Lecture 20: 2 April, 2026

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Data Mining and Machine Learning
January–April 2026

Conditional probabilities

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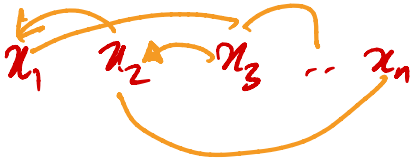
  - n parameters
- Can we strive for something in between?
  - "Local" dependencies between some variables

Probabilistic graphical models — Judea Pearl, Turing Award 2011

- Represent local dependencies using directed graph
- Each node has a local (conditional) probability table



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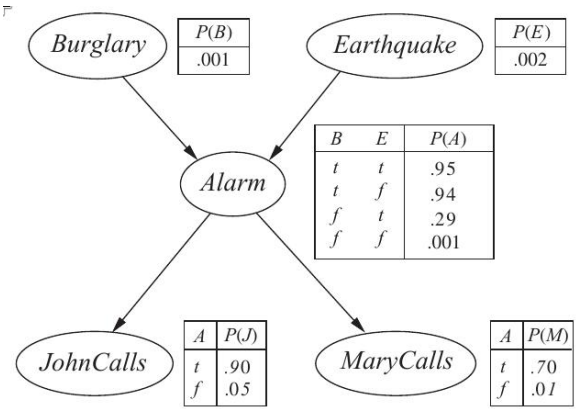

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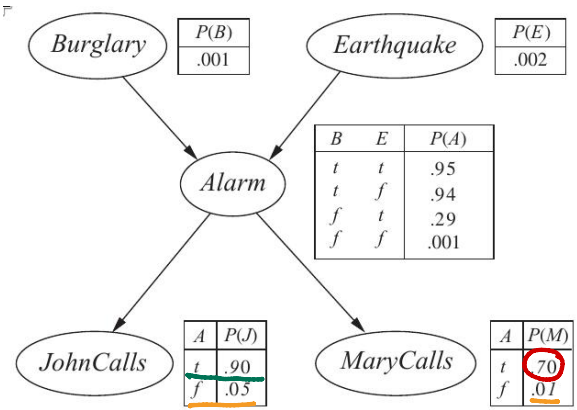

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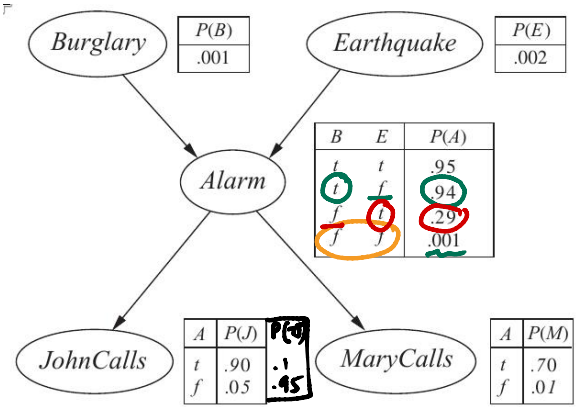

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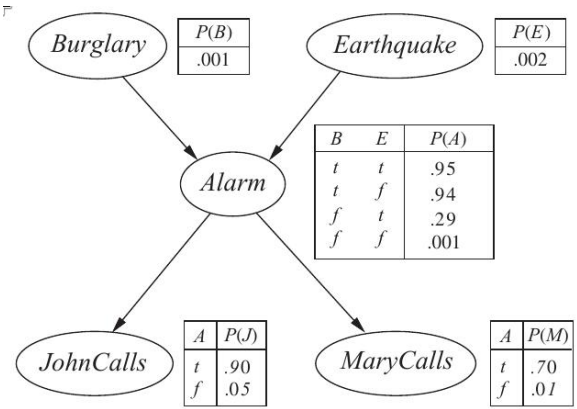
  - The alarm may also be triggered by an earthquake (California!)



