure includes descendants of the bottom node



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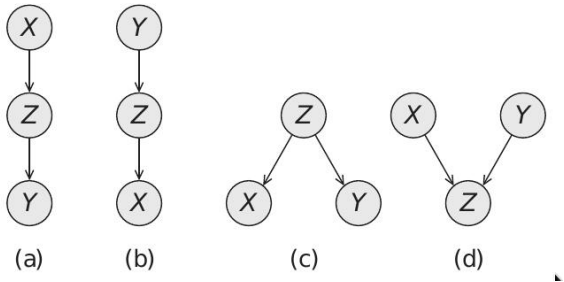


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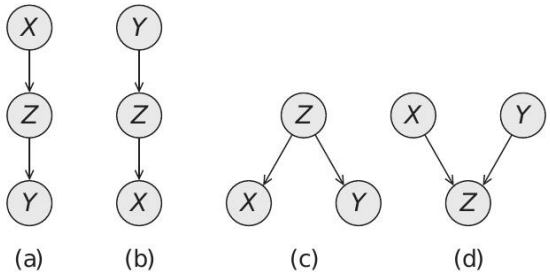
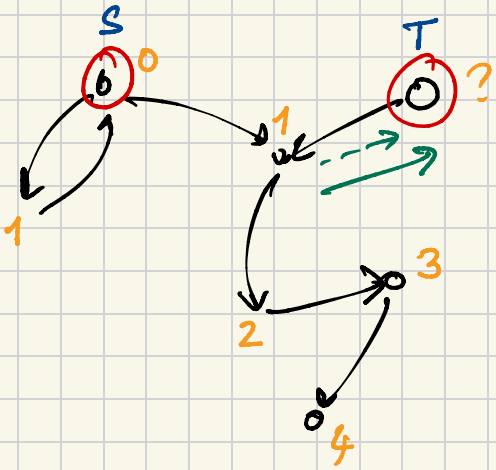



- $x$ and $y$ are D-separated given $z$ if all trails are blocked
- Variation of breadth first search (BFS) to check if $y$ is reachable from $x$ through some trail

Airline network

Reachability

Breadth First Search



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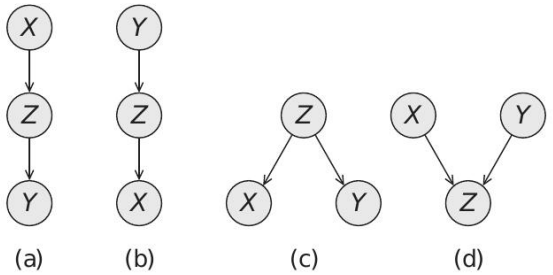
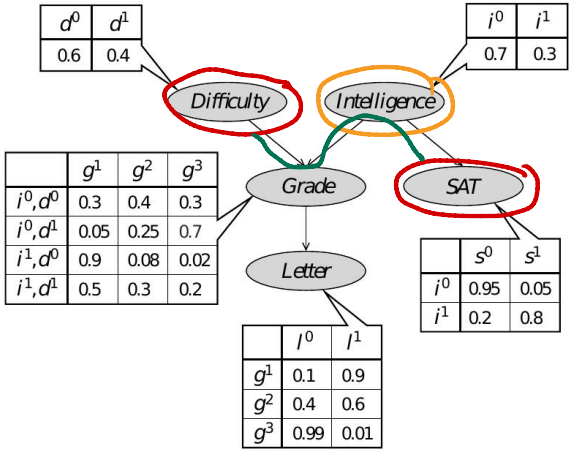



- $x$ and $y$ are D-separated given $z$ if all trails are blocked
- Variation of breadth first search (BFS) to check if $y$ is reachable from $x$ through some trail
- Extends to sets — each $x \in X$ is D-separated from each $y \in Y$

