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Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Data Mining and Machine Learning
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Conditional probabilities

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- Can we strive for something in between?
  - "Local" dependencies between some variables

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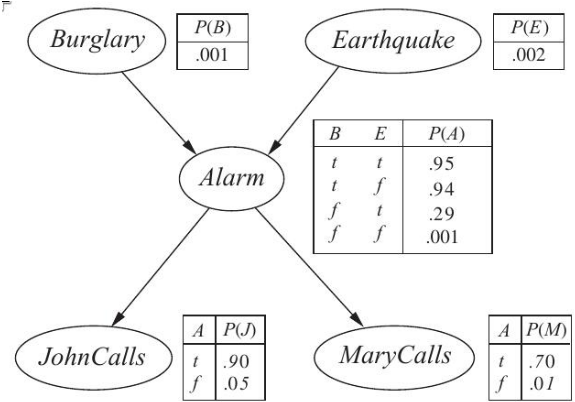
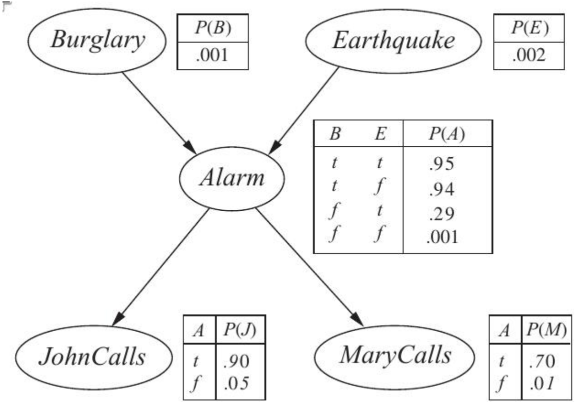
- Example: Burglar alarm
  - Pearl's house has a burglar alarm
  - Neighbours John and Mary call if they hear the alarm
  - John is prone to mistaking ambulances etc for the alarm
  - Mary listens to loud music and sometimes fails to hear the alarm
  - The alarm may also be triggered by an earthquake (California!)



Probabilistic graphical models

- Each node has a local (conditional) probability table



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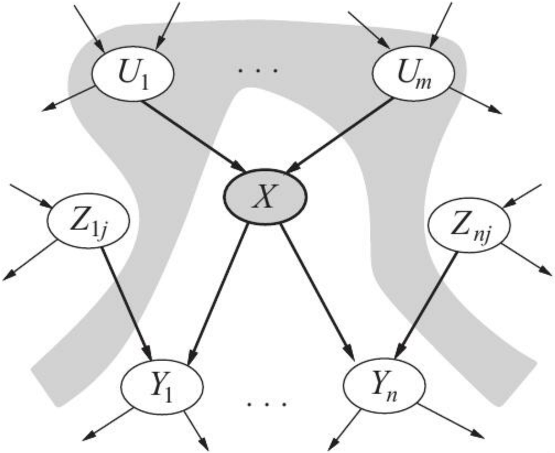

Probabilistic graphical models

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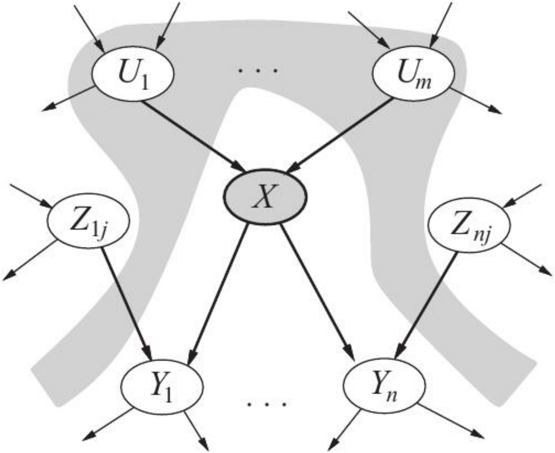
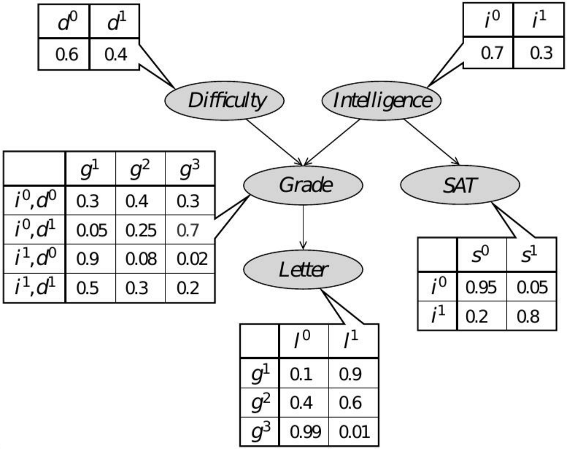
- Graph is a DAG, no cyclic dependencies



