Lecture 22: 4 April, 2023

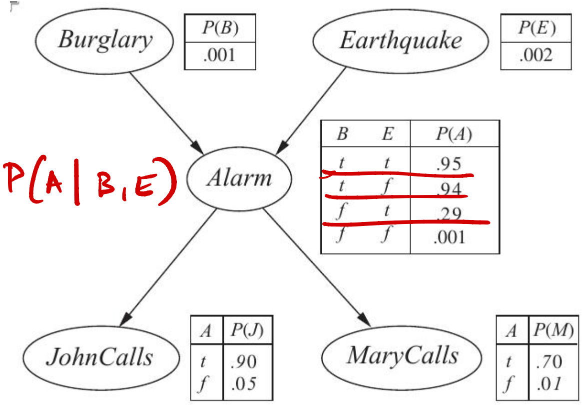
Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Data Mining and Machine Learning
January–April 2023

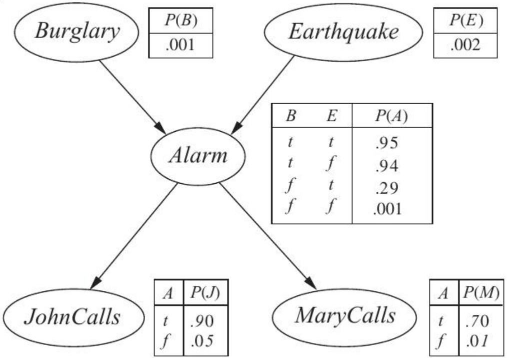
Probabilistic graphical models

- Underlying DAG, no cyclic dependencies
- Each node has a local (conditional) probability table



Conditional independence

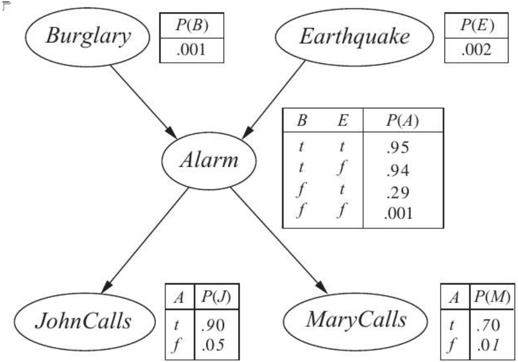
- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$



Conditional independence

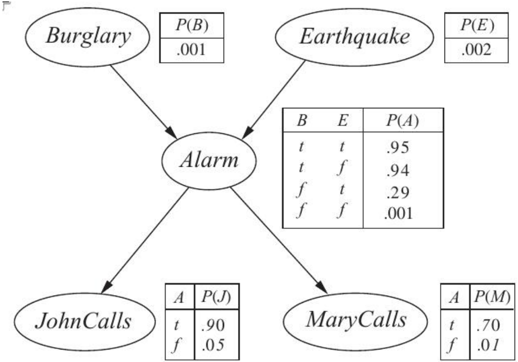
- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - x and y are independent given z
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$

$$P(x \mid y, z) = P(x \mid z)$$
$$P(y \mid x, z) = P(y \mid z)$$



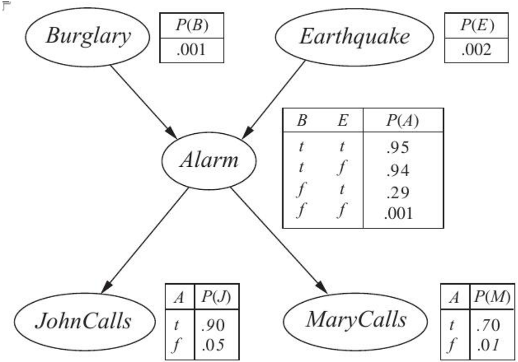
Conditional independence

- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - x and y are independent given z
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$
- Is *JohnCalls* independent of *MaryCalls* ($j \perp m$)?



