Lecture 22: 15 April, 2025

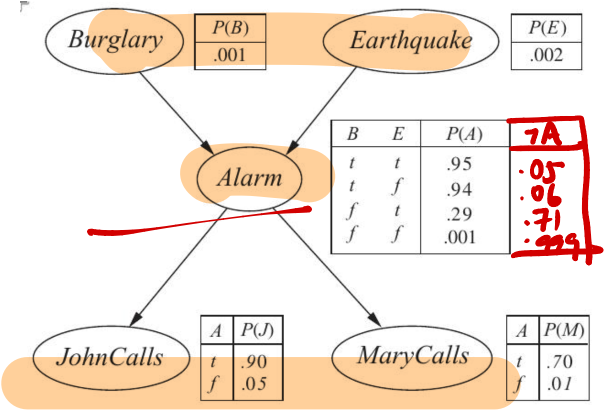
Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Data Mining and Machine Learning
January–April 2025

- Underlying DAG, no cyclic dependencies
- Each node has a local (conditional) probability table

$P(A \mid B, E)$



Evaluating a network

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- Using $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n) P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n) P(x_n)$ and writing variables in topolological sort order,

$$P(m, j, b) = \sum_{e=0}^{1} \sum_{a=0}^{1} P(m \mid a) P(j \mid a) P(a \mid b, e) P(b) P(e)$$

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- Why is computing $P(b,m,j)$ enough? Should we not compute $P(b \mid m,j)$?

① $P(b \mid m, j) = \dfrac{P(b, m, j)}{P(m, j)}$ ✓ ?

$P(A \wedge B) = P(A \mid B) \cdot P(B)$

② $P(\neg b \mid m, j) = \dfrac{P(\neg b, m, j)}{P(m, j)}$

$P(A \mid B) = \dfrac{P(A \wedge B)}{P(B)}$

$P(b \mid m, j) = \alpha \cdot P(b, m, j)$ ✓

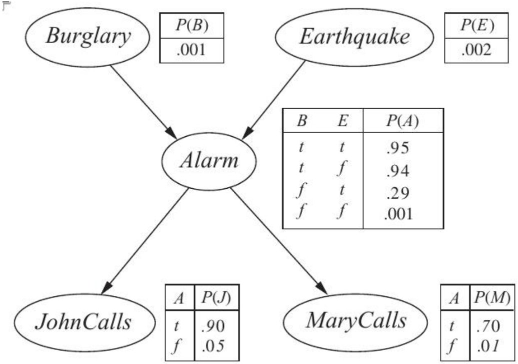
$P(\neg b \mid m, j) = \alpha \cdot P(\neg b, m, j)$ ✓

Conditional independence

- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$

$P(x = b \mid y = a) = P(x = b)$

For any b, a



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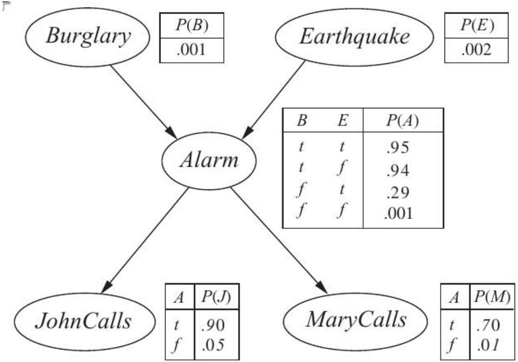
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - x and y are independent given z
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$

$P(x \mid y, z) = P(x \mid z)$

Is $M \perp B \mid A$?



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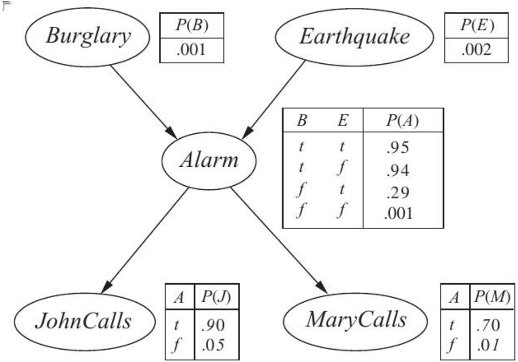