Probabilistic graphical models

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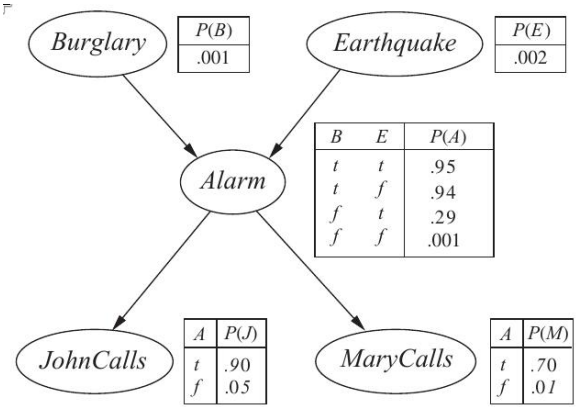

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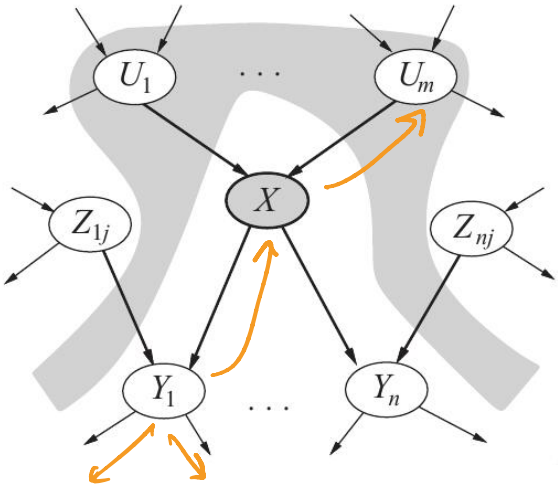

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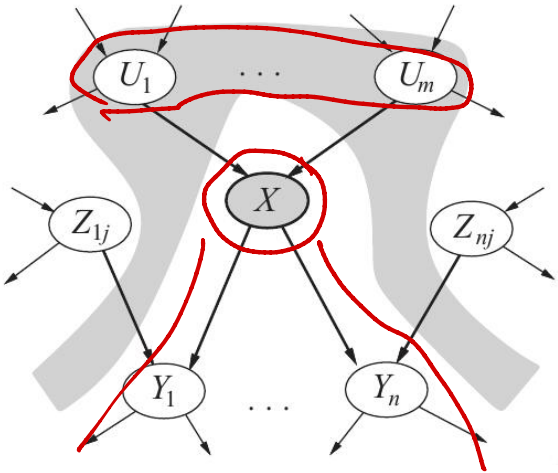
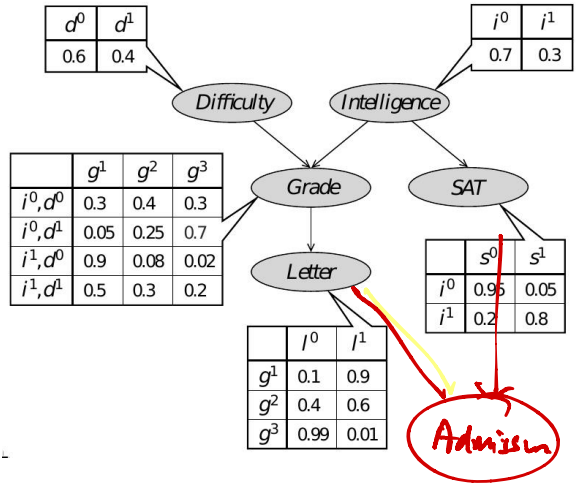


Probabilistic Graphical Models

- Example due to Nir Friedman and Daphne Koller
- Student asks teacher for a reference letter
- Teacher has forgotten the student, so letter is entirely based on student's grade in the course



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

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$$P(A|B) = \frac{P(A, B)}{P(B)}$$