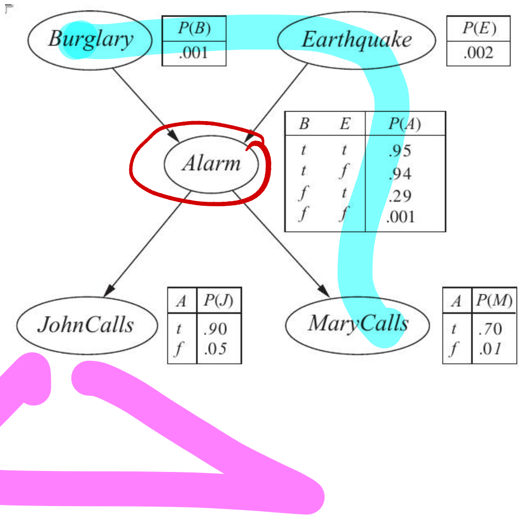
Conditional independence

- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - x and y are independent given z
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$
- Is *JohnCalls* independent of *MaryCalls* ($j \perp m$)?
  - No — value of j tells us something about value of m and vice versa



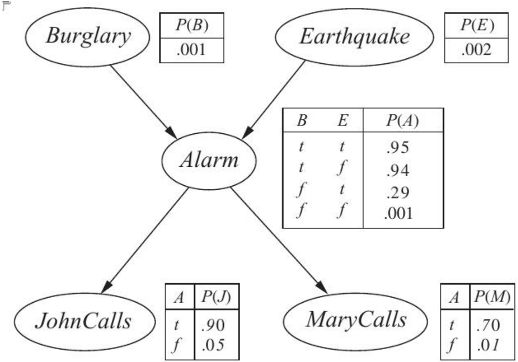
Conditional independence

- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - x and y are independent given z
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$
- Is *JohnCalls* independent of *MaryCalls* ($j \perp m$)?
  - No — value of j tells us something about value of m and vice versa
- Is *JohnCalls* independent of *MaryCalls* given *Alarm* ($j \perp m \mid a$)?



Conditional independence

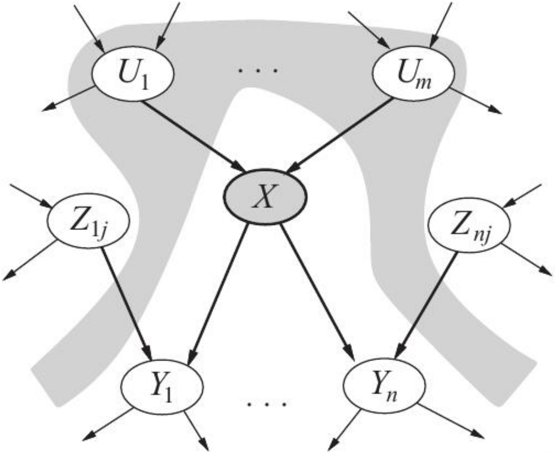
- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$
- $x \perp y \mid z$
  - x and y are independent given z
  - $P(x \wedge y \mid z) = P(x \mid z) \cdot P(y \mid z)$
- Is *JohnCalls* independent of *MaryCalls* ($j \perp m$)?
  - No — value of j tells us something about value of m and vice versa
- Is *JohnCalls* independent of *MaryCalls* given *Alarm* ($j \perp m \mid a$)?
  - Yes — by semantics of network, local independence



