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Data Mining and Machine Learning
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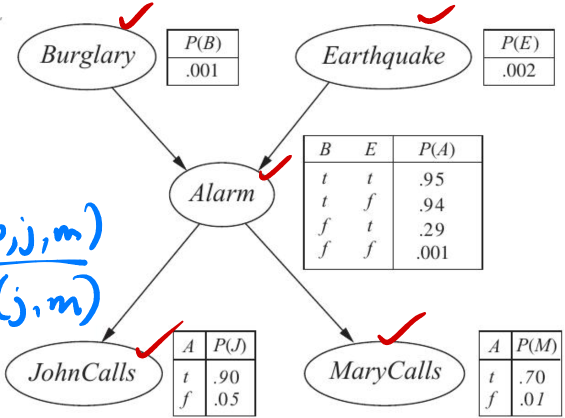
Approximate inference

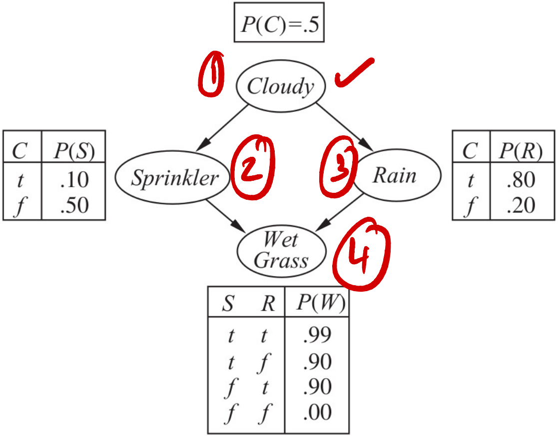
- Exact inference is NP-complete
- Generate random samples, count to estimate probabilities
- Respect conditional probabilities — generate in topological order
- Suppose we are interested in $P(b \mid j, m)$
- Samples with $\neg j$ or $\neg m$ are useless
- Can we sample more efficiently?

$$\frac{P(b,j,m)}{P(j,m)}$$



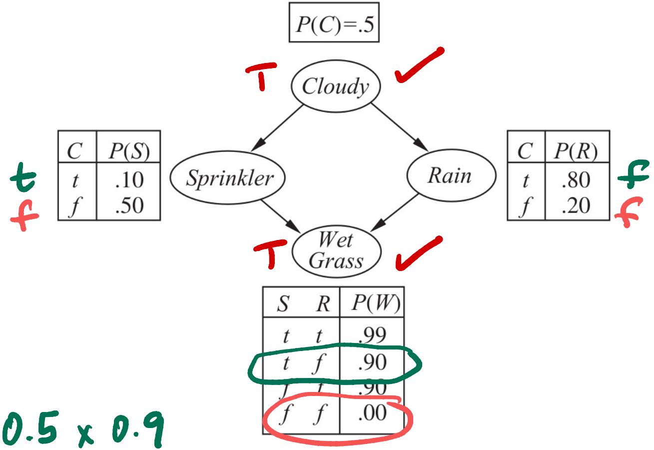
Rejection sampling

- $P(Rain \mid Cloudy, Wet\ Grass)$
- If we start with $\neg Cloudy$, sample is useless
- Immediately stop and reject this sample — rejection sampling
- General problem with low probability situation — many samples are rejected



Likelihood weighted sampling

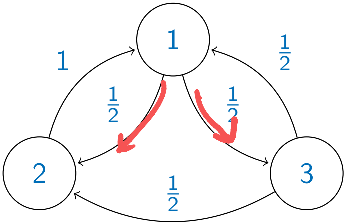
- $P(Rain \mid Cloudy, Wet\ Grass)$
- Fix evidence $Cloudy, Wet\ Grass$ true
- Then generate the other variables
- Compute likelihood of evidence
- Samples $s_1, s_2, \ldots, s_N$ with weights $w_1, w_2, \ldots w_N$
- $P(r \mid c, w) = \dfrac{\sum_{s_i \text{ has rain}} w_i}{\sum_{1 \le j \le N} w_j}$



0.5 x 0.9

Markov chains

- Finite set of states, with transition probabilities between states
- For us, a state will be an assignment of values to variables
- A three state Markov Chain



- Represent using a transition matrix — stochastic

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

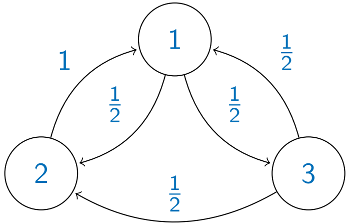
- $P[j]$ is probability of being in state j