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- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, x_3, \ldots, x_n)P(x_2, x_3, \ldots, x_n)$

$$P(B \mid M, J) = \frac{P(B, M, J)}{P(M, J)}$$

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- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, x_3, \ldots, x_n)P(x_2, x_3, \ldots, x_n)$
- Applied recursively, this gives us the chain rule
  $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n)P(x_n)$

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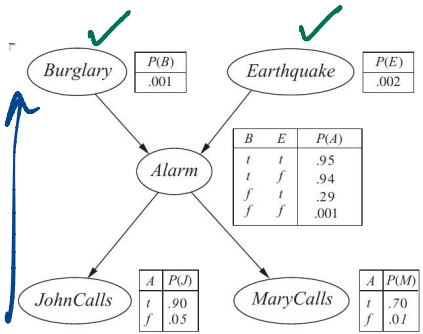

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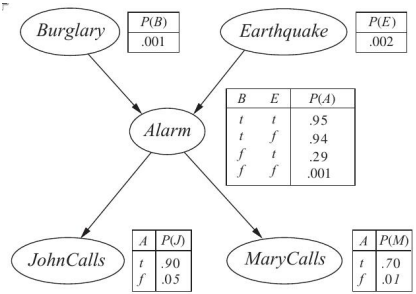

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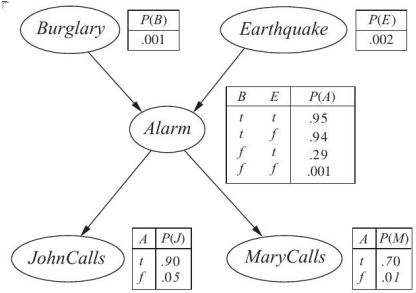

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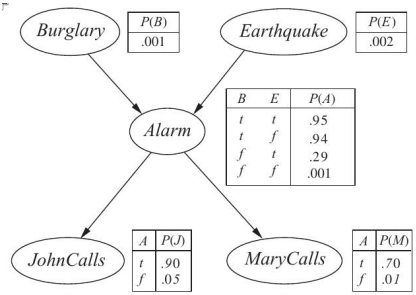

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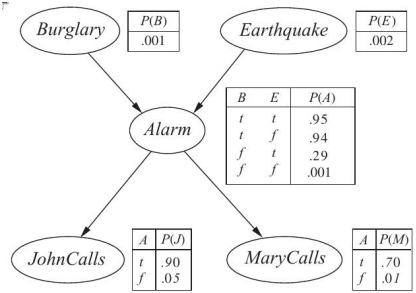
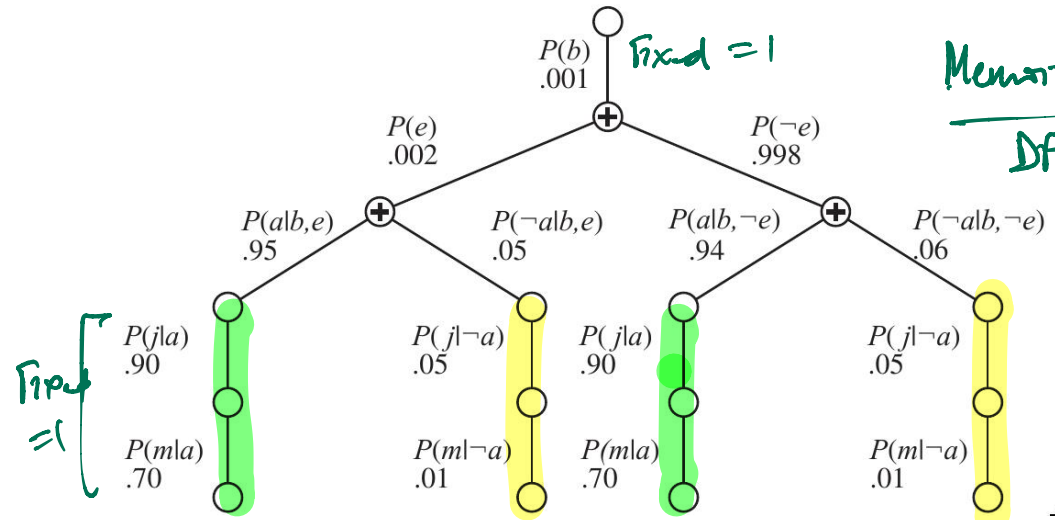