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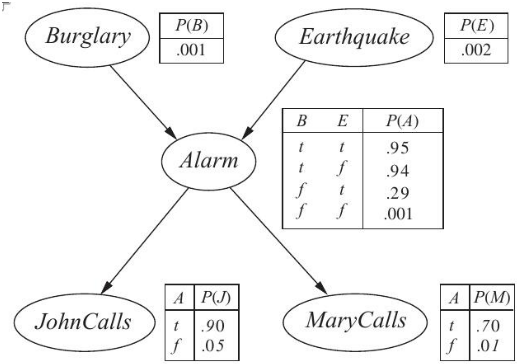

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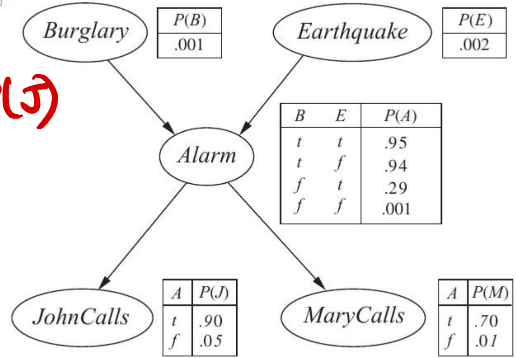
  - No — value of j tells us something about value of m and vice versa
- Is *JohnCalls* independent of *MaryCalls* given *Alarm* ($j \perp m \mid a$)?

$P(J|M) = P(J)$



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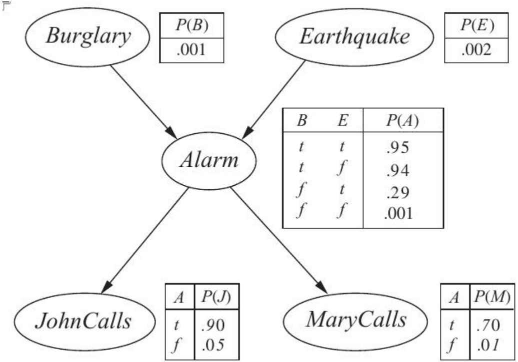
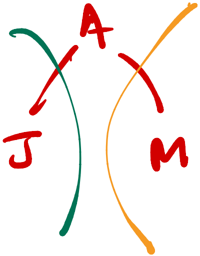
  - Yes — by semantics of network, local independence

