Neural Networks

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Data Mining and Machine Learning
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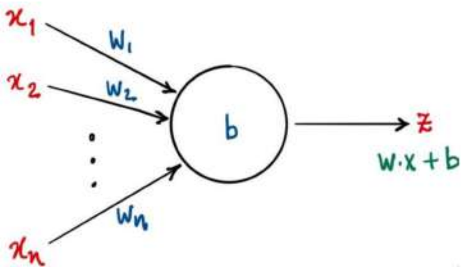
Linear separators and Perceptrons

- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)



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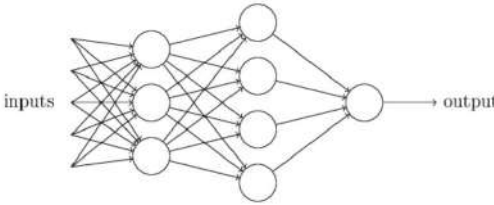

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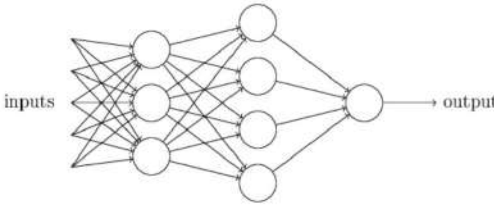

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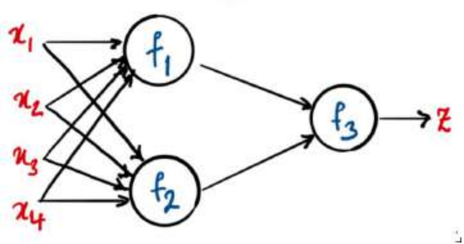

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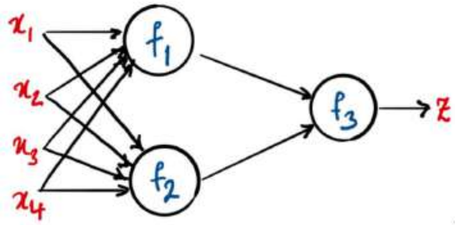

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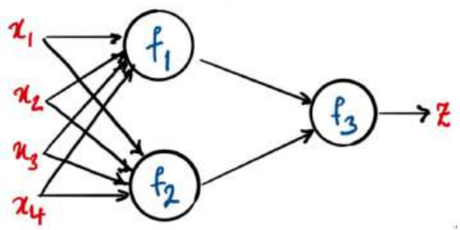

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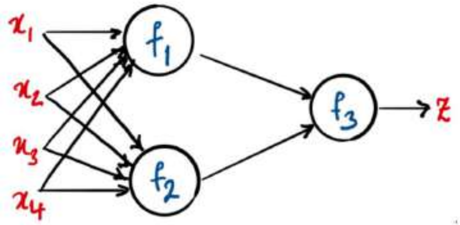
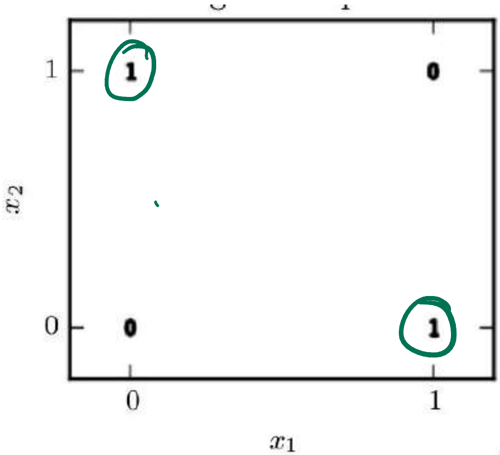
  - $f_3 = \sum_{i=1}^{4} (w_{3_1} w_{1_i} + w_{3_2} w_{2_i}) \cdot x_i$
    $+(w_{3_1} b_1 + w_{3_2} b_2 + b_3)$



- Cannot compute *exclusive-or* (XOR)
- $XOR(x_1, x_2)$ is true if exactly one of x_1, x_2 is true (not both)