Evaluation tree



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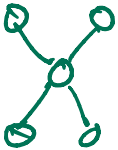

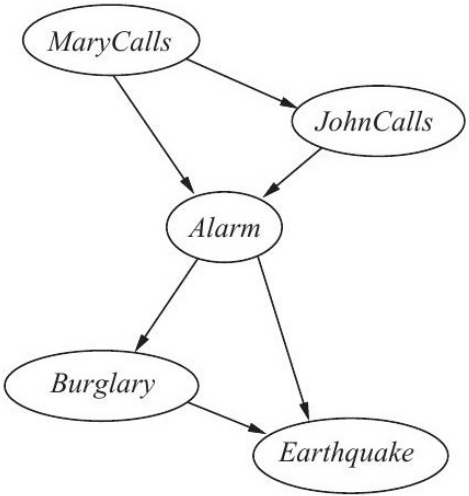

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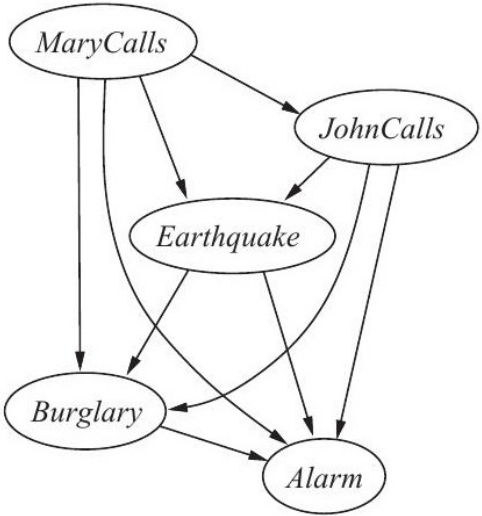

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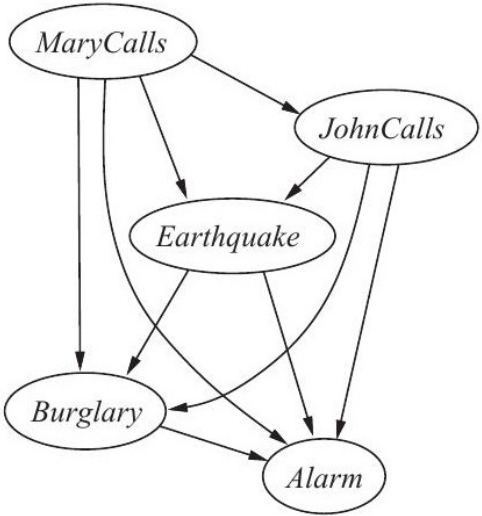
- Causal model (causes to effects) works better than diagnostic model (effects to causes)



Complexity of exact inference

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$P(b, m_j)$

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- Boolean formula in Conjunctive Normal Form (CNF)
  - Boolean variables $\{v_1, v_2, \ldots, v_n\}$
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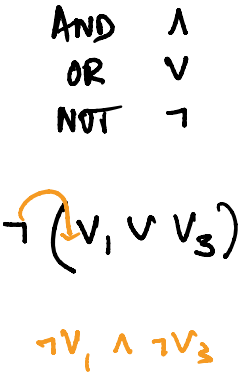

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  - A CNF formula is a conjunction of clauses $C_1 \wedge C_2 \wedge \cdots \wedge C_m$