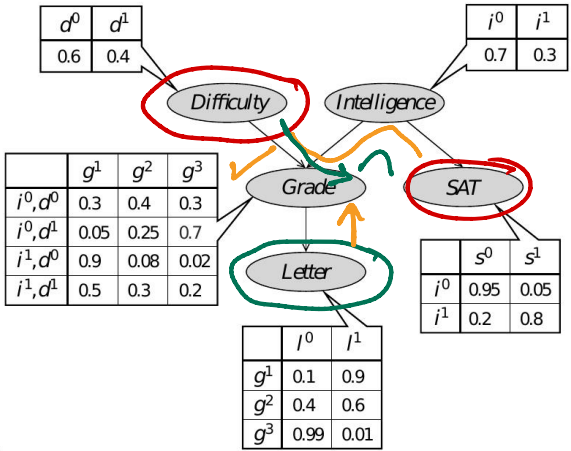
# Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence



# Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence
- Is SAT independent of Difficulty given Letter?
  - No, Difficulty – Grade – Intelligence – SAT trail is open
  - Letter is known, hence something about Grade is known (V-structure)
  - Intelligence is not known

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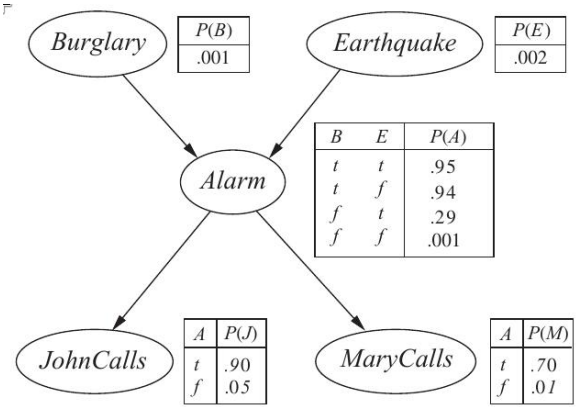
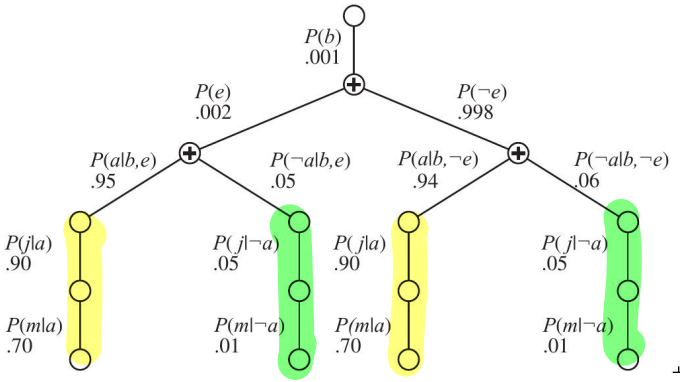
# Computing with probabilistic graphical models

- John and Mary call Pearl. What is the probability that there has been a burglary?
- Want $P(b \mid m, j)$
- $\dfrac{P(b, m, j)}{P(m, j)}$
- Use chain rule to evaluate joint probabilities
- Reorder variables appropriately, topological order of graph



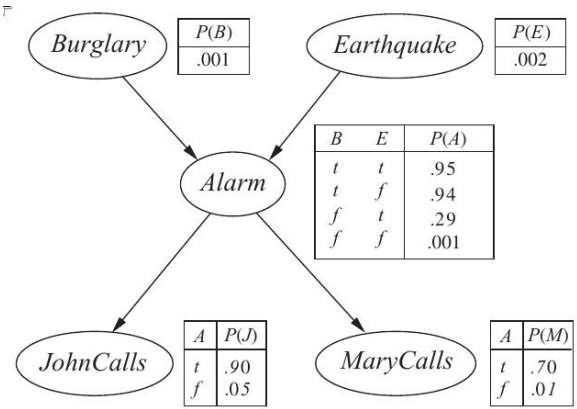
# Computing with probabilistic graphical models

- $P(m, j, b) = P(b) \sum_{e=0}^{1} P(e) \sum_{a=0}^{1} P(a \mid b, e) P(m \mid a) P(j \mid a)$
- Construct the computation tree
- Use dynamic programming to avoid duplicated computations
- However, exact inference is NP-complete, in general
- Instead, approximate inference through sampling