Student example

- Example due to Nir Friedman and Daphne Koller
- Student asks teacher for a reference letter
- Teacher has forgotten the student, so letter is entirely based on student's grade in the course



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

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- Applied recursively, this gives us the chain rule
  $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n)P(x_n)$

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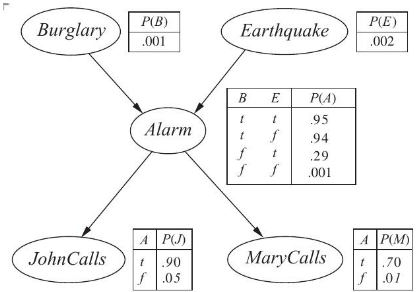

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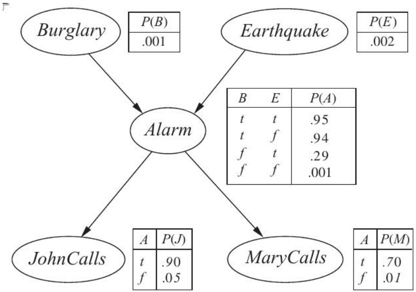

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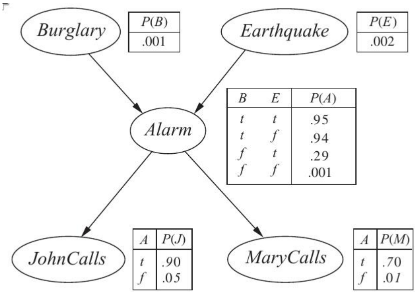
- $P(m, j, b) =$
  $$\sum_{a=0}^{1} \sum_{e=0}^{1} P(m \mid a) P(j \mid a) P(a \mid b, e) P(b) P(e)$$



P(B)
.001

P(E)
.002

B	E	P(A)
t	t	.95
t	f	.94
f	t	.29
f	f	.001

A	P(J)
t	.90
f	.05

A	P(M)
t	.70
f	.01

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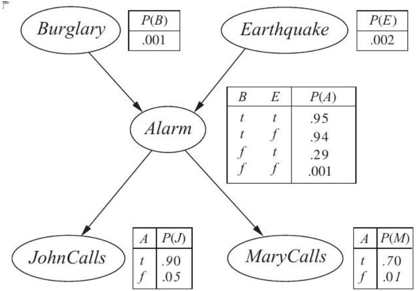
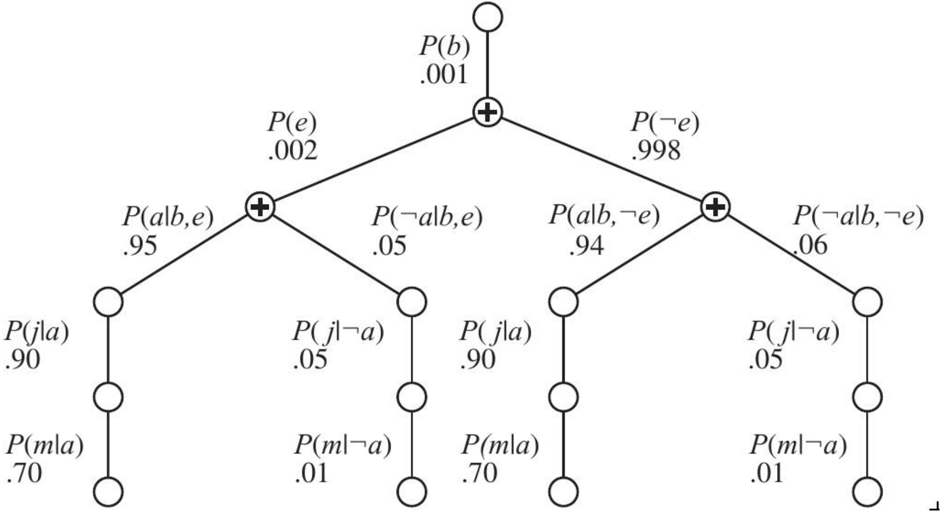