Limits of linearity

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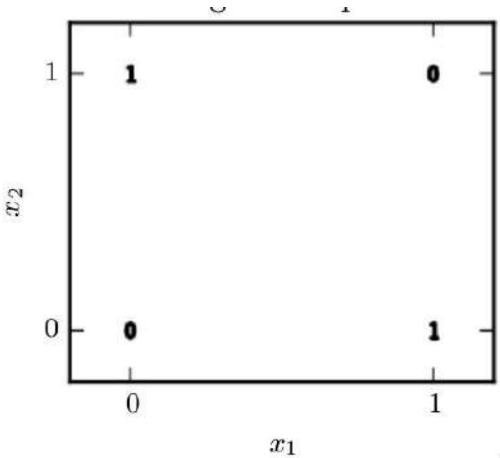

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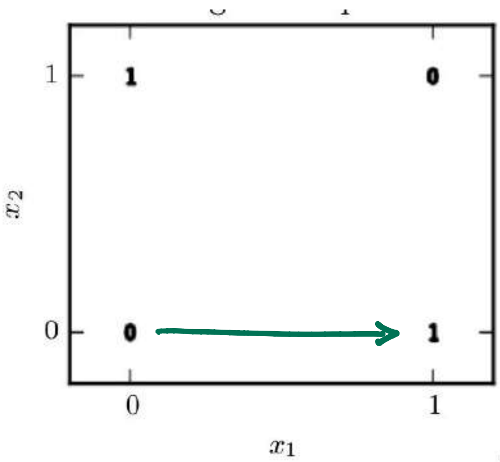

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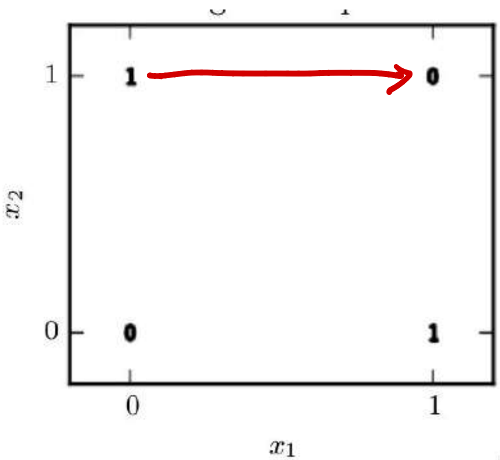

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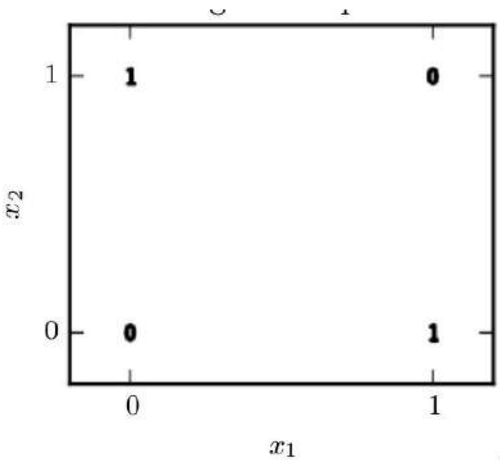
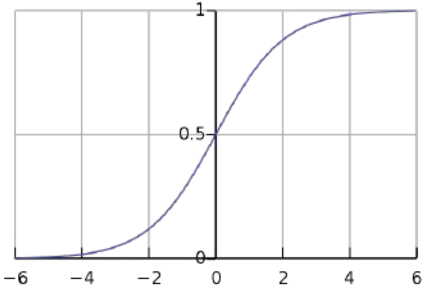
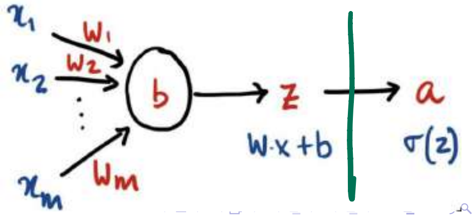
- Observed by Minsky and Papert, 1969, first "AI Winter"



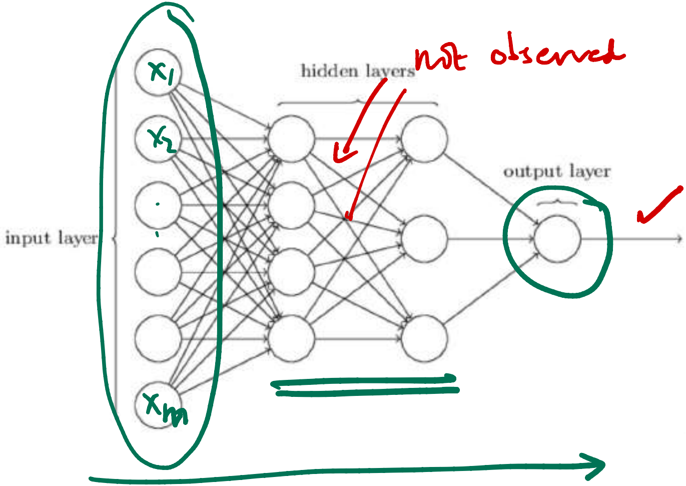
- Transform linear output z through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$





Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer



Structure of a neural network

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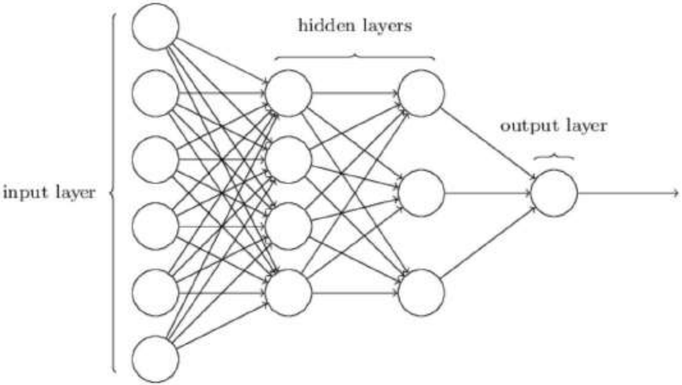

Structure of a neural network

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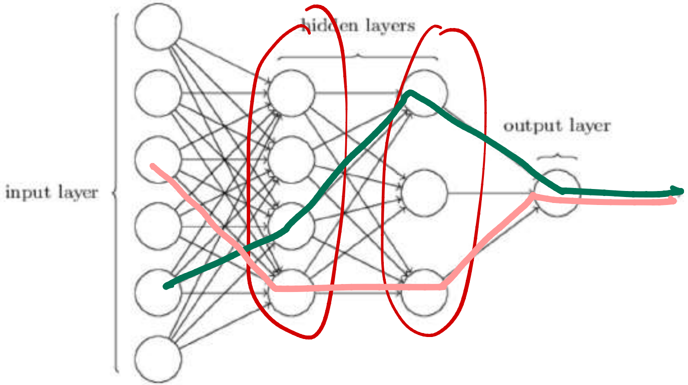

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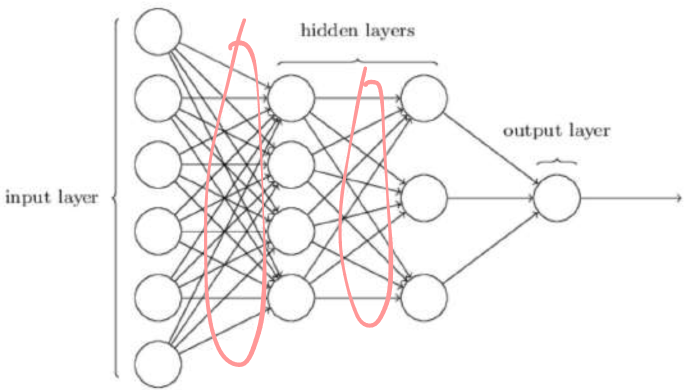
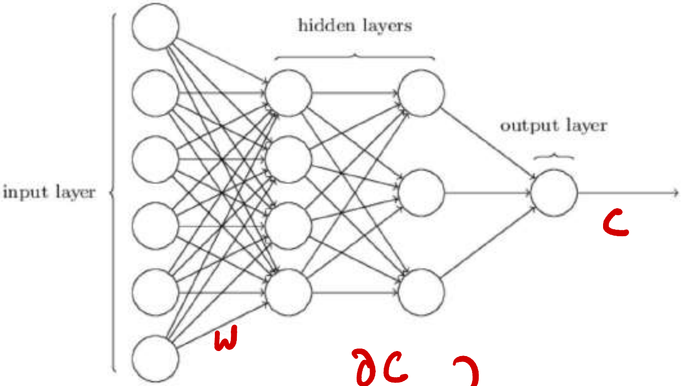


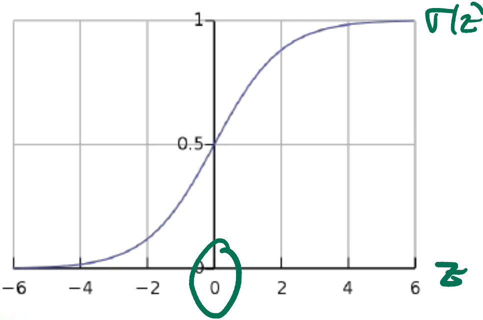
- Acyclic
- Input layer, hidden layers, output layer
- Assumptions
  - Hidden neurons are arranged in layers
  - Each layer is fully connected to the next
  - Set weight to zero to remove an edge