Conjunction $\wedge$

$D_1 \vee D_2 \vee D_4$

$\downarrow$

$v_1 \wedge \neg v_2 \wedge v_4$

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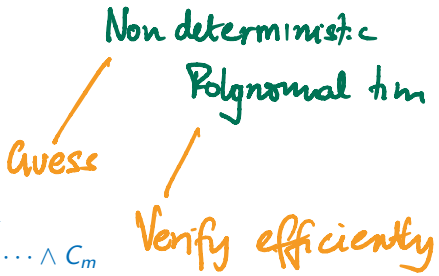
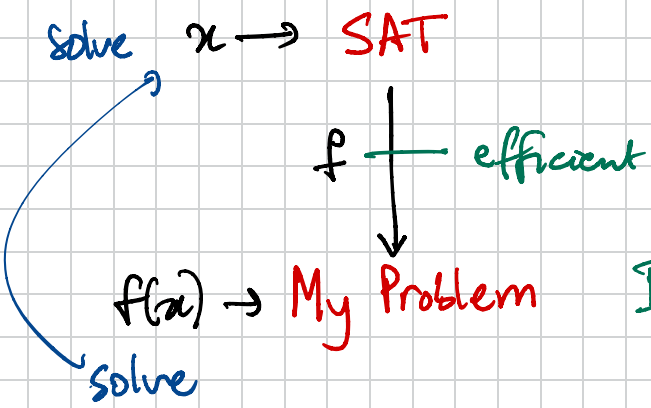


- SAT — given a formula in CNF, is there an assignment to variables that makes the formula true?
- 3-SAT — SAT where each clause has exactly 3 literals
- Both SAT and 3-SAT are NP-complete
  - No known efficient algorithm — try all possible valuations



Inference

Formula

P.G.M

Bayesian networks

Reducing 3-SAT to exact inference

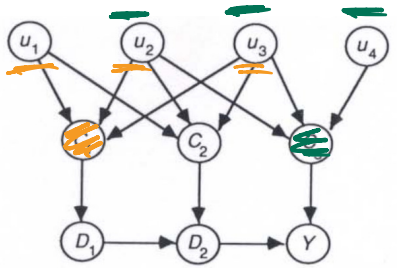
- Convert a 3-CNF formula into a Bayesian network

$C_1 \vee C_2 \vee C_3$

$V_1, \ldots, V_k$

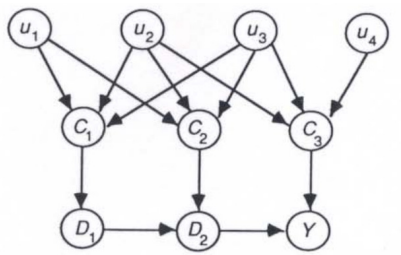
$v_1 \ldots v_k$

$\neg v_1 \quad \neg v_k$



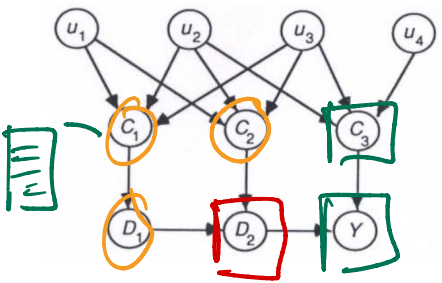
Reducing 3-SAT to exact inference

- Convert a 3-CNF formula into a Bayesian network
- Top layer: one node for each literal u_i



Reducing 3-SAT to exact inference

- Convert a 3-CNF formula into a Bayesian network
- Top layer: one node for each literal u_i
- Middle layer: one node for each clause C_j
  - Parents are three variables whose literals are in C_j
  - Conditional probability table for C_j has 8 rows, for all possible valuations of 3 variables
  - $P(C_j = 1) = 0$ for row where each input literal is false, $P(C_j = 1) = 1$ for remaining 7 rows



Reducing 3-SAT to exact inference

- Convert a 3-CNF formula into a Bayesian network
- Top layer: one node for each literal u_i
- Middle layer: one node for each clause C_j
  - Parents are three variables whose literals are in C_j
  - Conditional probability table for C_j has 8 rows, for all possible valuations of 3 variables
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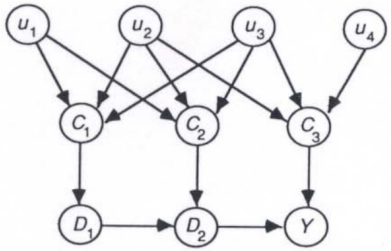
- Bottom row builds up $C_1 \wedge \cdots \wedge C_m$ one clause at a time