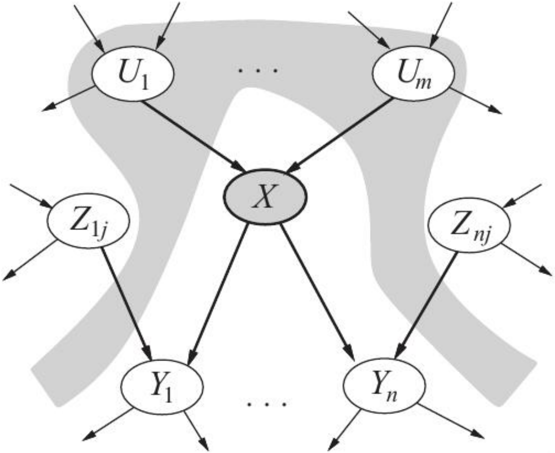


Probabilistic graphical models

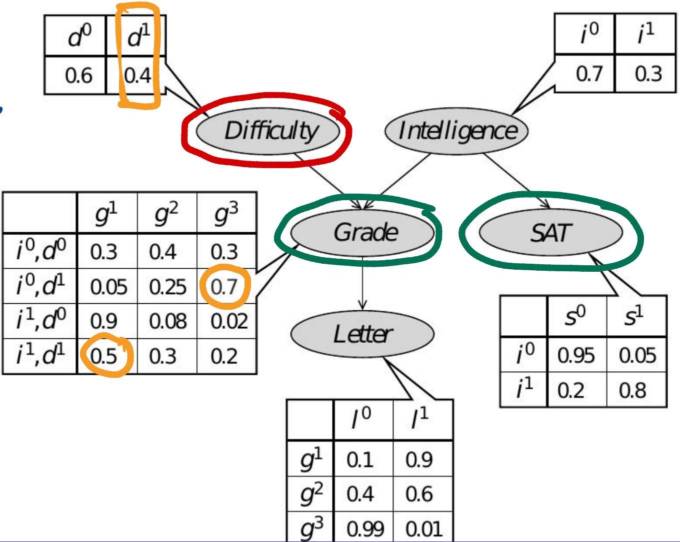
- Fundamental assumption

  A node is conditionally independent of non-descendants, given its parents





- *SAT* ⊥ *Grade* | ~~*Difficulty*~~ ? Intelligence



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
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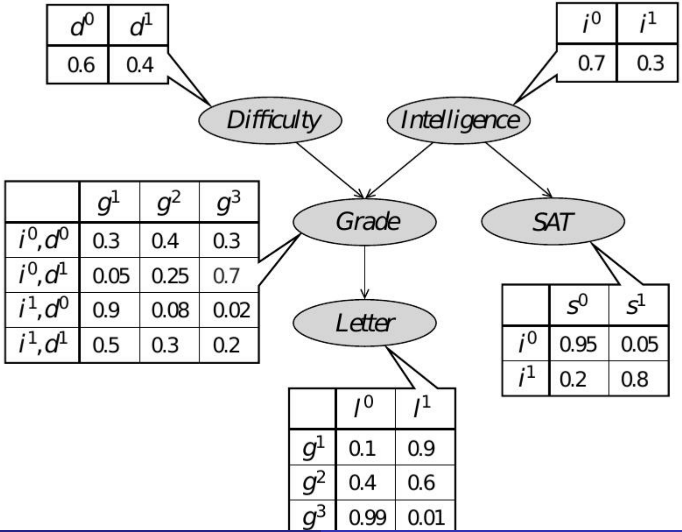
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

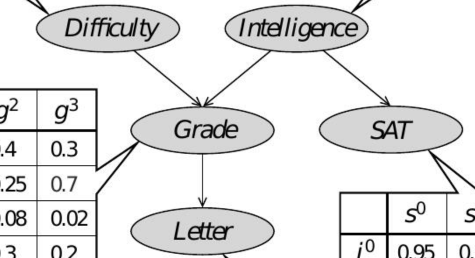
	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

- *SAT* $\perp$ *Grade* | *Difficulty* ?
  - No



Student example

- *SAT* $\perp$ *Grade* | *Difficulty* ?
  - No
- Can we calculate conditional independence from the graph?



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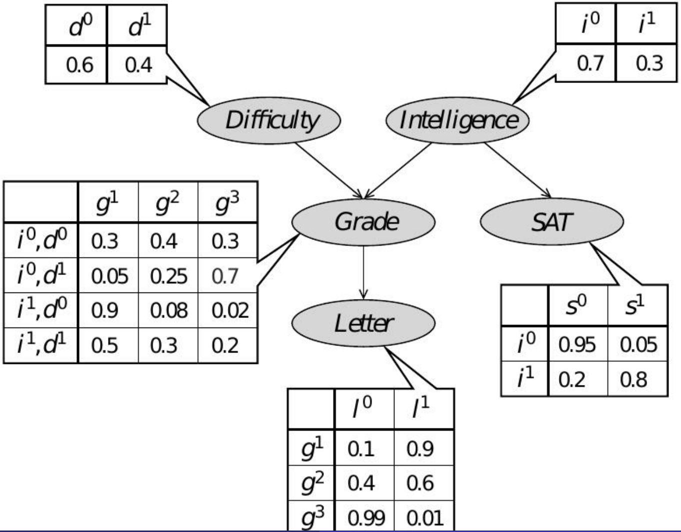
Student example

- $SAT \perp Grade \mid Difficulty$?
  - No
- Can we calculate conditional independence from the graph?
- In general, check if $X \perp Y \mid Z$ for sets of variables X, Y, Z



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- How does dependence "flow" through a network?

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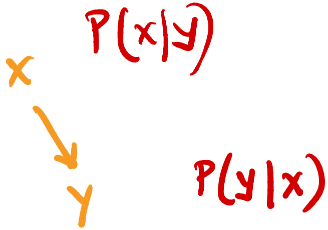

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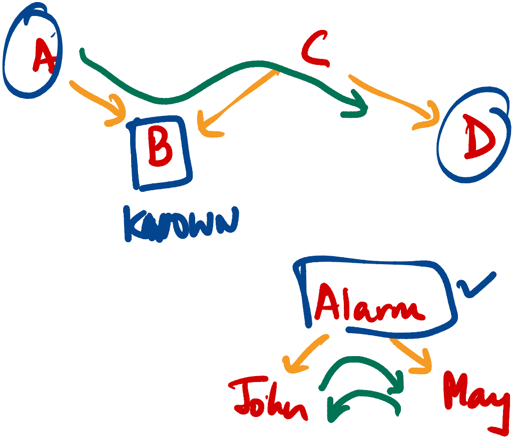
- Examine trails between nodes
  - Paths in the underlying undirected graph



