Lecture 21: 18 April, 2022

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Data Mining and Machine Learning
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Conditional probabilities

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- Can we strive for something in between?
  - "Local" dependencies between some variables

Probabilistic graphical models

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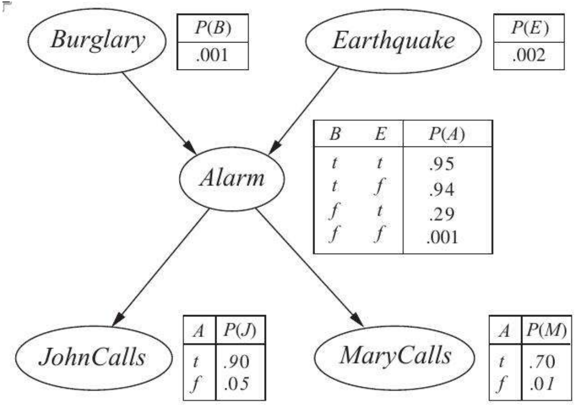
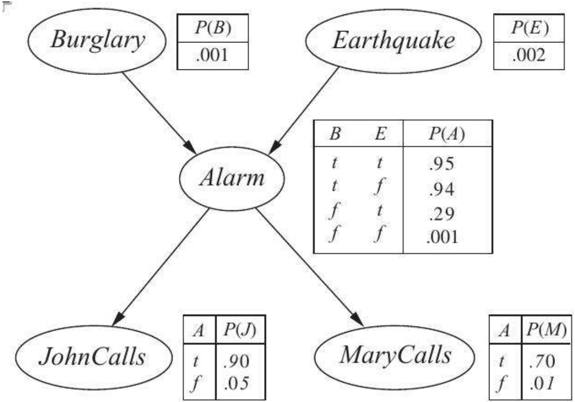
- Example: Burglar alarm
  - Pearl's house has a burglar alarm
  - Neighbours John and Mary call if they hear the alarm
  - John is prone to mistaking ambulances etc for the alarm
  - Mary listens to loud music and sometimes fails to hear the alarm
  - The alarm may also be triggered by an earthquake (California!)



- Each node has a local (conditional) probability table



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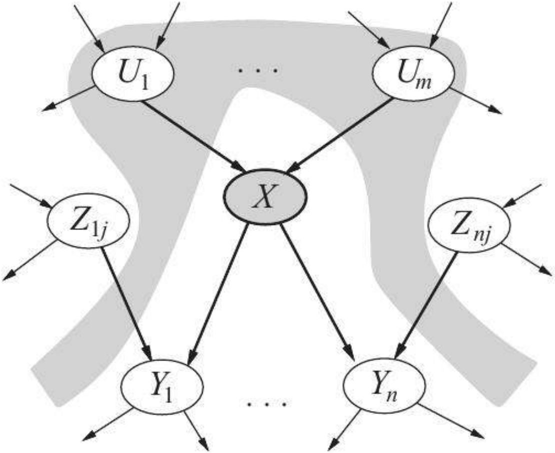
- Fundamental assumption:
  A node is conditionally independent of non-descendants, given its parents



Probabilistic graphical models

- Each node has a local (conditional) probability table
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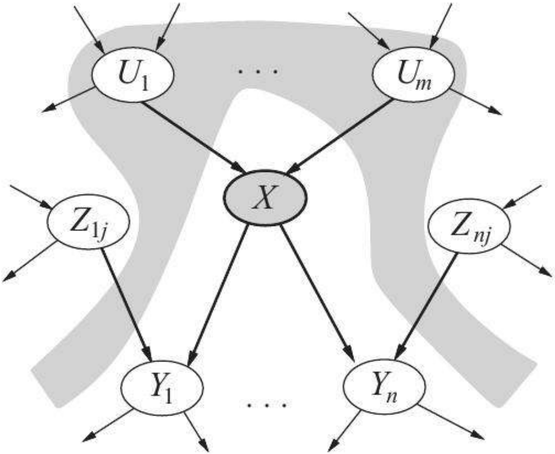
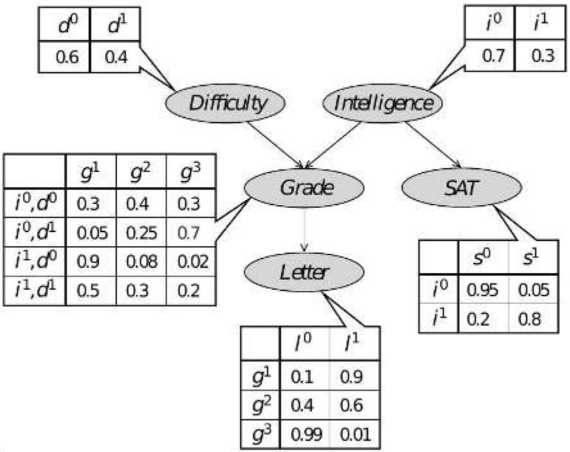
- Graph is a DAG, no cyclic dependencies