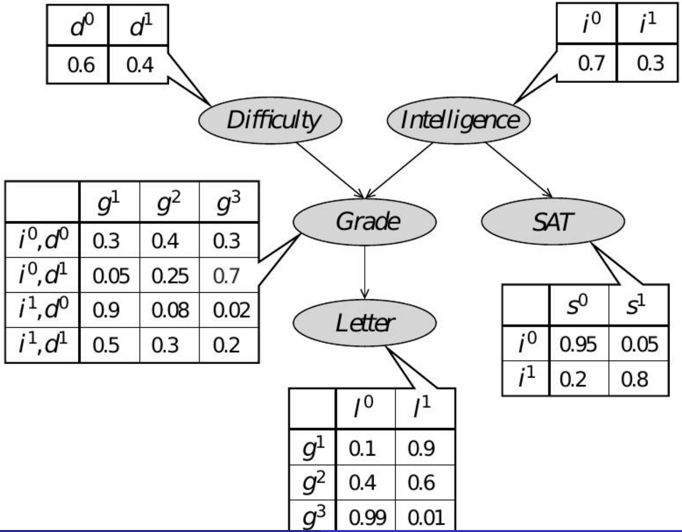
Probabilistic graphical models

- Fundamental assumption

  A node is conditionally independent of non-descendants, given its parents

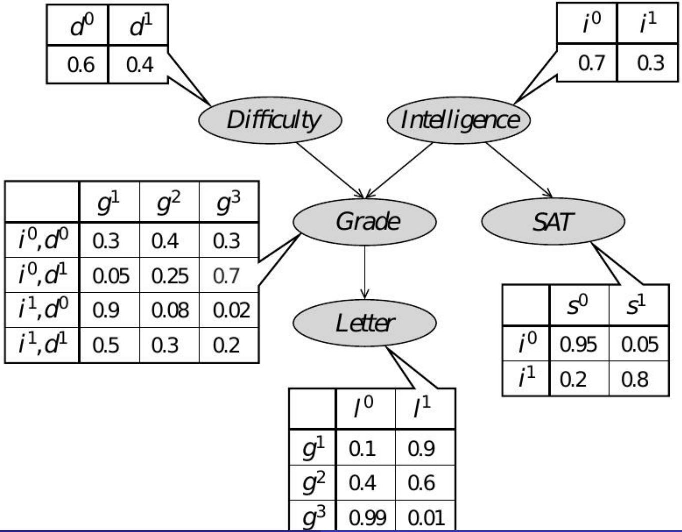


- $SAT \perp Grade \mid Difficulty$?



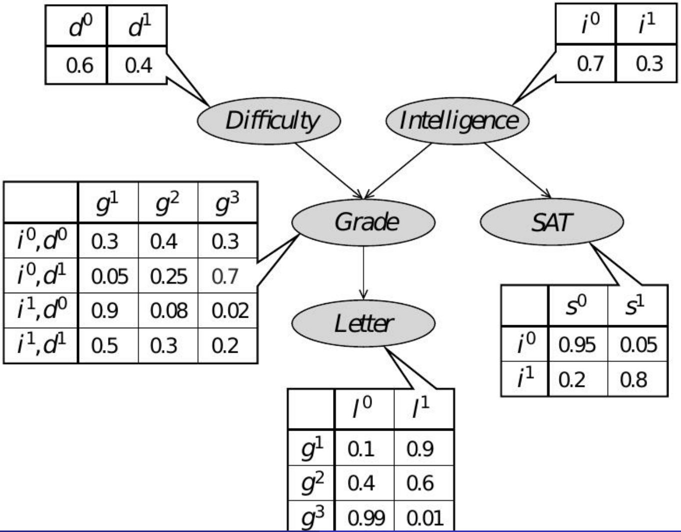
Student example

- *SAT* $\perp$ *Grade* | *Difficulty* ?
  - No

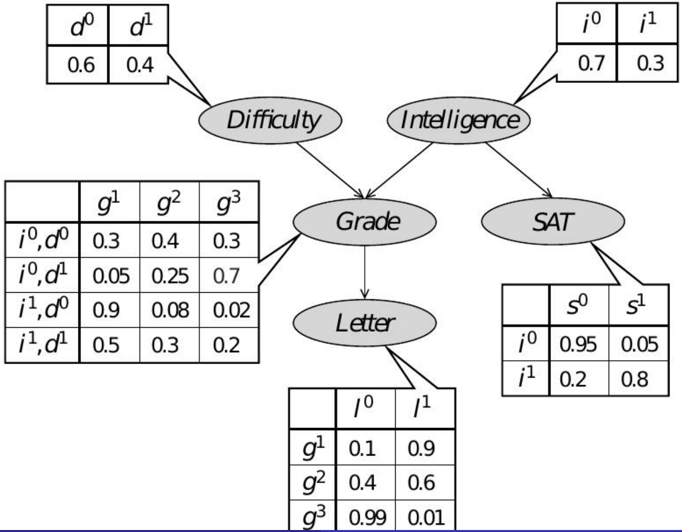


Student example

- *SAT* $\perp$ *Grade* | *Difficulty* ?
  - No
- Can we calculate conditional independence from the graph?