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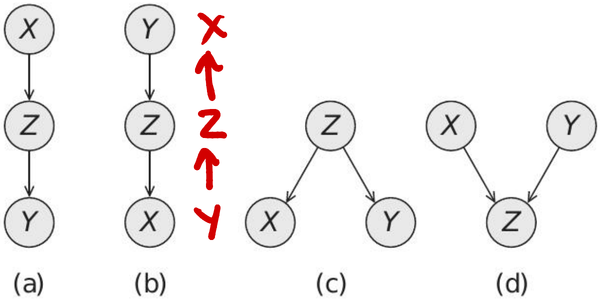
- Basic trails — (undirected) paths of length 2
  - Four basic trails



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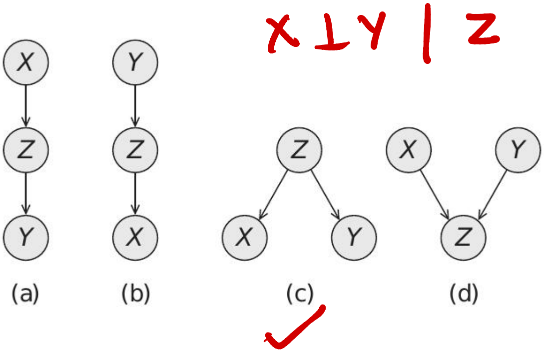
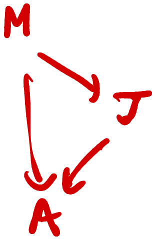
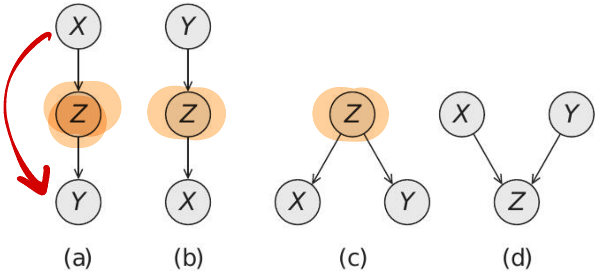


Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks





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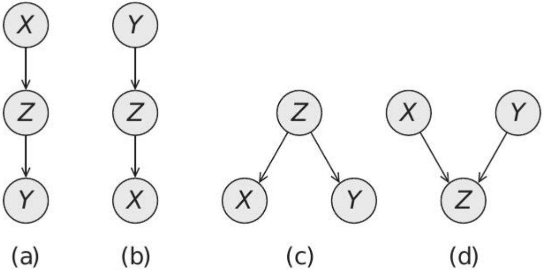
- In (d), knowing Z allows influence to flow



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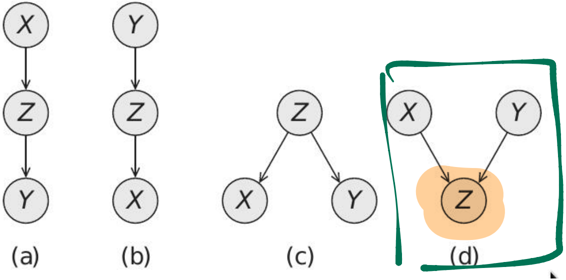

Know Z enable flow

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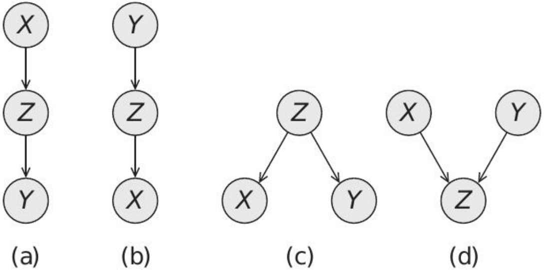