# Approximate inference

- Generate random samples $(b, e, a, m, j)$, count to estimate probabilities



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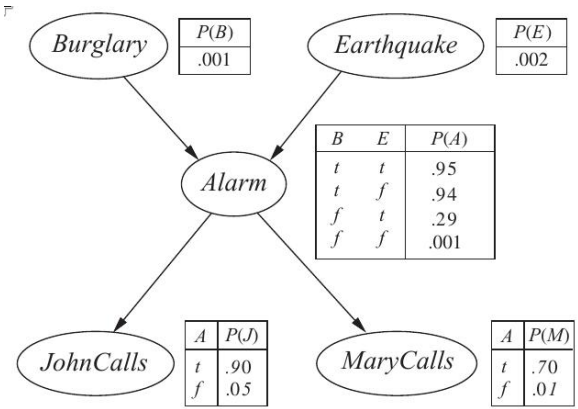
- Generate random samples $(b, e, a, m, j)$, count to estimate probabilities
- Random samples should respect conditional probabilities



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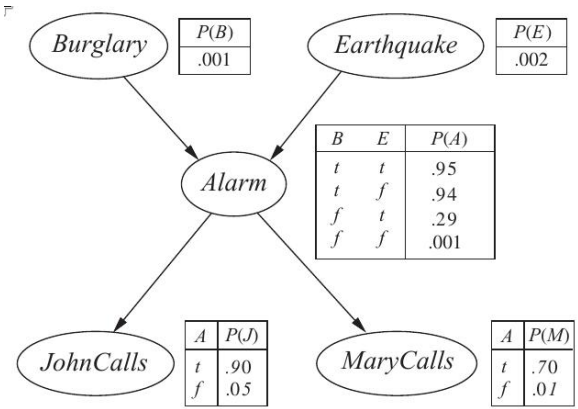
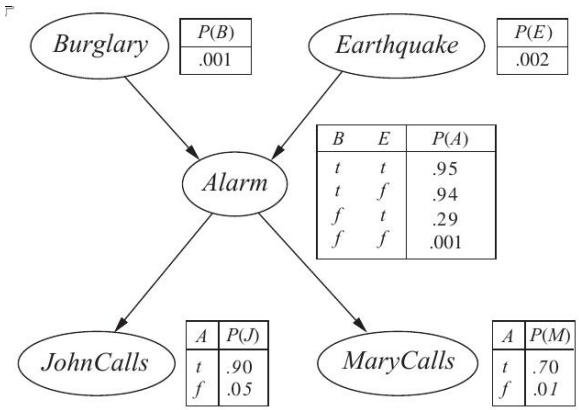


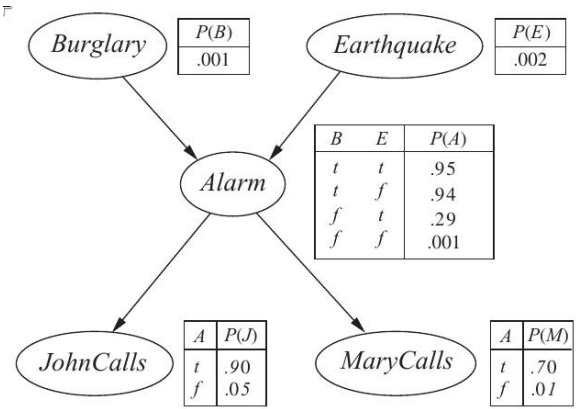
# Approximate inference

- Generate random samples $(b, e, a, m, j)$, count to estimate probabilities
- Random samples should respect conditional probabilities
- Fix parents of $x$ before generating $x$
- Generate in topological order
  - Generate $b$, $e$ with probabilities $P(b)$ and $P(e)$
  - Generate $a$ with probability $P(a \mid b, e)$
  - Generate $j$, $m$ with probabilities $P(j \mid a)$, $P(m \mid a)$