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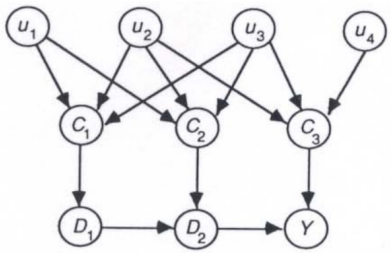
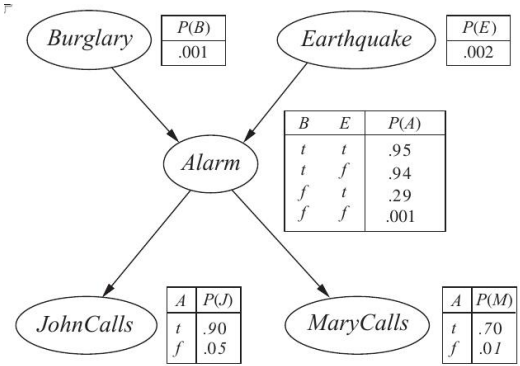
- $P(Y = 1) > 0$ iff original 3-CNF formula is satisfiable



Conditional independence

- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$



Conditional independence

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- $x \perp y \mid z$
  - x and y are independent given z
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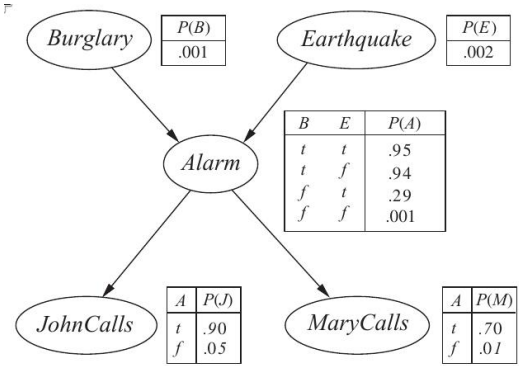

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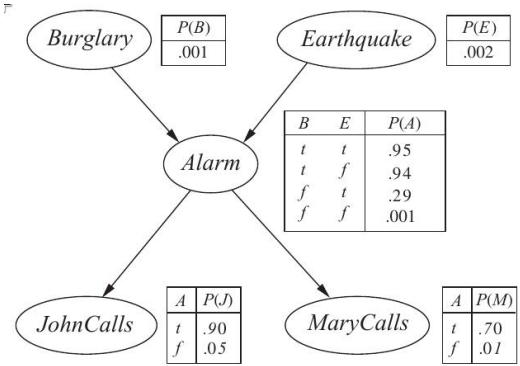
- Is *JohnCalls* independent of *MaryCalls* ($j \perp m$)?



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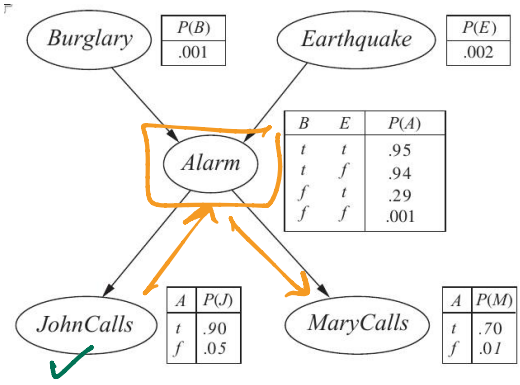