- $SAT \perp Grade \mid Difficulty$?
  - No
- Can we calculate conditional independence from the graph?
- In general, check if $X \perp Y \mid Z$ for sets of variables X, Y, Z



Conditional independence

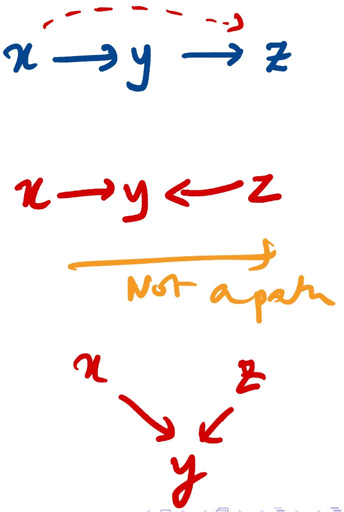
- How does dependence “flow” through a network?

Conditional independence

- How does dependence "flow" through a network?
- For neighbouring nodes, dependence flows both ways
  - If $x \to y$, knowing x tells us about y and vice versa

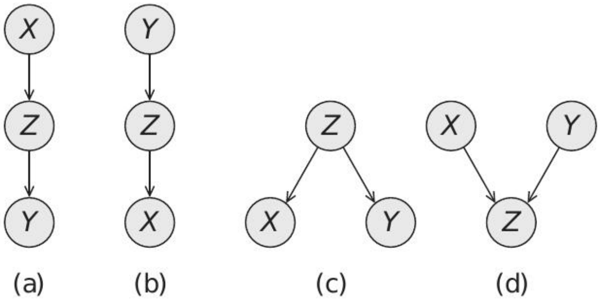
Conditional independence

- How does dependence "flow" through a network?
- For neighbouring nodes, dependence flows both ways
  - If $x \to y$, knowing x tells us about y and vice versa
- Examine trails between nodes
  - Paths in the underlying undirected graph

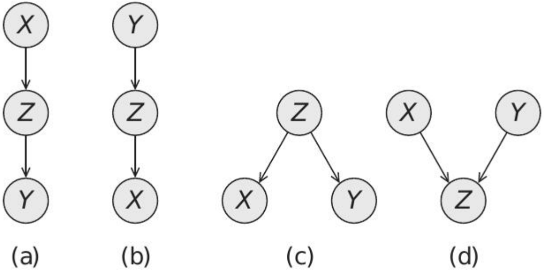


Conditional independence

- How does dependence "flow" through a network?
- For neighbouring nodes, dependence flows both ways
  - If $x \rightarrow y$, knowing x tells us about y and vice versa
- Examine trails between nodes
  - Paths in the underlying undirected graph
- Basic trails — (undirected) paths of length 2
  - Four basic trails

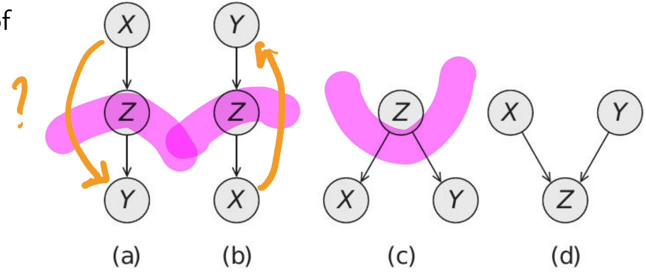


Basic trails



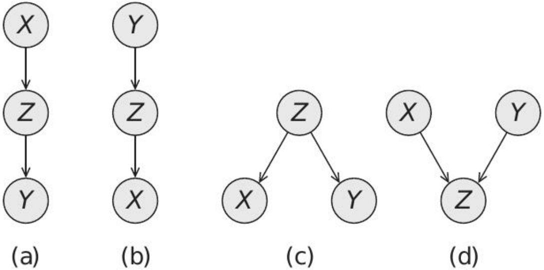
Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks



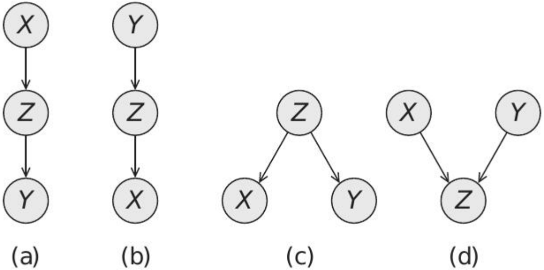
Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks
- In (d), knowing Z allows influence to flow