# Approximate inference

- We are interested in $P(b \mid j, m)$



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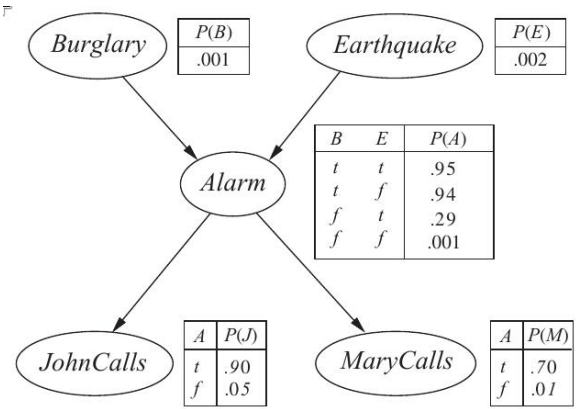

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- We are interested in $P(b \mid j, m)$
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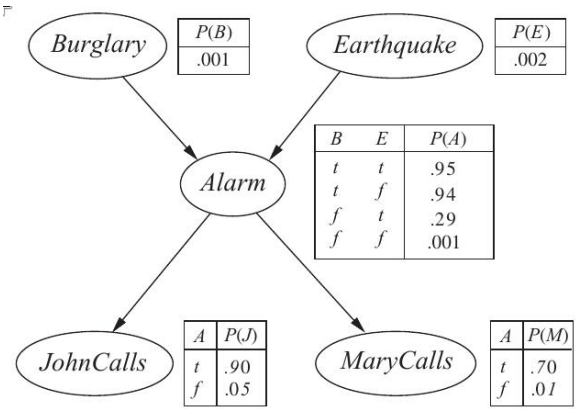
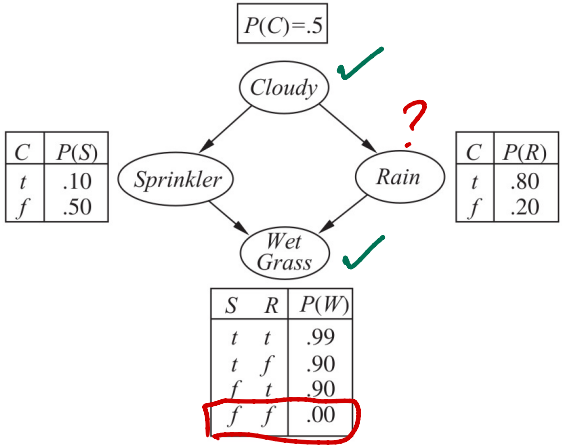
- Can we sample more efficiently?



# Rejection sampling

- $P(Rain \mid Cloudy, Wet\ Grass)$



# Rejection sampling

- $P(Rain \mid Cloudy, Wet\ Grass)$
- Topological order
  - Generate *Cloudy*
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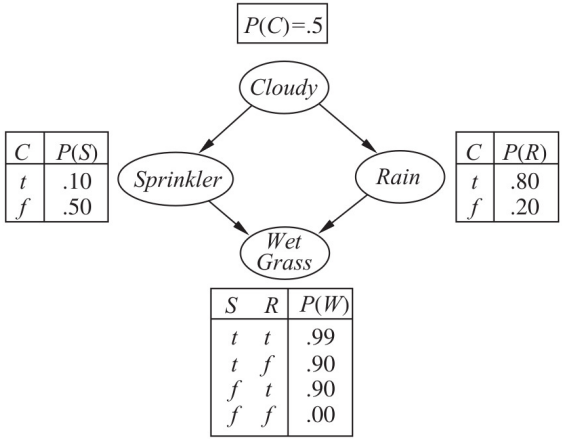

# Rejection sampling

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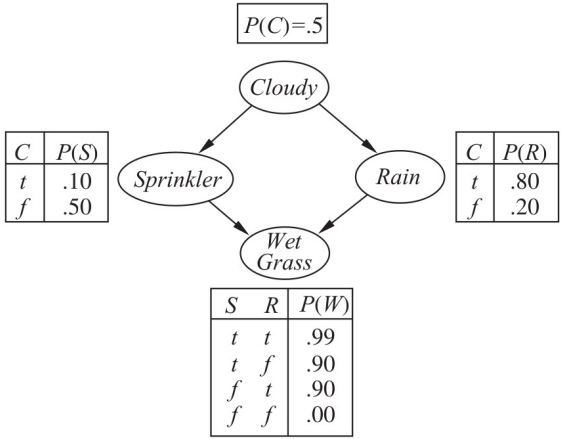

