# Lecture 19: 3 April, 2025

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Data Mining and Machine Learning
January–April 2025

# Support Vector Machines

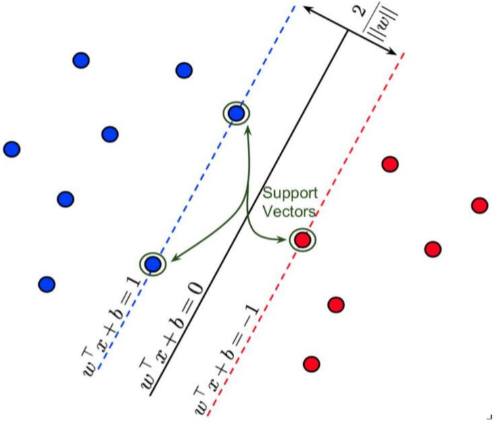
Minimize $\frac{|w|}{2}$

Subject to

$w_1x_1^i + w_2x_2^i + \cdots w_nx_n^i + b > 1$, if $y_i = 1$

$w_1x_1^i + w_2x_2^i + \cdots w_nx_n^i + b < -1$, if $y_i = -1$

- The constraints are linear
- The objective function is not linear
  $|w| = \sqrt{w_1^2 + w_2^2 + \cdots + w_n^2}$
- This is a quadratic optimization problem, not linear programming

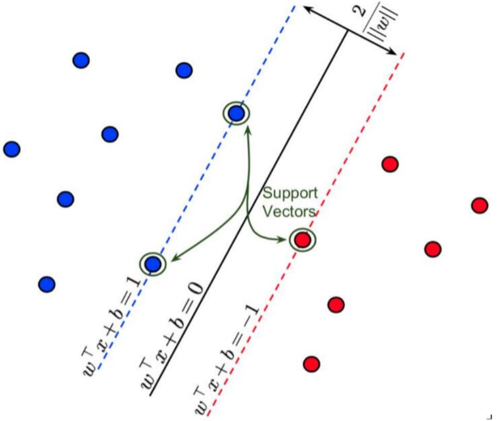


# Support Vector Machines

- Convex optimization theory
- Can be solved using computational techniques
- Solution expressed in terms of Lagrange multipliers $\alpha_1, \alpha_2, \ldots, \alpha_N$
- $\alpha_i$ is non-zero iff $x_i$ is a support vector
- Final classifier for new input $z$

$$\text{sign}\left[\sum_{i \in sv} y_i \alpha_i (x_i \cdot z) + b\right]$$

- $sv$ is set of support vectors



# Soft margin optimization

Minimize $\frac{|w|}{2} + \sum_{i=1}^{N} \xi_i^2$

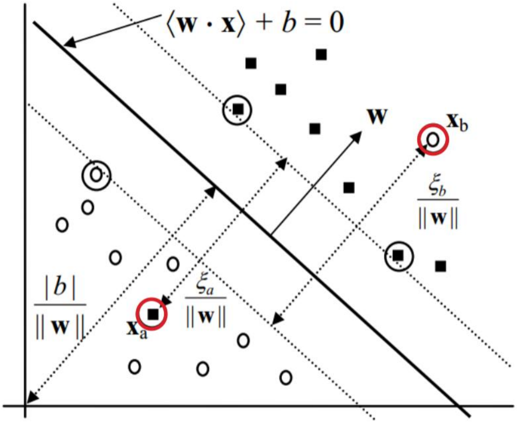
Subject to

$\xi_i \geq 0$

$w \cdot x_i + b > 1 - \xi_i$, if $y_i = 1$

$w \cdot x_i + b < -1 + \xi_i$, if $y_i = -1$

- Constraints include requirement that error terms are non-negative
- Again the objective function is quadratic

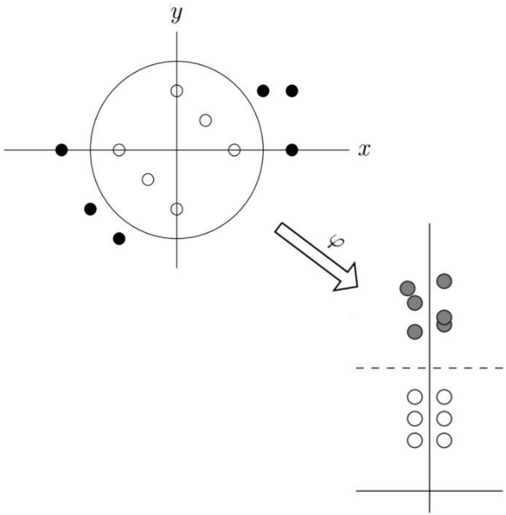


# Kernels

- $K$ is a kernel for transformation $\varphi$ if $K(x,z) = \varphi(x) \cdot \varphi(z)$
- If we have a kernel, we don't need to explicitly compute transformed points
- All dot products can be computed implicitly using the kernel on original data points

$$\sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{N} y_i y_j \alpha_i \alpha_j \; \varphi(x_i) \cdot \varphi(x_j)$$

$$\text{sign}\left[ \sum_{i \in sv} y_i \alpha_i (\varphi(x_i) \cdot \varphi(z)) + b \right]$$