Ergodicity

- Markov chain A is ergodic if there is some t_0 such that for every P, for all $t > t_0$, for every j, $(P^\top A^t)[j] > 0$.
- Ergodic Markov chain has a stationary distribution π, $\pi^\top A = \pi$
- For *any* starting distribution P, $\lim_{t \to \infty} P^\top A^t = \pi$
- Stationary distribution represents fraction of visits to each state in a long enough execution
- Sufficient conditions for ergodicity
  - Irreducible (strong connected)
  - Aperiodic (paths of all lengths between states)

$P^T \cdot A$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$



finite

π

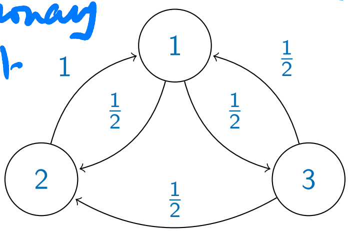
Approximate inference using Markov chains

- Bayesian network has variables $V_1, V_2, \ldots, V_n$
- Each assignment of values to the variables is a state
- Set up a Markov chain based on these states
- Stationary distribution should assign to state s the probability $P(s)$ in the Bayesian network
- How to reverse engineer the transition probabilities to achieve this?

$\pi(s) = P(s)$

stationary dist

In G



$\sum_a \sum_e (b, m, j, a, e)$

Reversible Markov chains

- Ergodic Markov chain with stationary distribution π

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  - In steady state, probability of being in state j and then moving to k same as probability of being in state k and then moving to j

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$k \rightarrow j$

- Derivation of balance equations
  - Given an evolution $x_1 x_2 \ldots$, for large n, $P[x_n = j \mid x_{n-1} = k] = P[x_{n-1} = j \mid x_n = k]$

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  - $P[x_{n-1} = j \mid x_n = k] = \underbrace{P[x_n = k \mid x_{n-1} = j]}_{p_{jk}} \cdot \dfrac{P[x_{n-1} = j]}{P[x_n = k]}$ $\dfrac{\pi_j}{\pi_k}$

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  - $P[x_{n-1} = j \mid x_n = k] = P[x_n = k \mid x_{n-1} = j] \cdot \dfrac{\pi_j}{\pi_k}$, in steady state
  - $p_{kj} = p_{jk}\dfrac{\pi_j}{\pi_k}$, so $\pi_j \cdot p_{jk} = \pi_k \cdot p_{kj}$

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- $\sum_k a_j \cdot p_{jk} = \sum_k a_k \cdot p_{kj}$
- $a_j \sum_k p_{jk} = \sum_k a_k \cdot p_{kj}$
- $a_j \cdot 1 = \sum_k a_k \cdot p_{kj}$
- $a^\top = a^\top A$, so $a^\top$ is the stationary distribution of A

$\overset{a}{[\ \cdot\ ]} = \overset{a}{[\ \cdot\ ]}\overset{A}{[\ \updownarrow\ ]}$

Gibbs sampling

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- Allow such a move only when s_j, s_k differ at exactly one position
  - $s_j = (x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$
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  - $s_k = (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$
- Sampling algorithm
  - Current state is $s_j = (x_1, x_2, \ldots, x_n)$
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- Need to compute $P[y_i \mid x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$

Markov blanket

- Recall $MB(X)$ — Markov blanket of X
  - $Parents(X)$
  - $Children(X)$
  - $Parents\ of\ Children(X)$
- $X \perp \neg MB(X) \mid MB(X)$

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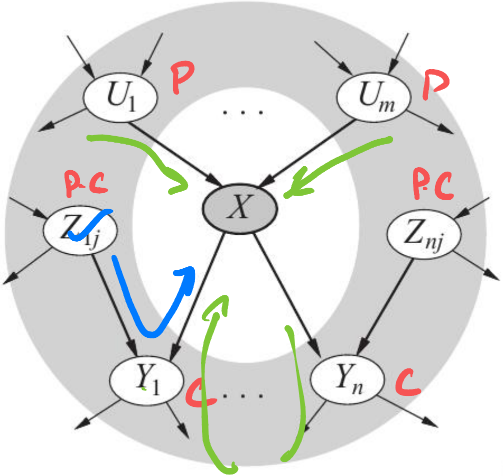
Rain

WetGrass T



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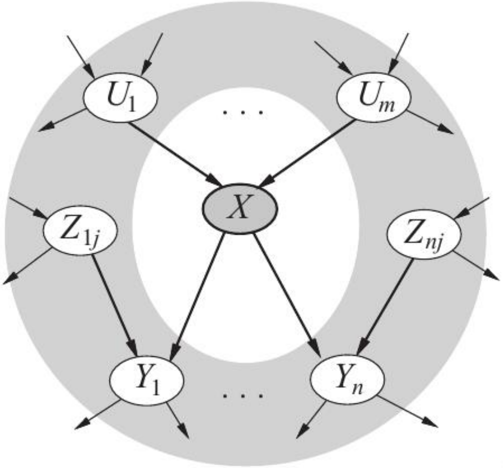