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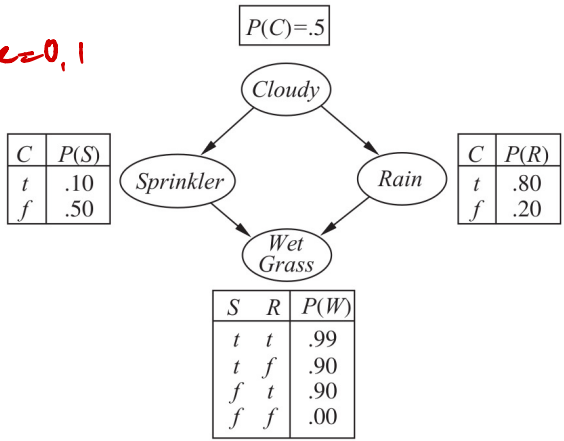

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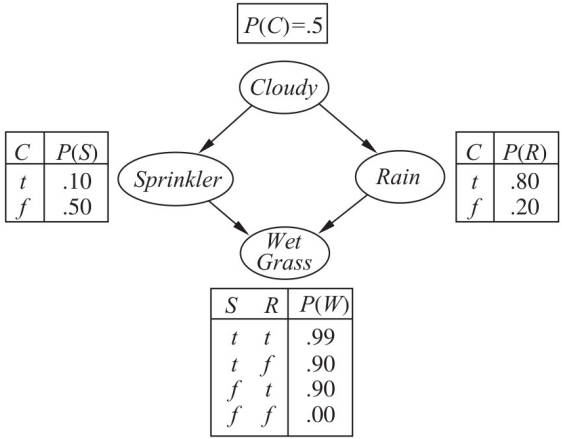

# Likelihood weighted sampling

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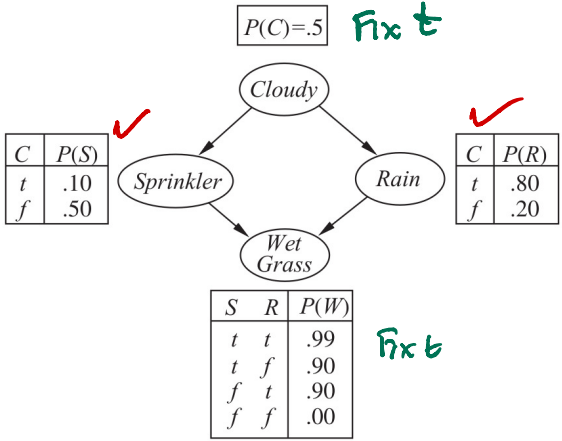

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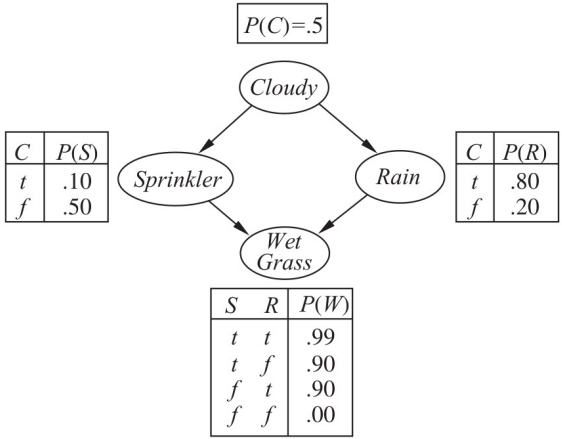

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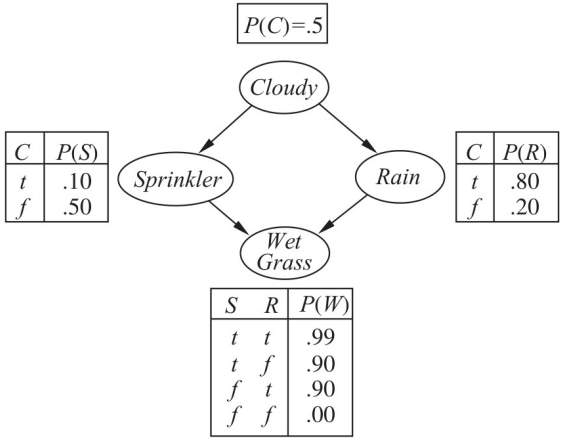