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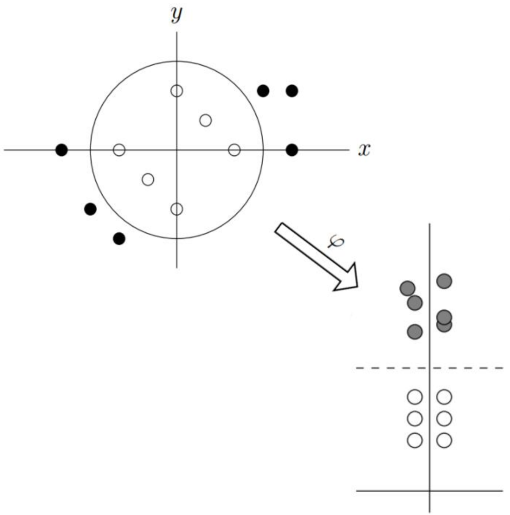
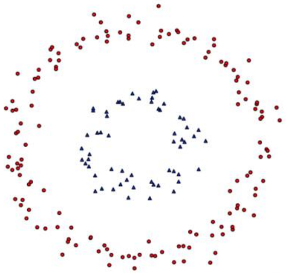
$$\sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{N} y_i y_j \alpha_i \alpha_j \; K(x_i, x_j)$$

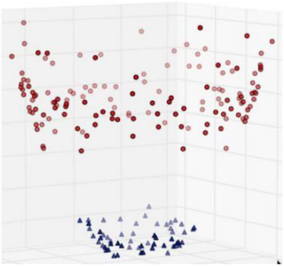
$$\text{sign}\left[\sum_{i \in sv} y_i \alpha_i K(x_i, z) + b\right]$$



# Known kernels

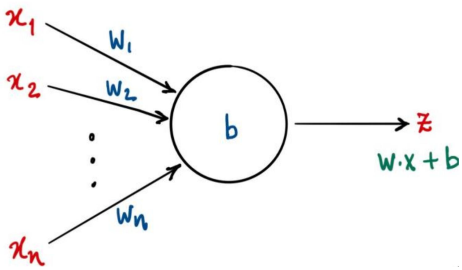
- Fortunately, there are many known kernels
- Polynomial kernels
$$K(x,z) = (1 + x \cdot z)^k$$
- Any $K(x,z)$ representing a similarity measure
- Gaussian radial basis function — similarity based on inverse exponential distance
$$K(x,z) = e^{-c|x-z|^2}$$





# Linear separators and Perceptrons

- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No $(-1)$
- What if we cascade perceptrons?
- Result is still a linear separator
  - $f_1 = w_1 \cdot x + b_1, f_2 = w_2 \cdot x + b_2$
  - $f_3 = w_3 \cdot \langle f_1, f_2 \rangle + b_3$
  - $f_3 = w_3 \cdot \langle w_1 \cdot x + b_1, w_2 \cdot x + b_2 \rangle + b_3$
  - $f_3 = \sum_{i=1}^{4} \left( w_{3_1} w_{1_i} + w_{3_2} w_{2_i} \right) \cdot x_i$
    $+ \left( w_{3_1} b_1 + w_{3_2} b_2 + b_3 \right)$



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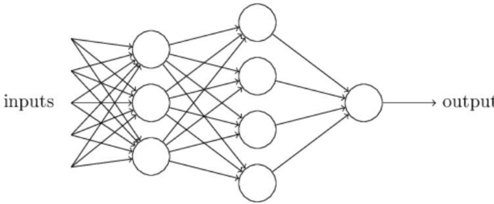