Student example

- Example due to Nir Friedman and Daphne Koller
- Student asks teacher for a reference letter
- Teacher has forgotten the student, so letter is entirely based on student's grade in the course



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

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- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2, x_3, \ldots, x_n)$

A B

$P(x_2 \mid x_3 - x_n) \cdot P(x_3 - x_n)$

$P(x_3 \mid x_4 - x_n) P(x_4 - \ldots$

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- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2, x_3, \ldots, x_n)$
- Applied recursively, this gives us the chain rule
  $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n)P(x_n)$

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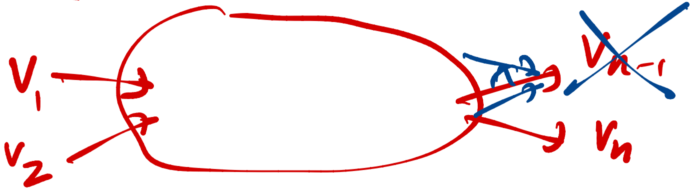
- Use topological ordering in a Bayesian network

DAG must have a node with no incoming (outgoing) edges

Suppose not



V_{n-1}

Evaluating a network

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- $P(m, j, a, b, e) =$
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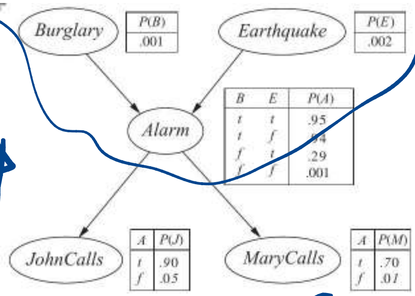

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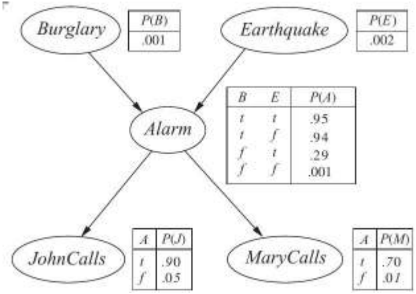

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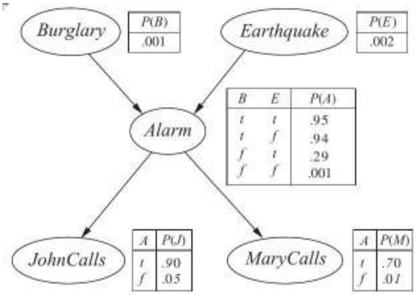
- $P(m, j, b) =$
  $$\sum_{a=0}^{1}\sum_{e=0}^{1} P(m \mid a)P(j \mid a)P(a \mid b, e)P(b)P(e)$$