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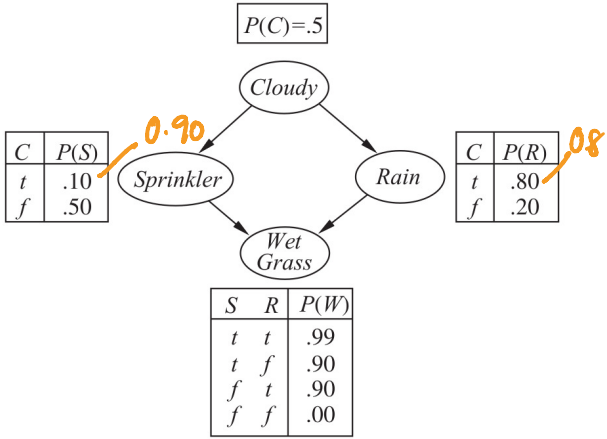

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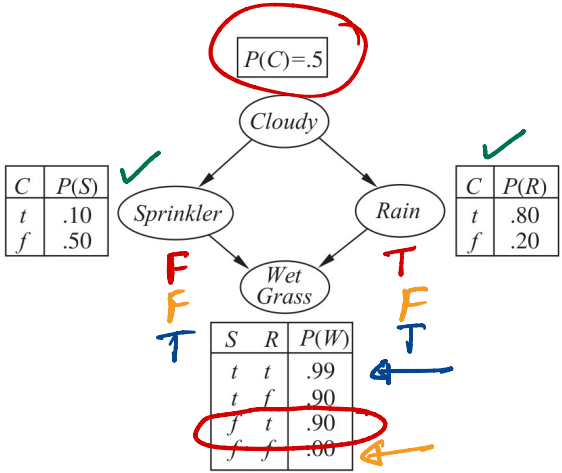

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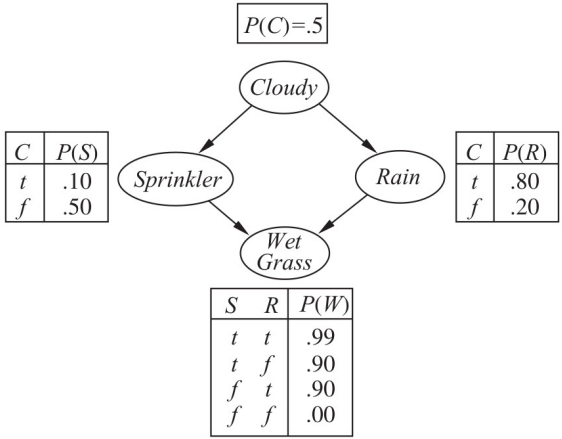

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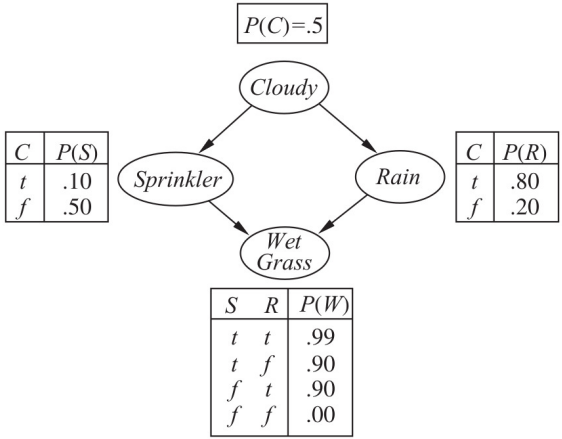