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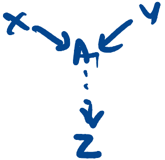
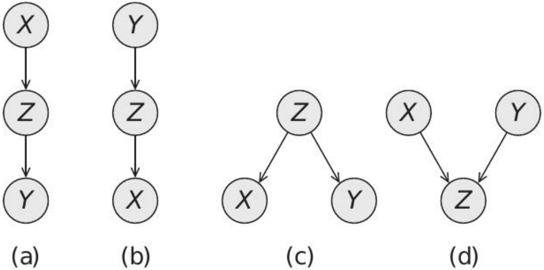
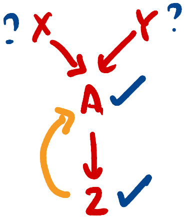
  - Simplest form of V-structure



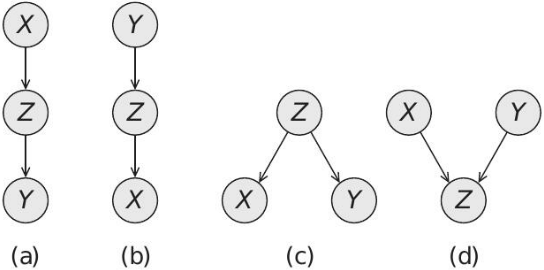




$X \perp Y \mid Z$

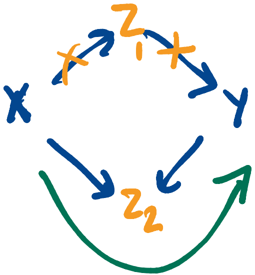
D-Separation

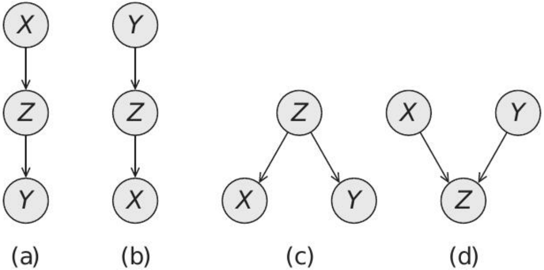
- Check if $X \perp Y \mid Z$



D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y





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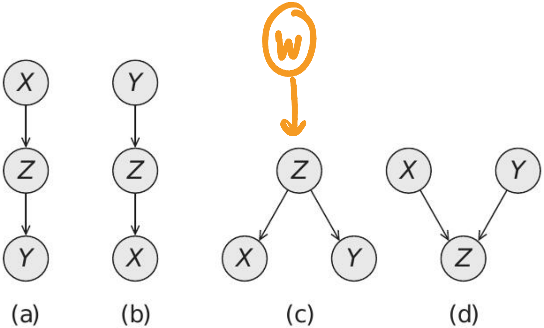

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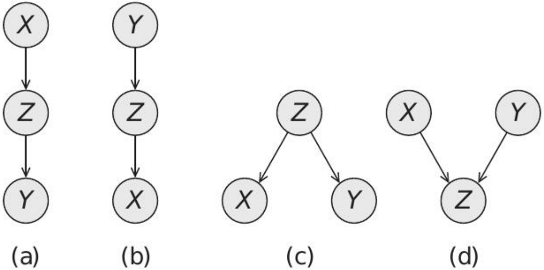

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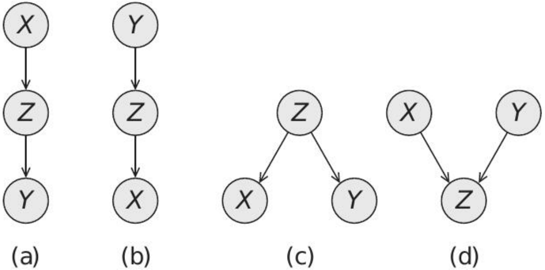

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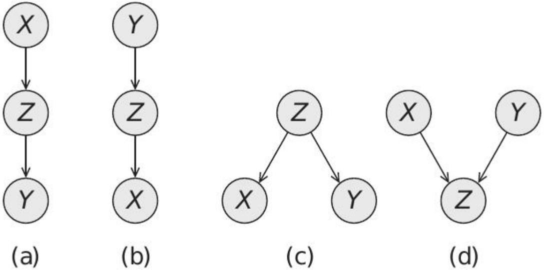
  - In general, V-structure includes descendants of the bottom node