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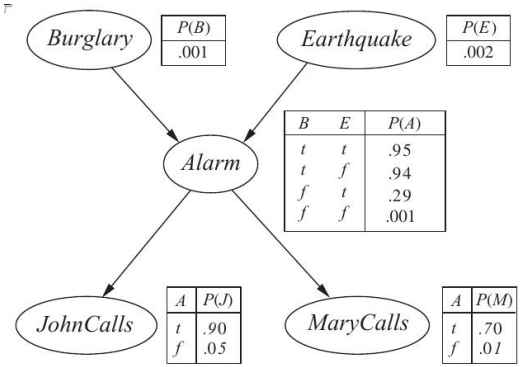

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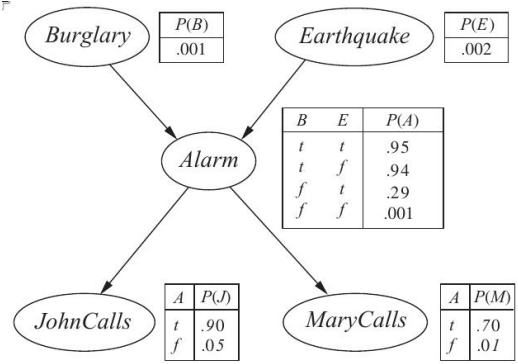
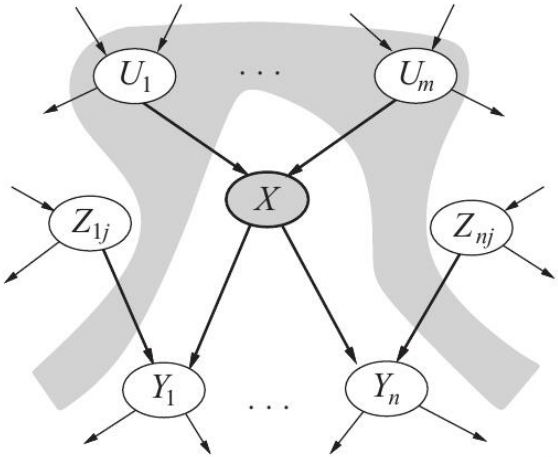
  - Yes — by semantics of network, local independence



Probabilistic graphical models

- Fundamental assumption

  A node is conditionally independent of non-descendants, given its parents

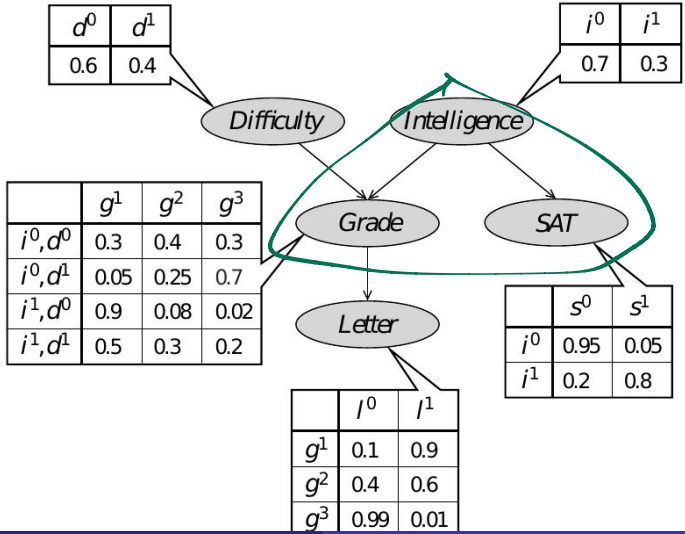


Student example

- $SAT \perp Grade \mid Difficulty$?