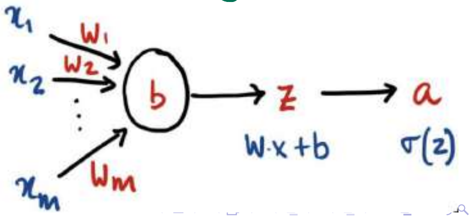


$\frac{\partial C}{\partial w}$?

Non-linear activation

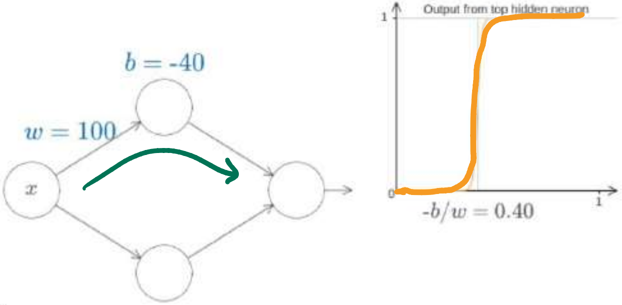
- Transform linear output z through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$ $\dfrac{1}{1+1} = \dfrac{1}{2}$
- Step is at $z = 0$
  - $z = wx + b$, so step is at $x = \cancel{-w/b}$ $-\dfrac{b}{w}$
  - Increasing w makes step steeper





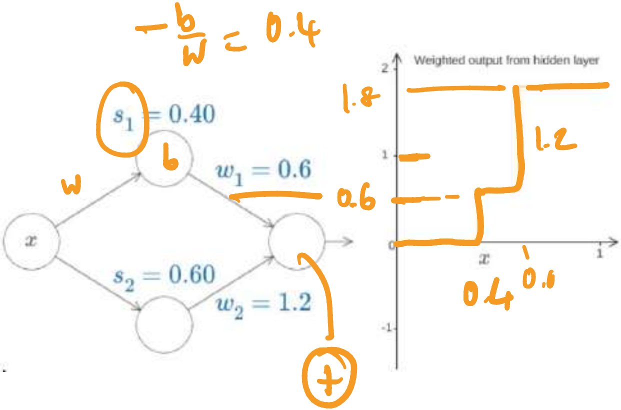
Universality

- Create a step at $x = -\cancel{w/b}$ b/w



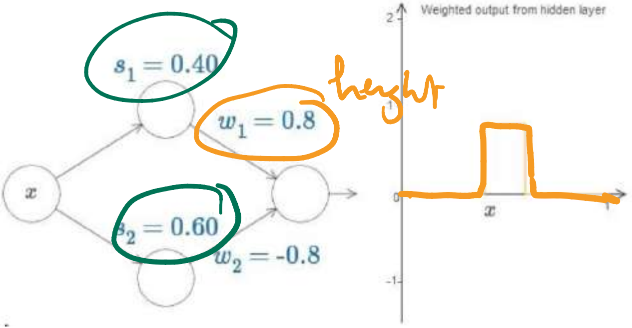
Universality

- Create a step at $x = -w/b$
- Cascade steps



Universality

- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box



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- Create a step at $x = -w/b$
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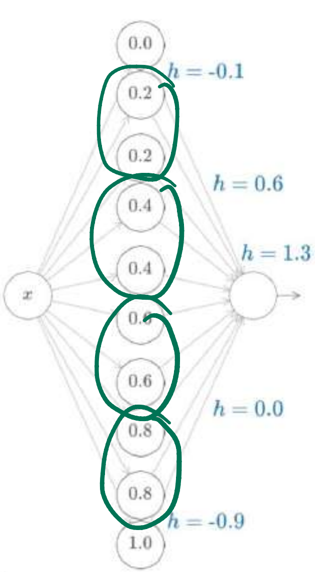
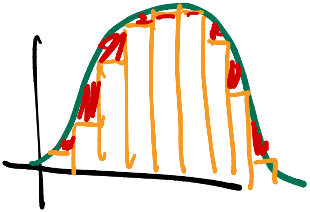
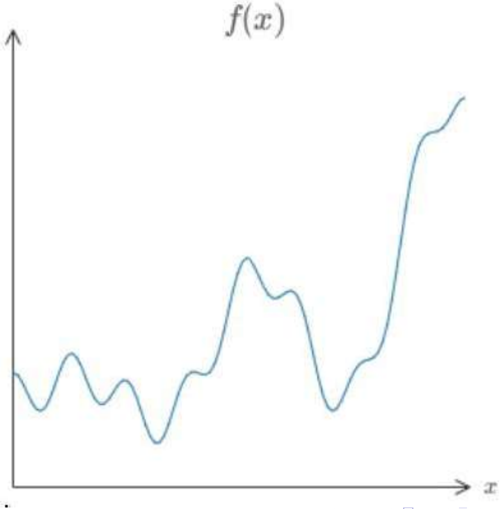
- Create many boxes