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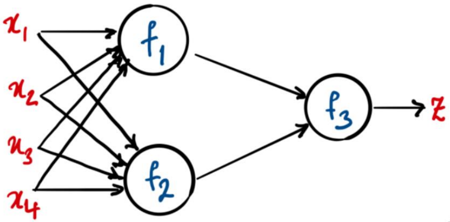
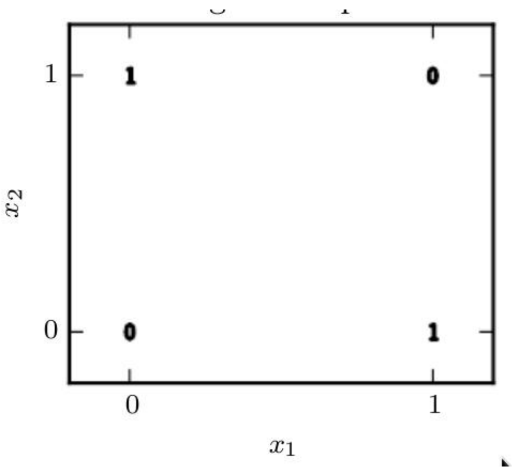
    $+(w_{3_1} b_1 + w_{3_2} b_2 + b_3)$



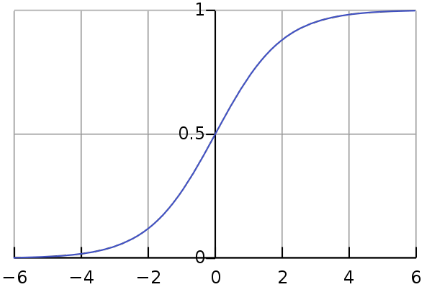
# Limits of linearity

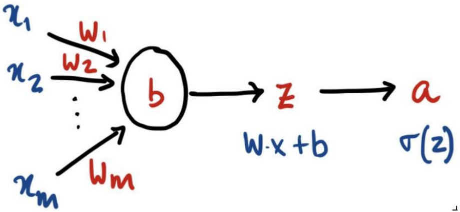
- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of $x_1$, $x_2$ is true (not both)
- Suppose $XOR(x_1, x_2) = ux_1 + vx_2 + b$
- $x_2 = 0$: As $x_1$ goes from $0$ to $1$, output goes from $0$ to $1$, so $u > 0$
- $x_2 = 1$: As $x_1$ goes from $0$ to $1$, output goes from $1$ to $0$, so $u < 0$
- Observed by Minsky and Papert, 1969, first "AI Winter"



# Non-linear activation

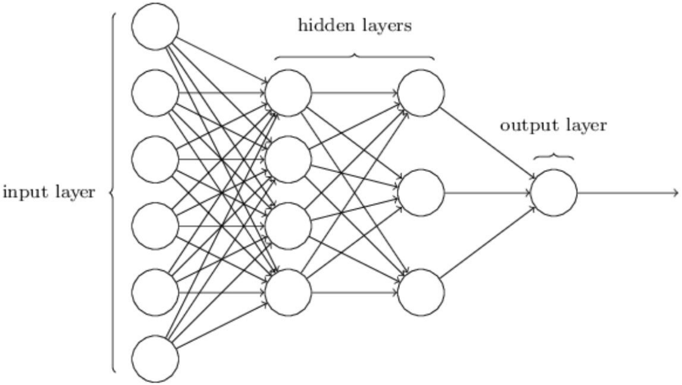
- Transform linear output $z$ through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$



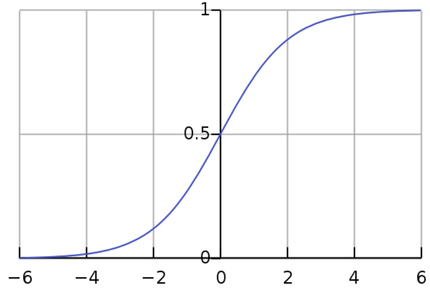
# Structure of a neural network

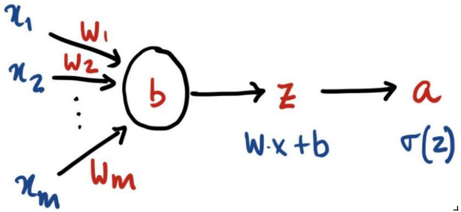
- Acyclic
- Input layer, hidden layers, output layer
- Assumptions
  - Hidden neurons are arranged in layers
  - Each layer is fully connected to the next
  - Set weight to zero to remove an edge



# Non-linear activation

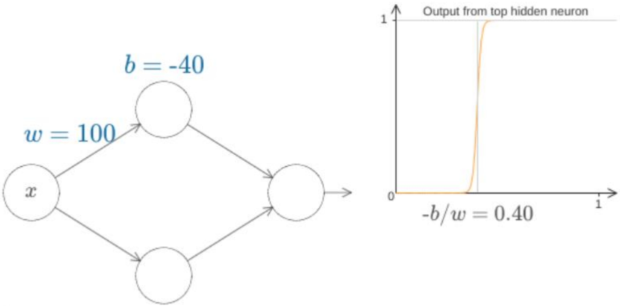
- Transform linear output $z$ through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$
- Step is at $z = 0$
  - $z = wx + b$, so step is at $x = -b/w$
  - Increasing $w$ makes step steeper
  - Shift step by adjusting $b$



# Universality

- Create a step at $x = -b/w$
  - Use $s$ to denote $-b/w$
  - Step position
- Cascade steps
- Subtract steps to create a box
- Create many boxes
- Approximate any function
- Need only one hidden layer!