Intelligence

By assumption



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

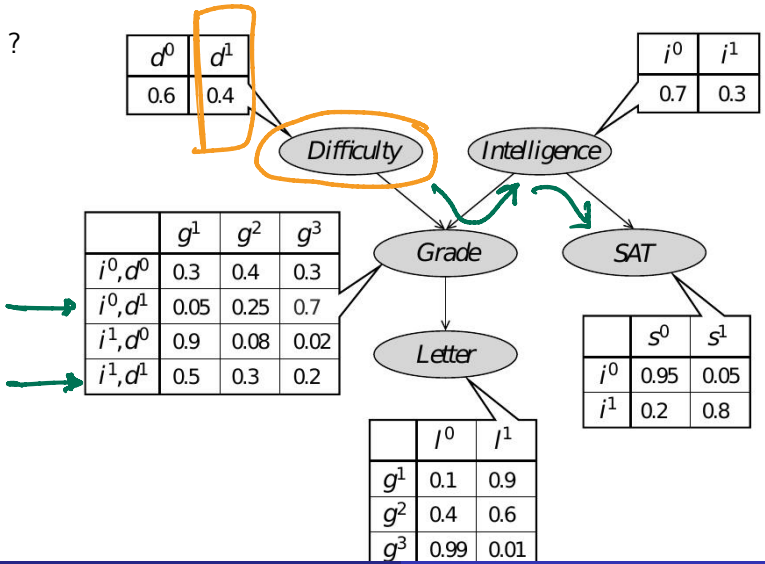
	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

Student example

- $SAT \perp Grade \mid Difficulty$?



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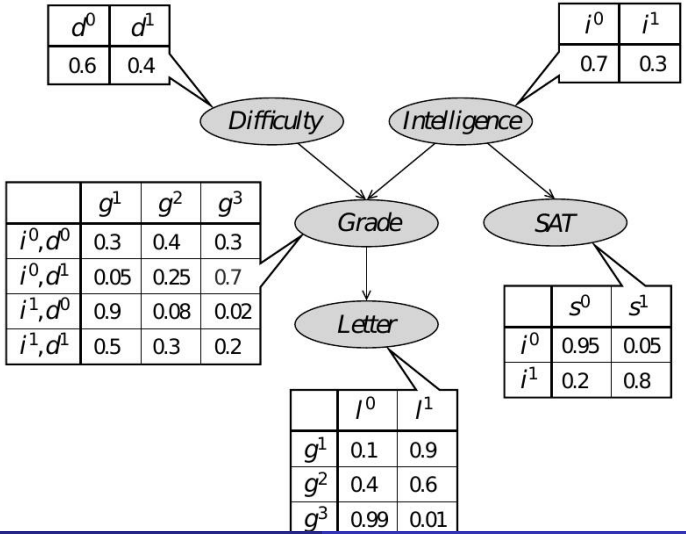
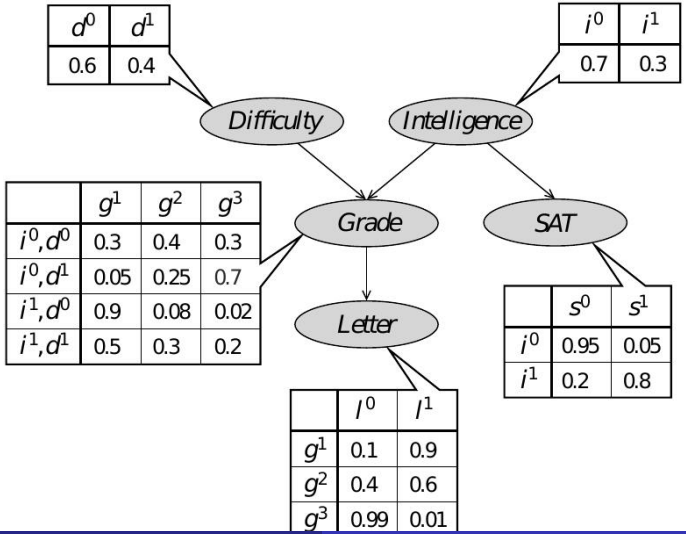
  - No



Student example

- $SAT \perp Grade \mid Difficulty$?
  - No
- Can we calculate conditional independence from the graph?

YES / NO



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i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

Student example

- $SAT \perp Grade \mid Difficulty$?
  - No
- Can we calculate conditional independence from the graph?
- In general, check if $X \perp Y \mid Z$ for sets of variables X, Y, Z

Graph theoretic analysis