Universality

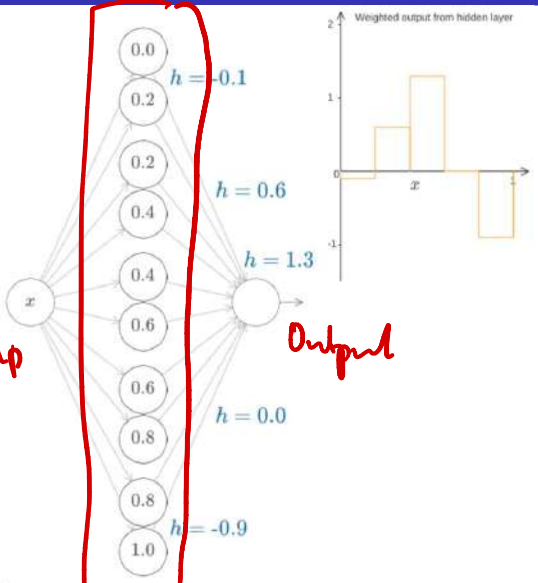
- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box
- Create many boxes
- Approximate any function



Universality

- Create a step at $x = -w/b$
- Cascade steps
- Subtract steps to create a box
- Create many boxes
- Approximate any function
- Need only one hidden layer!

Inp



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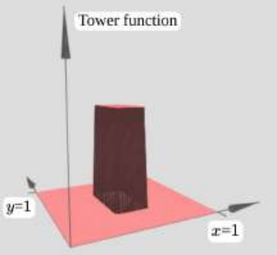
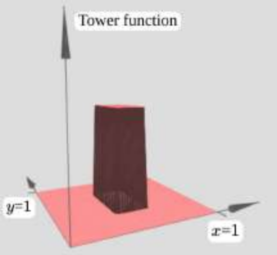
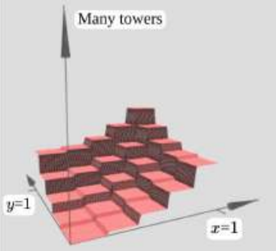


Non-linear activation

- With non-linear activation, network of neurons can approximate any function
  - Can build “rectangular” blocks
  - Combine blocks to capture any classification boundary





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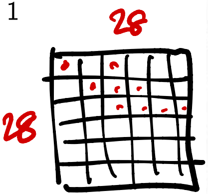

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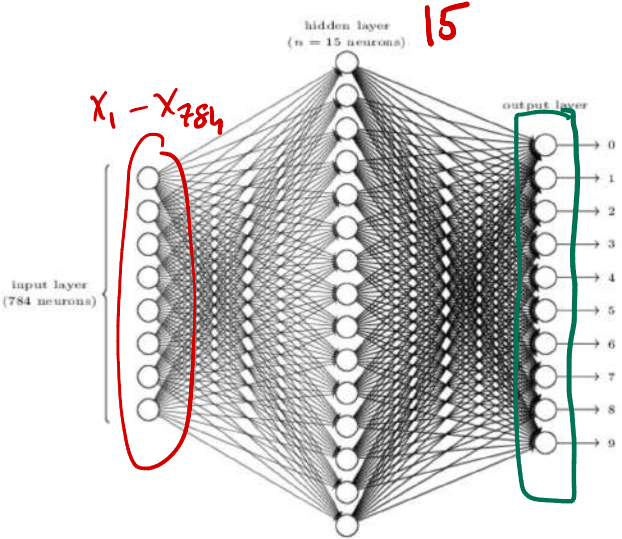
- Input $x = (x_1, x_2, \ldots, x_{784})$