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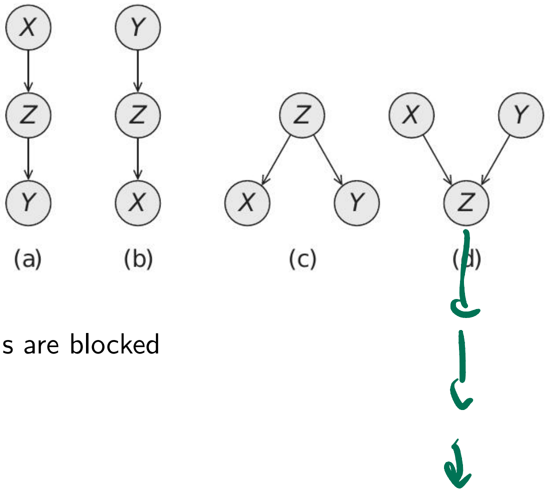
- x and y are D-separated given z if all trails are blocked



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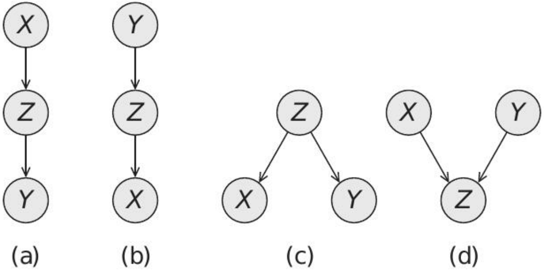
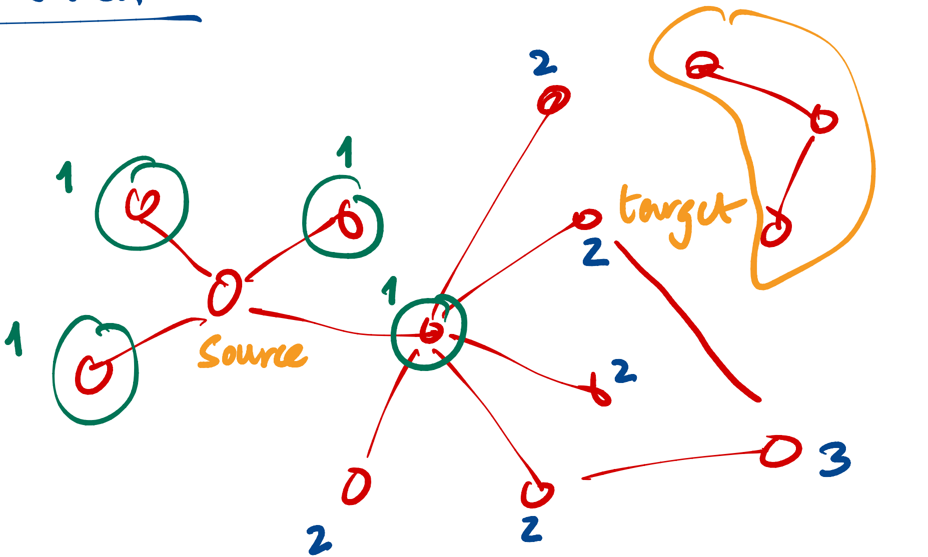


- x and y are D-separated given z if all trails are blocked
- Variation of breadth first search (BFS) to check if y is reachable from x through some trail

Breadth first search



BFS

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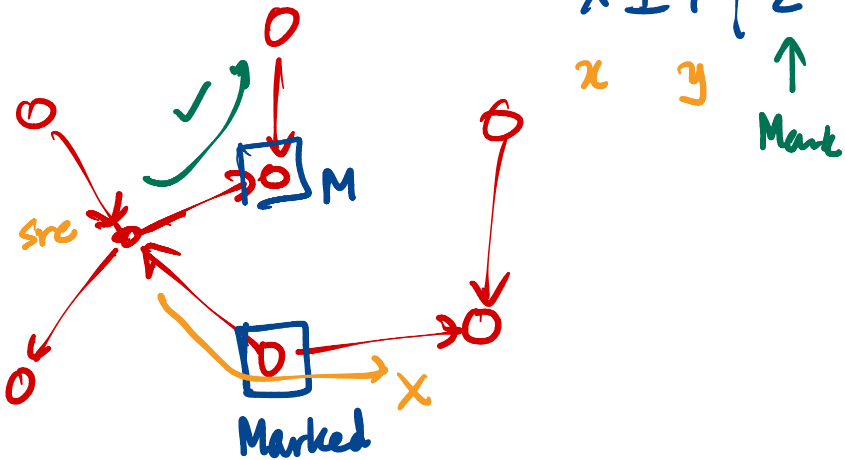