$P(b)(t_1 + t_2 + t_3 + t_4)$

$t_1 P(b) + t_2 P(b) \ldots + t_4 P(b)$

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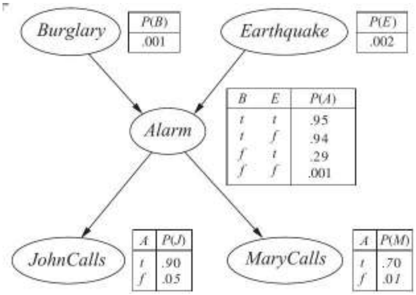
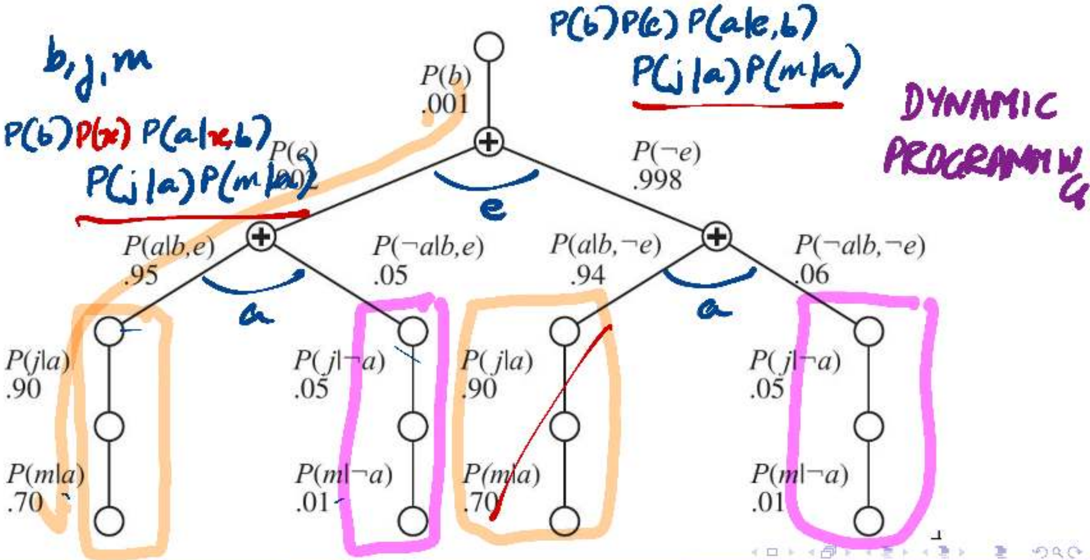
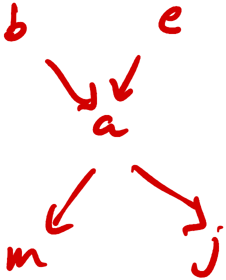


Evaluation tree



Designing the Bayesian network

- Need to choose node ordering wisely to get a compact Bayesian network



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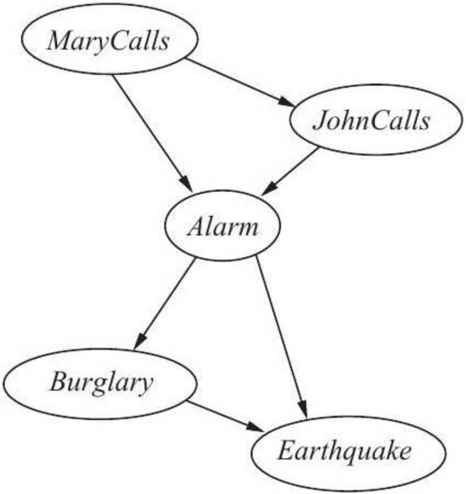

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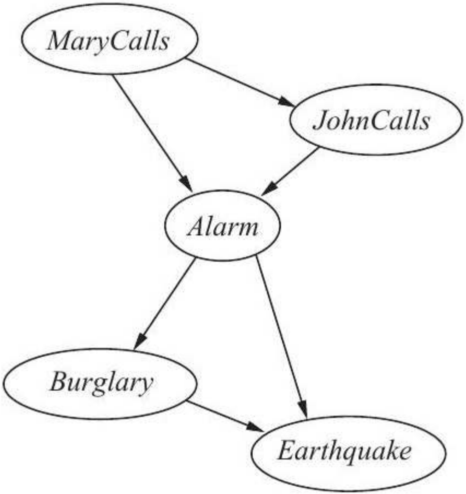

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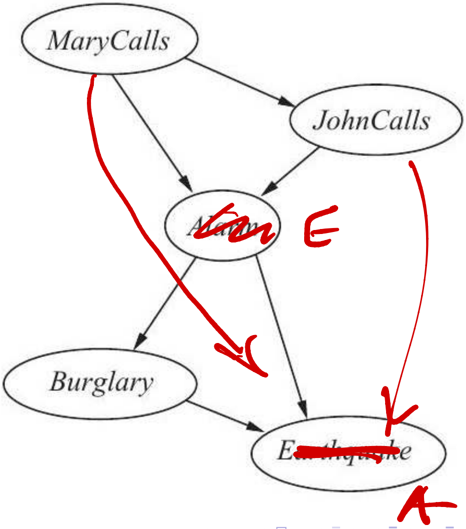
- Causal model (causes to effects) works better than diagnostic model (effects to causes)



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$x_3 \vee \neg x_7 \vee x_9 \vee x_{11}$

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  - A CNF formula is a conjunction of clauses, $C_1 \wedge C_2 \wedge \cdots \vee C_m$

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_3 \vee \neg x_1)$$

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$x_1 \vee \neg x_4 \vee x_9$