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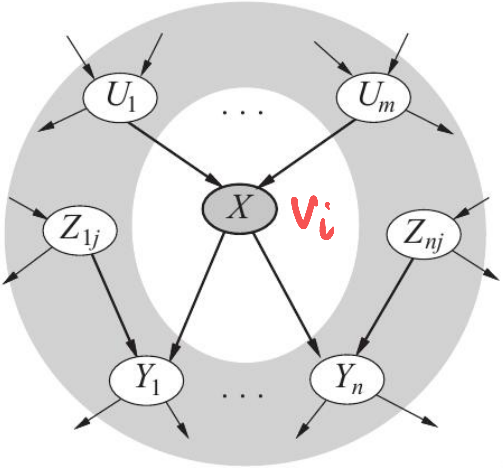
- $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ fix $MB(V_i)$



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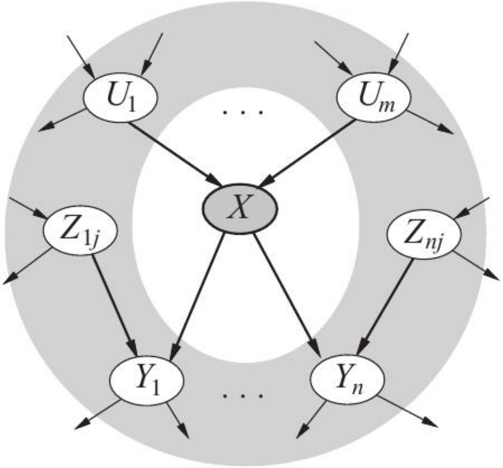
- $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ fix $MB(V_i)$
- Can compute $P[y_i \mid x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ given conditional probability tables in the network



Gibbs sampling

- Move from $s_j = (x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ to $s_k = (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$

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- $p_{jk} = \frac{1}{n} P[y_i \mid \bar{x}]$

  pick i

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- $p_{jk} = \dfrac{1}{n} P[y_i \mid \bar{x}] \;\; = \dfrac{1}{n} \dfrac{P(s_k)}{P(\bar{x})}$

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- Let $\bar{x} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
- $p_{jk} = \frac{1}{n} P[y_i \mid \bar{x}] = \frac{1}{n} \frac{P(s_k)}{P(\bar{x})}$
- Likewise $p_{kj} = \frac{1}{n} P[x_i \mid \bar{x}] = \frac{1}{n} \frac{P(s_j)}{P(\bar{x})}$

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- Likewise $p_{kj} = \dfrac{1}{n} P[x_i \mid \bar{x}] = \dfrac{1}{n} \dfrac{P(s_j)}{P(\bar{x})}$
- Therefore, $\dfrac{p_{jk}}{p_{kj}} = \dfrac{P(s_k)}{P(s_j)}$, so $\underline{P(s_j)} \cdot \overset{A}{p_{jk}} = \underline{P(s_k)} \cdot \overset{A}{p_{kj}}$ and this chain is reversible

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- Therefore, $\frac{p_{jk}}{p_{kj}} = \frac{P(s_k)}{P(s_j)}$, so $P(s_j) \cdot p_{jk} = P(s_k) \cdot p_{kj}$ and this chain is reversible
- By our previous observation about any vector $a^\top$ satisfying balance equations, we must have $(P(s_1), P(s_2), \ldots, P(s_n)) = (\pi_1, \pi_2, \ldots, \pi_n)$ for the current Markov chain

Gibbs sampling

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- We have created a reversible Markov chain whose stationary distribution provides the true probabilities of the original Bayesian network!

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- Gibbs sampling is a special case of the more general Metropolis-Hastings algorithm

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  - $\dots$
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  - $\ldots$
  - Then generate y_n, given $y_1, y_2, \ldots, y_{n-1}$
- Standard Gibbs sampler — again a reversible Markov chain

Approximate inference using Markov chains

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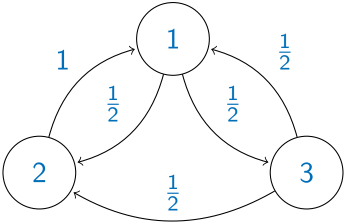
- Use Gibbs sampling to set up a reversible Markov chain
- Stationary distribution will assign to each state s its probability $P(s)$ in the Bayesian network



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