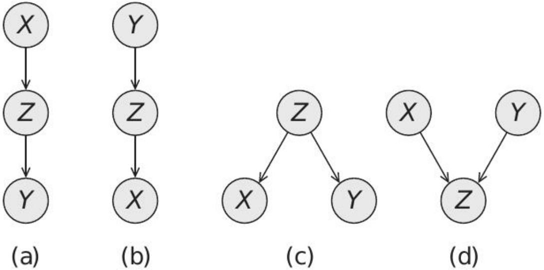
Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks
- In (d), knowing Z allows influence to flow
  - Z: Car does not start
    X: Low Battery, Y: No Fuel



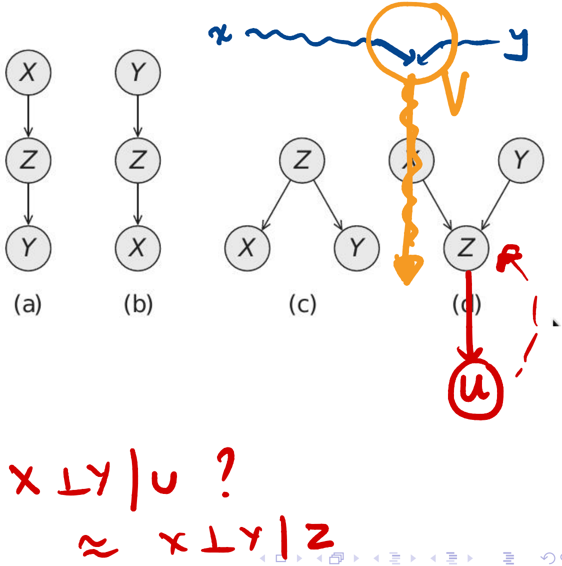
Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks
- In (d), knowing Z allows influence to flow
  - Z: Car does not start
    X: Low Battery, Y: No Fuel
  - Z: Grass is wet
    X: Overnight rain, Y: Sprinkler ran



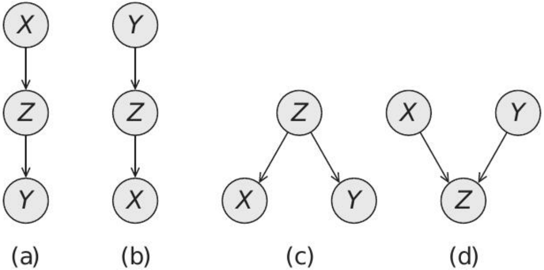
Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks
- In (d), knowing Z allows influence to flow
  - Z: Car does not start
    X: Low Battery, Y: No Fuel
  - Z: Grass is wet
    X: Overnight rain, Y: Sprinkler ran
  - Simplest form of V-structure



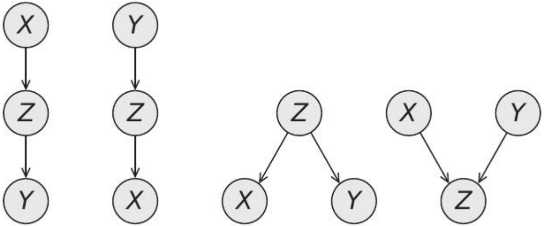
D-Separation

- Check if $X \perp Y \mid Z$

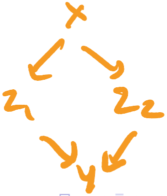


D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y

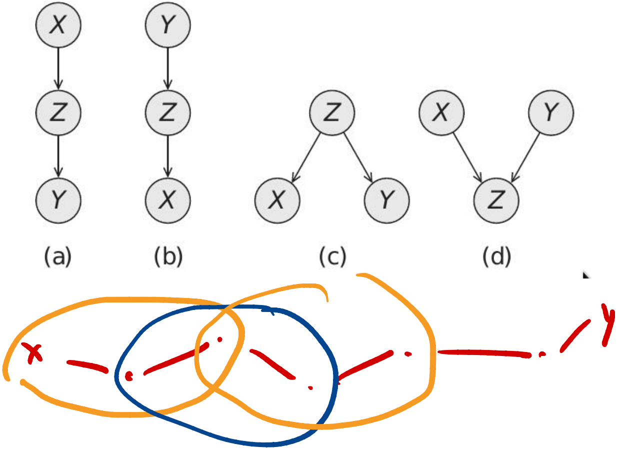


(a) (b) (c) (d)