0-1 0-1

Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$



Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$
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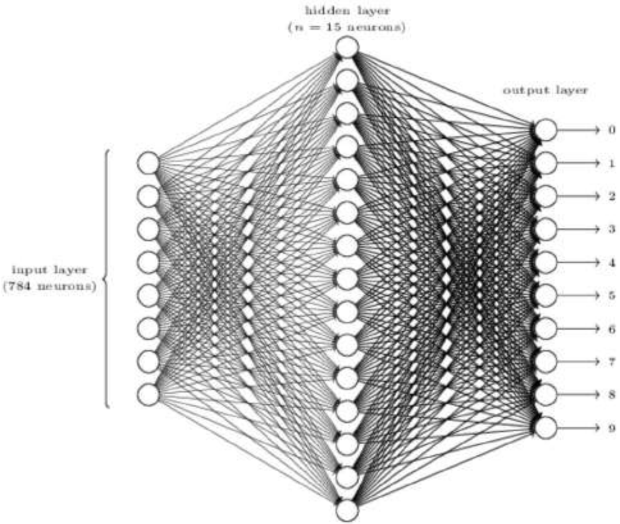

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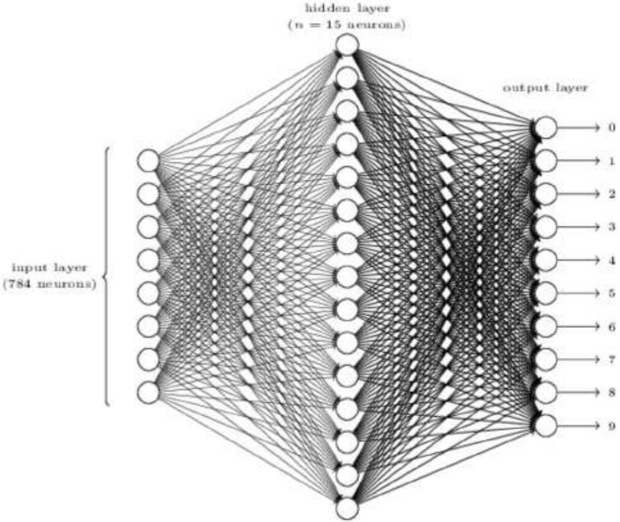

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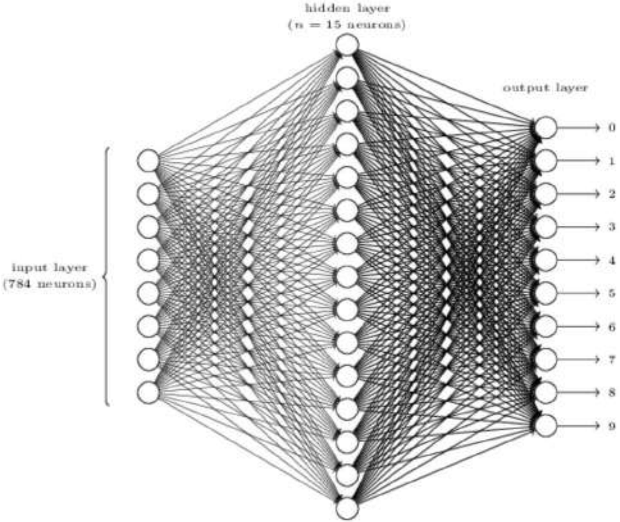

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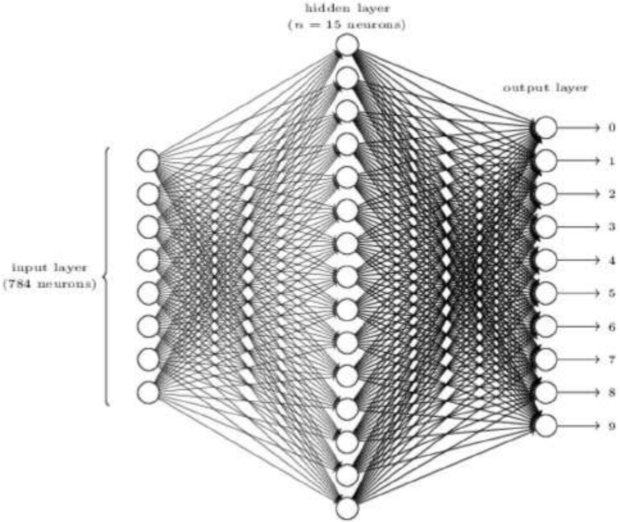