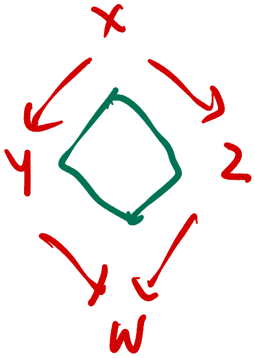

Undirected cycles are possible

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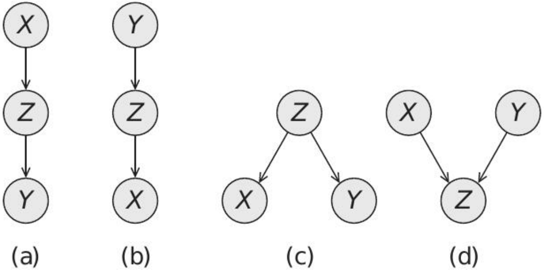


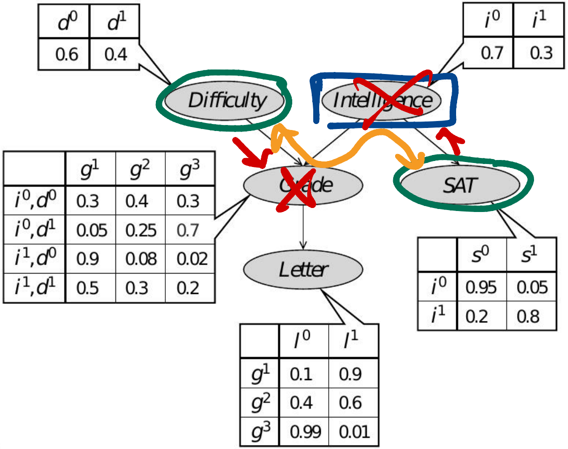
- x and y are D-separated given z if all trails are blocked

$x \perp y \mid z$

- Variation of breadth first search (BFS) to check if y is reachable from x through some trail
- Extends to sets — each $x \in X$ is D-separated from each $y \in Y$

Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence



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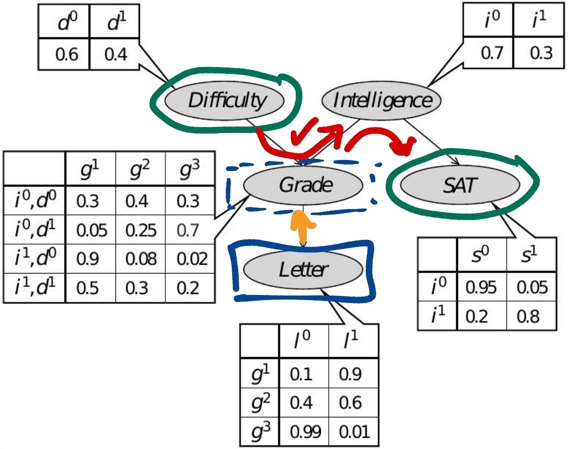
Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence
- Is SAT independent of Difficulty given Letter?
  - No, Difficulty – Grade – Intelligence – SAT trail is open
  - Letter is known, hence something about Grade is known (V-structure)
  - Intelligence is not known



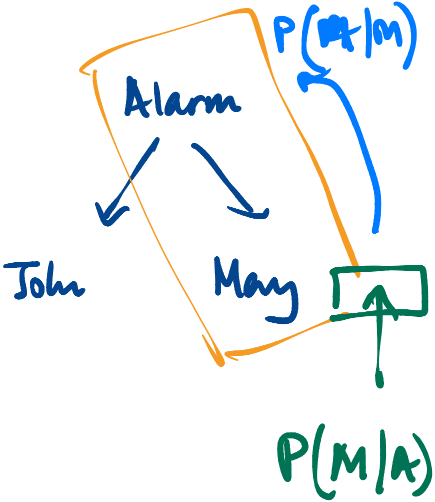
d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01



$$P(A \mid M)$$

$$P(J \mid A=1) \cdot P(A=1 \mid M)$$

$$P(J = A = 0) \quad P(A=0 \mid M)$$