3 literals

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NP-Complete problem $\xrightarrow{\text{transform}}$ My Problem

Solution $\xleftarrow{\text{implies}}$ Solution

$f \xrightarrow{\text{quick}}$ Network N_f

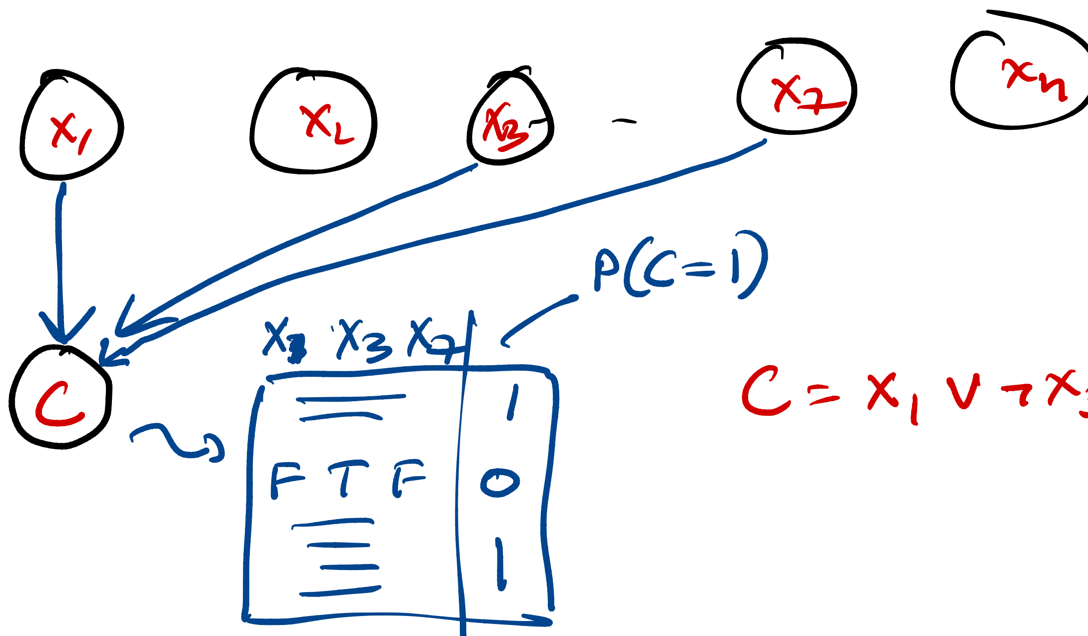
X $\Downarrow$ quick! $\qquad\qquad$ $\downarrow$ quick X