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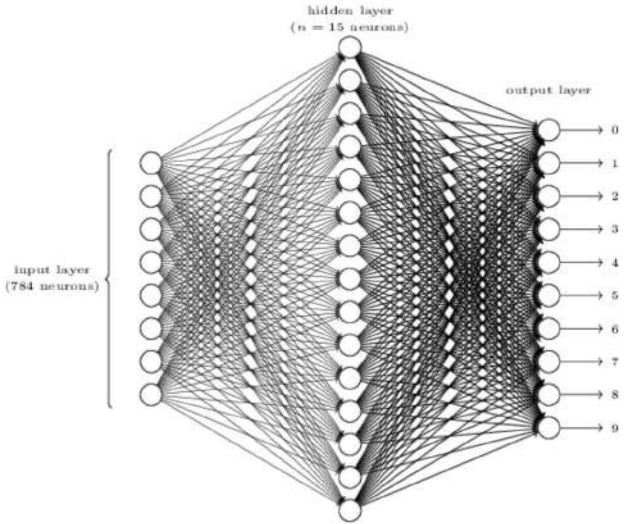
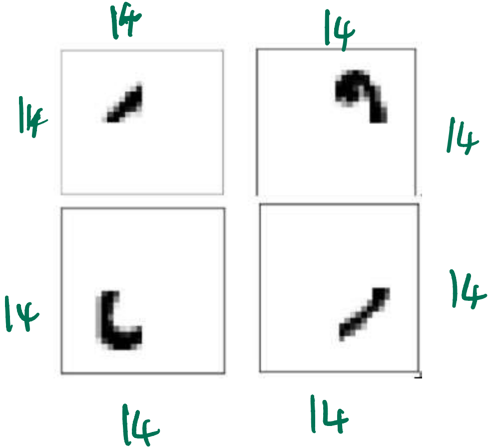
  - Softmax, $\arg\max_j \dfrac{e^{a_j}}{\sum_j e^{a_j}}$
    - "Smooth" version of $\arg\max$



Example: Extracting features

- Hidden layers extract features
  - For instance, patterns in different quadrants



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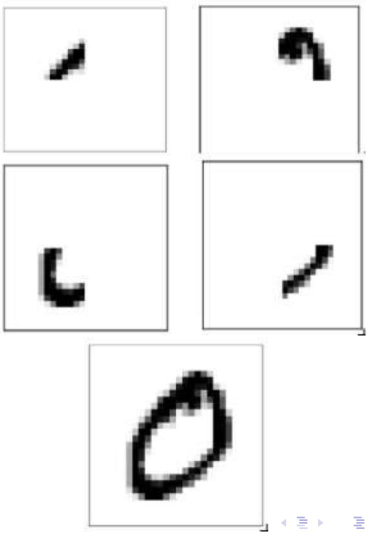

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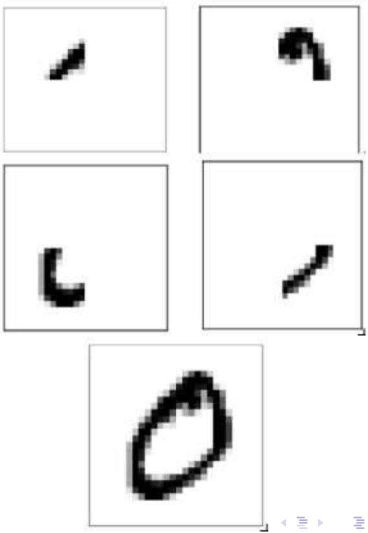

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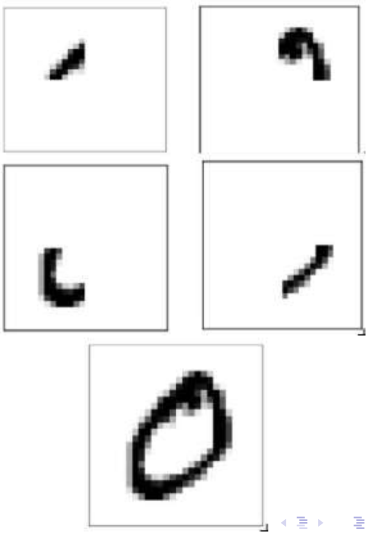
  - Explainability



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