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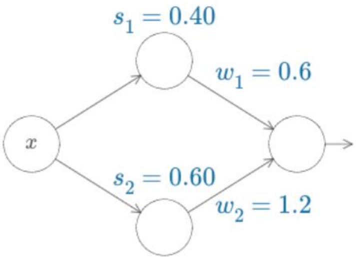


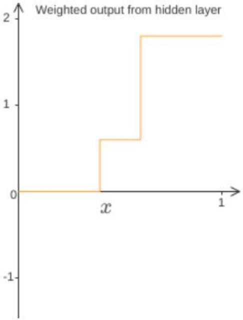

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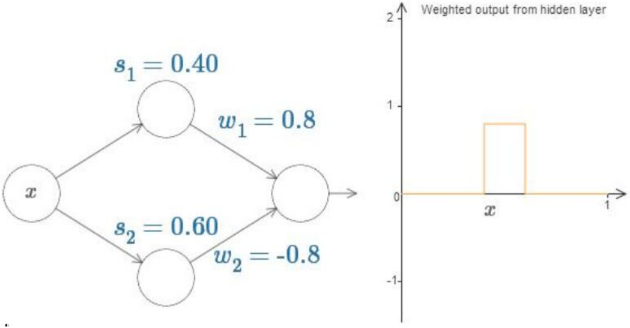

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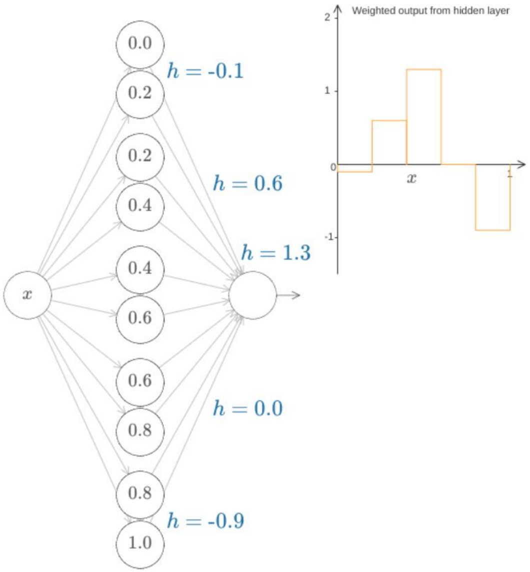

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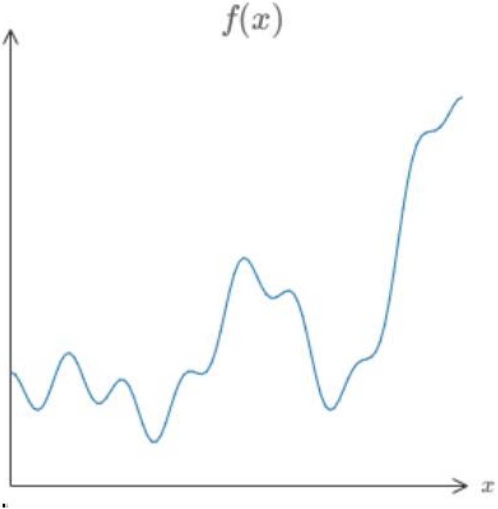

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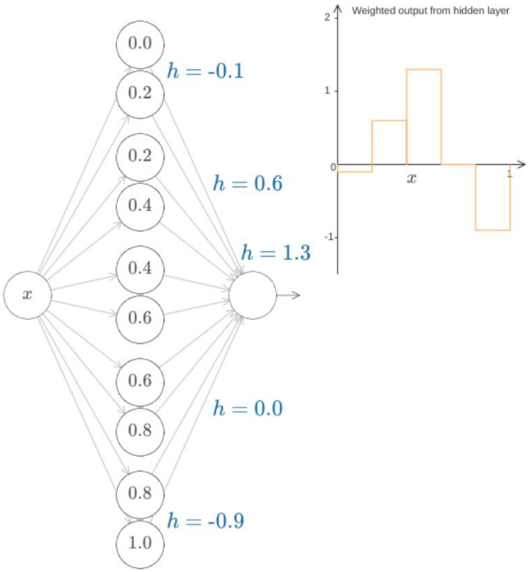

# Non-linear activation

- With non-linear activation, network of neurons can approximate any function
  - Can build "rectangular" blocks
  - Combine blocks to capture any classification boundary