Evaluation tree



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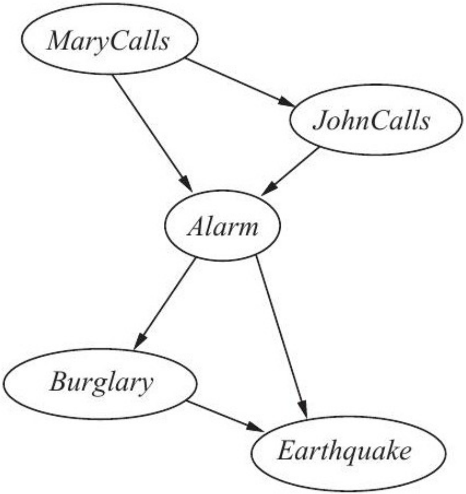

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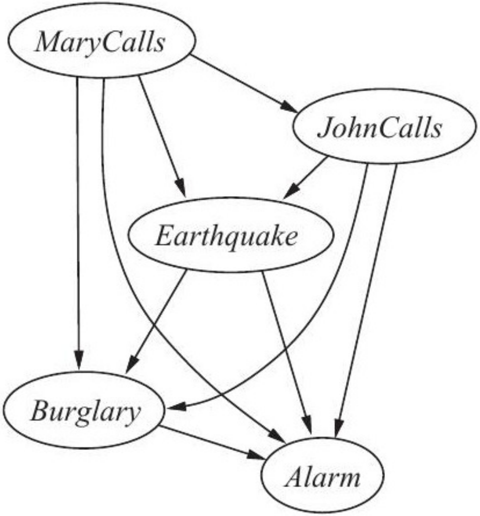

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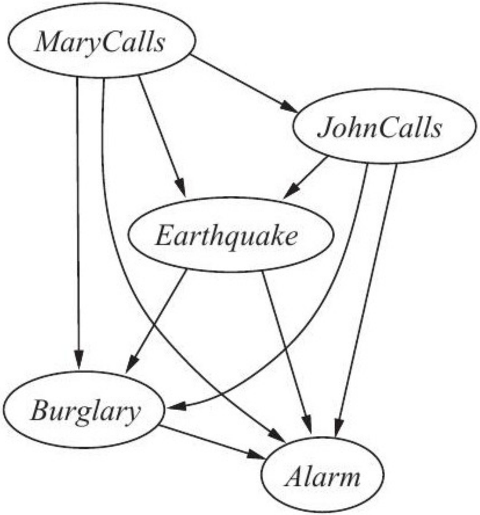
- Causal model (causes to effects) works better than diagnostic model (effects to causes)



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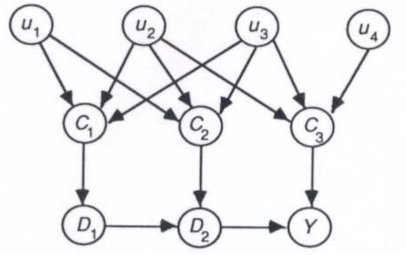
- SAT — given a formula in CNF, is there an assignment to variables that makes the formula true?
- 3-SAT — SAT where each clause has exactly 3 literals
- Both SAT and 3-SAT are NP-complete
  - No known efficient algorithm — try all possible valuations

Reducing 3-SAT to exact inference

- Convert a 3-CNF formula into a Bayesian network



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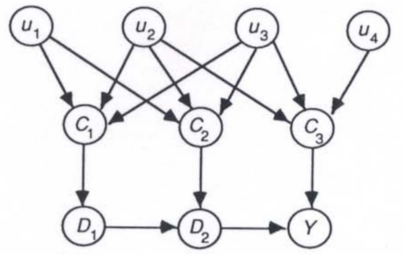

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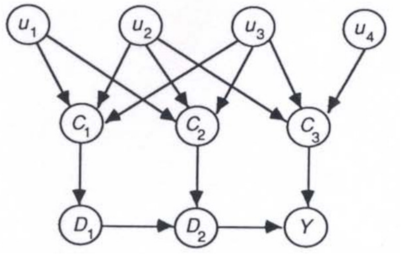
- Convert a 3-CNF formula into a Bayesian network
- Top layer: one node for each variable u_i
- Middle layer: one node for each clause C_j
  - Parents are three variables whose literals are in C_j
  - Conditional probability table for C_j has 8 rows, for all possible valuations of 3 variables
  - $P(C_j = 1) = 0$ for row where each input literal is false, $P(C_j = 1) = 1$ for remaining 7 rows



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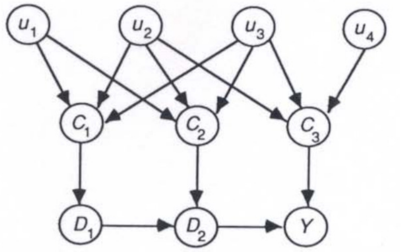

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- $P(Y = 1) > 0$ iff original 3-CNF formula is satisfiable