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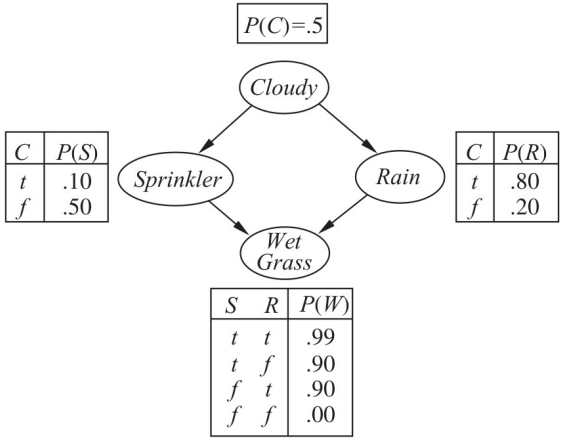
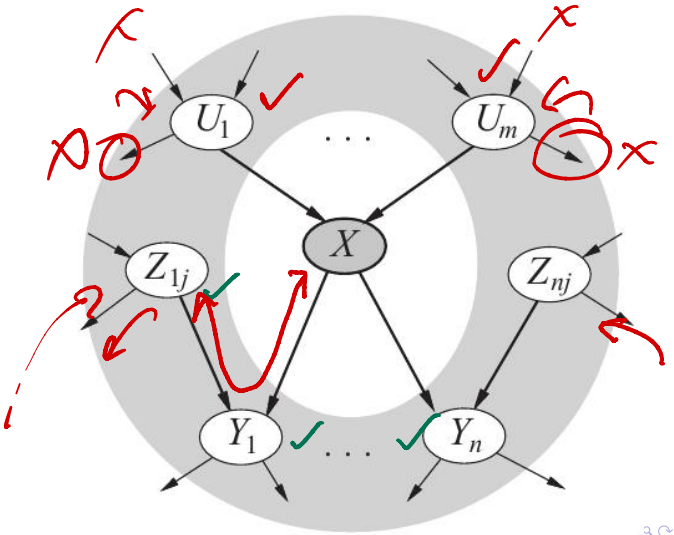


# Markov blanket

- $MB(X)$ — Markov blanket of $X$



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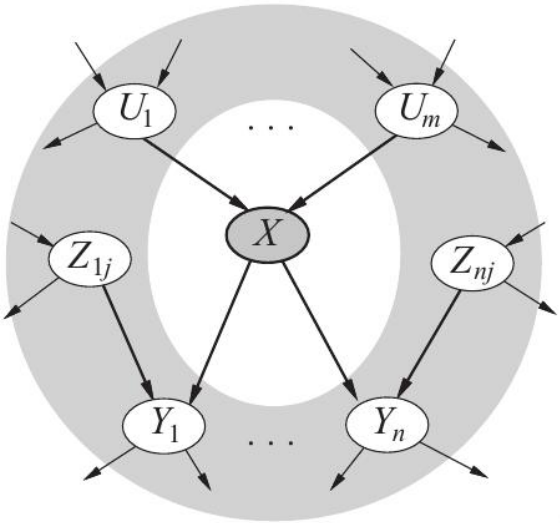

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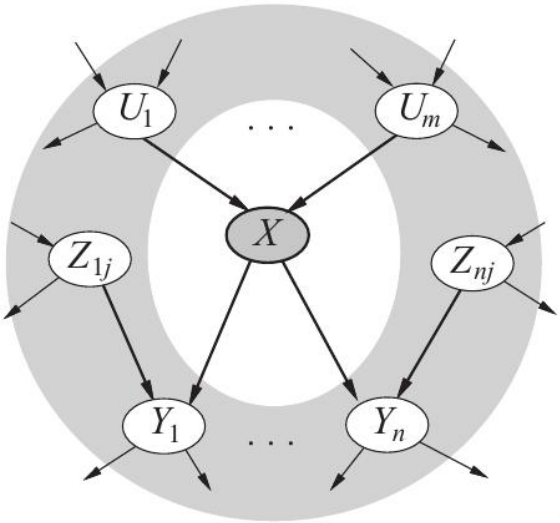