Solution for f $\xleftarrow{\text{quick}}$ solve N_f

Transforming 3-CNF to Bayesian network inference

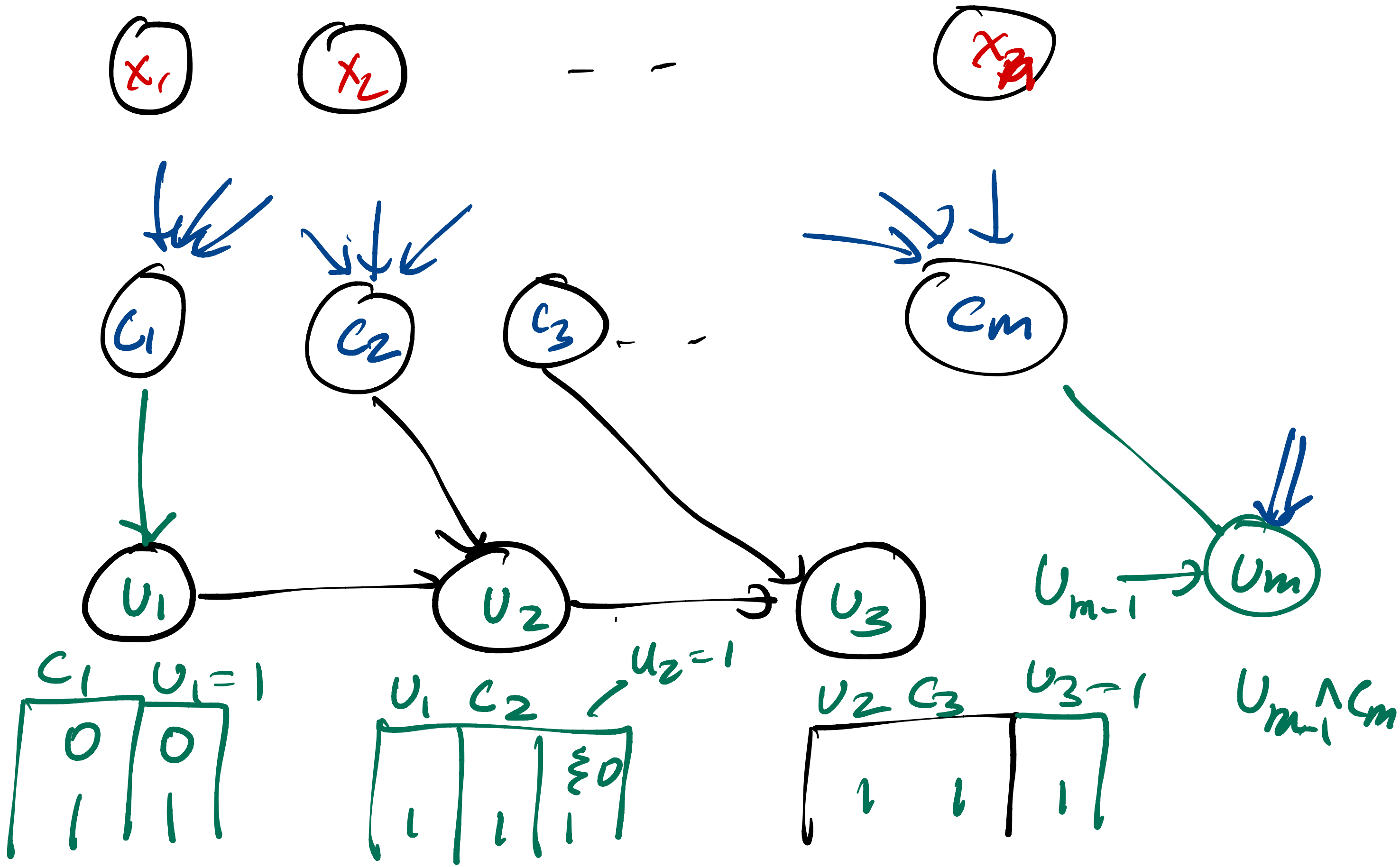
Variables $x_1, x_2, \ldots, x_n$

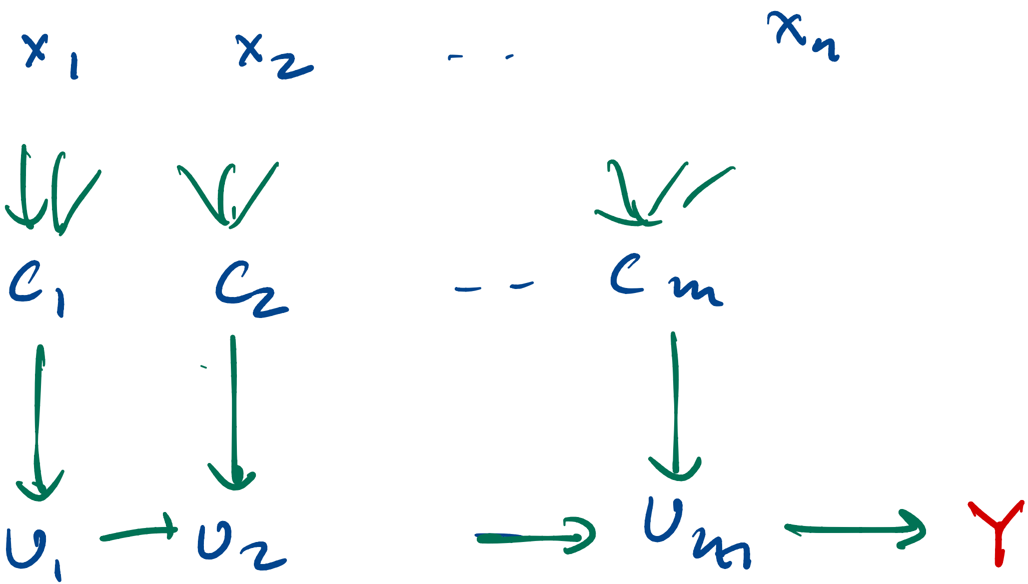
Clause $l_1 \vee l_2 \vee l_3$



$$C = x_1 \vee \neg x_3 \vee x_7$$

$C_1 \wedge C_2 \ldots \wedge C_m$





$Y = 1$ if $u_m = 1$

$P(Y=1) > 0$?