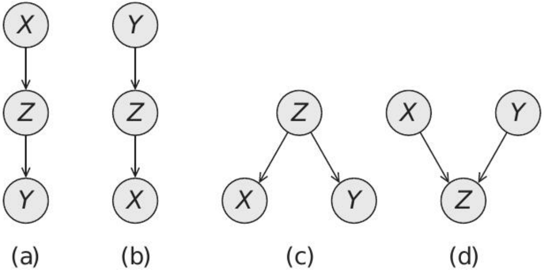
D-Separation

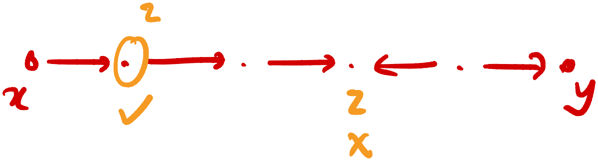
- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
  - Each undirected path from X to Y is a sequence of basic trails



D-Separation

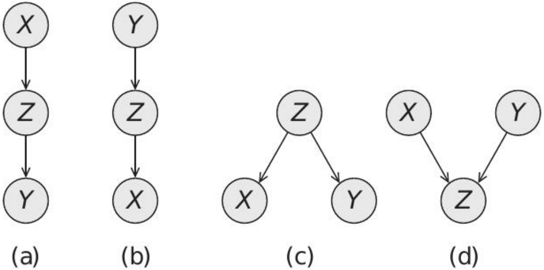
- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
  - Each undirected path from X to Y is a sequence of basic trails
  - For (a), (b), (c), need Z present





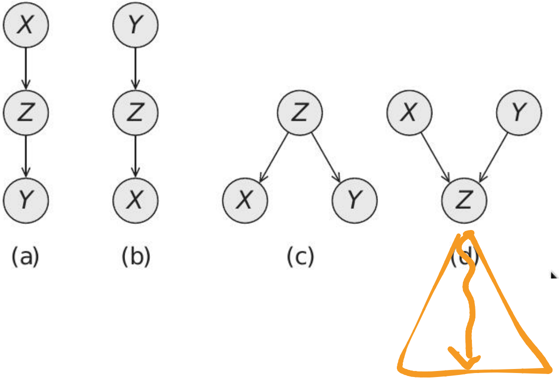
D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
    - Each undirected path from X to Y is a sequence of basic trails
    - For (a), (b), (c), need Z present
    - For (d), need Z absent



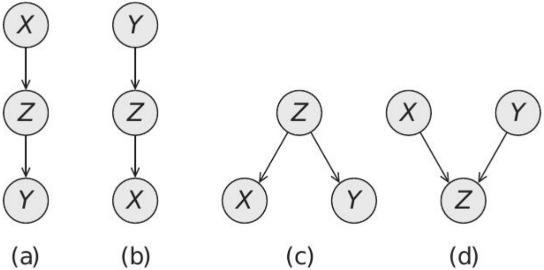
D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
  - Each undirected path from X to Y is a sequence of basic trails
  - For (a), (b), (c), need Z present
  - For (d), need Z absent
  - In general, V-structure includes descendants of the bottom node



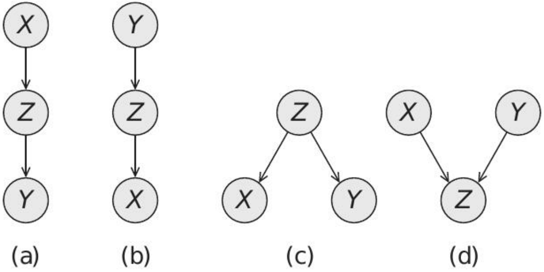
D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
  - Each undirected path from X to Y is a sequence of basic trails
  - For (a), (b), (c), need Z present
  - For (d), need Z absent
  - In general, V-structure includes descendants of the bottom node
- x and y are D-separated given z if all trails are blocked



D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
  - Each undirected path from X to Y is a sequence of basic trails
  - For (a), (b), (c), need Z present
  - For (d), need Z absent
  - In general, V-structure includes descendants of the bottom node



- x and y are D-separated given z if all trails are blocked
- Variation of breadth first search (BFS) to check if y is reachable from x through some trail

Breadth First Search

Reachability in a graph

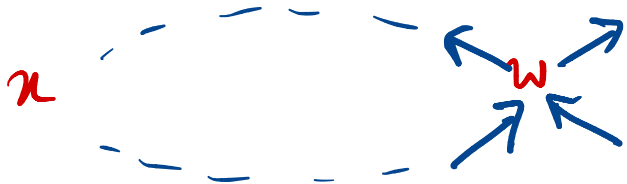
Is y reachable from x?