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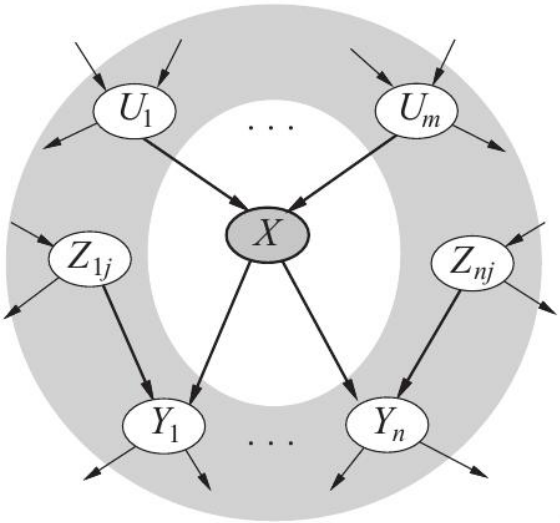

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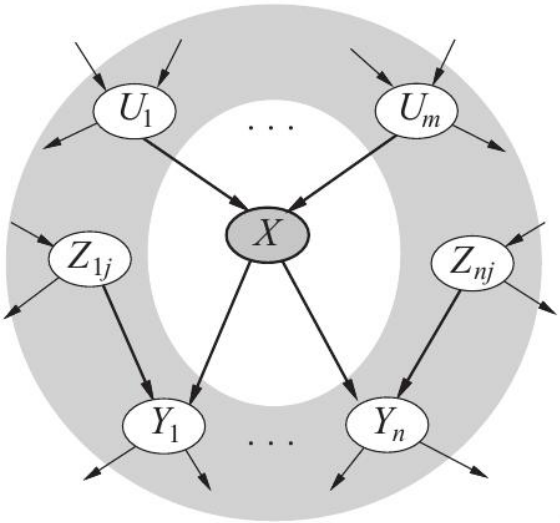

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- Why does this work at all?

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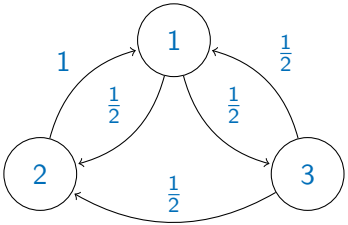

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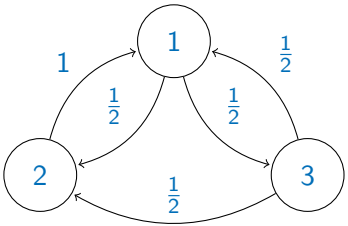


- Represent using a transition matrix — stochastic

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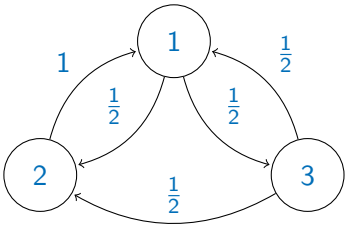


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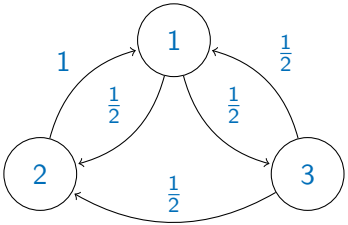


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- Start in state 1, so initially $P = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

# Markov chains . . .

- After one step:

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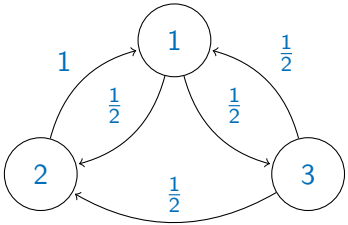

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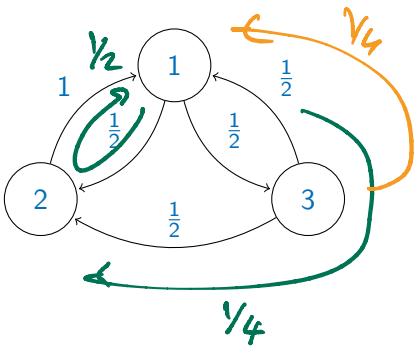

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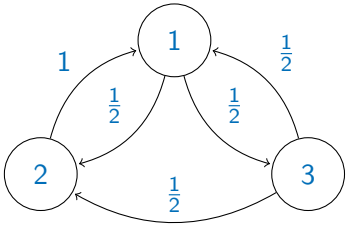

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- Continuing our example,

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{bmatrix} \to \begin{bmatrix} \frac{9}{16} & \frac{5}{16} & \frac{1}{8} \end{bmatrix}$$