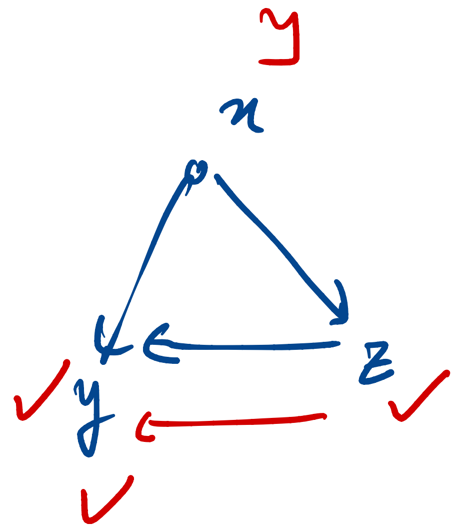
- Mark all neighbours of x as reachable
- Mark nbrs of nbrs

⋮

No new marks

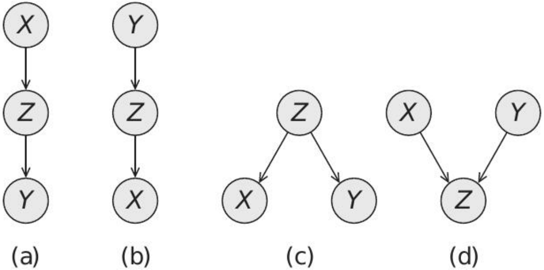
Is y marked?


x
w


y
x
y
z

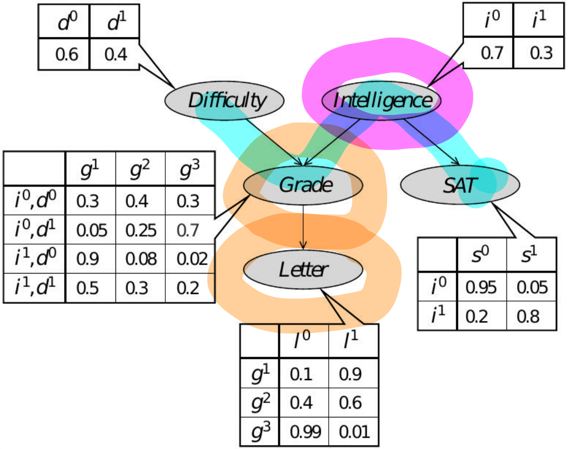
D-Separation

- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y
  - Each undirected path from X to Y is a sequence of basic trails
  - For (a), (b), (c), need Z present
  - For (d), need Z absent
  - In general, V-structure includes descendants of the bottom node

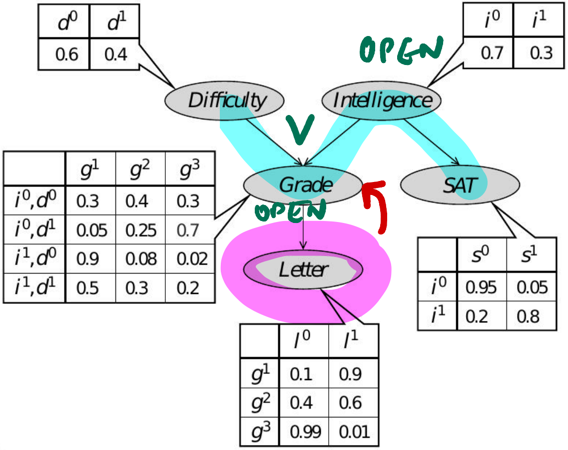


- x and y are D-separated given z if all trails are blocked
- Variation of breadth first search (BFS) to check if y is reachable from x through some trail
- Extends to sets — each $x \in X$ is D-separated from each $y \in Y$

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence



- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence
- Is SAT independent of Difficulty given Letter?
  - No, Difficulty – Grade – Intelligence – SAT trail is open
  - Letter is known, hence something about Grade is known (V-structure)
  - Intelligence is not known



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01