Domain Theory

Corrected and expanded version

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This text is based on the chapter *Domain Theory* in the **Handbook for Logic in Computer Science**, volume 3, edited by S. Abramsky, Dov M. Gabbay, and T. S. E. Maibaum, published by Clarendon Press, Oxford in 1994. While the numbering of all theorems and definitions has been kept the same, we have included comments and corrections which we have received over the years. For ease of reading, small typographical errors have simply been corrected. Where we felt the original text gave a misleading impression, we have included additional explanations, clearly marked as such.

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We will be grateful to receive further comments or suggestions. Please send them to `A.Jung@cs.bham.ac.uk`

So far, we have received comments and/or corrections from Francesco Consentino, Joseph D. Darcy, Mohamed El-Zawawy, Weng Kin Ho, Klaus Keimel, Olaf Klinke, Xuhui Li, Homeira Pajoohesh, Dieter Spreen, and Dominic van der Zypen.

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Contents

1 Introduction and Overview

1.1 Origins

Let us begin with the problems which gave rise to Domain Theory:

1. **Least fixpoints as meanings of recursive definitions.** Recursive definitions of procedures, data structures and other computational entities abound in programming languages. Indeed, recursion is the basic effective mechanism for describing infinite computational behaviour in finite terms. Given a recursive definition:

$$X = \ldots X \ldots \tag{1}$$

   How can we give a non-circular account of its meaning? Suppose we are working inside some mathematical structure D. We want to find an element $d \in D$ such that substituting d for x in (1) yields a valid equation. The right-hand-side of (1) can be read as a function of X, semantically as $f\colon D \to D$. We can now see that we are asking for an element $d \in D$ such that $d = f(d)$—that is, for a *fixpoint* of f. Moreover, we want a *uniform canonical* method for constructing such fixpoints for arbitrary structures D and functions $f\colon D \to D$ within our framework. Elementary considerations show that the usual categories of mathematical structures either fail to meet this requirement at all (sets, topological spaces) or meet it in a trivial fashion (groups, vector spaces).

2. **Recursive domain equations.** Apart from recursive definitions of computational objects, programming languages also abound, explicitly or implicitly, in recursive definitions of *datatypes*. The classical example is the type-free λ-calculus [Bar84]. To give a mathematical semantics for the λ-calculus is to find a mathematical structure D such that terms of the λ-calculus can be interpreted as elements of D in such a way that application in the calculus is interpreted by function application. Now consider the self-application term $\lambda x.xx$. By the usual condition for type-compatibility of a function with its argument, we see that if the second occurrence of x in xx has type D, and the whole term xx has type D, then the first occurrence must have, or be construable as having, type $[D \longrightarrow D]$. Thus we are led to the requirement that we have

$$[D \longrightarrow D] \cong D.$$

   If we view $[. \longrightarrow .]$ as a *functor* $F\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$ over a suitable category $\mathbf{C}$ of mathematical structures, then we are looking for a fixpoint $D \cong F(D, D)$. Thus recursive datatypes again lead to a requirement for fixpoints, but now lifted to the functorial level. Again we want such fixpoints to exist uniformly and canonically.

This second requirement is even further beyond the realms of ordinary mathematical experience than the first. Collectively, they call for a novel mathematical theory to serve as a foundation for the semantics of programming languages.

A first step towards Domain Theory is the familiar result that every monotone function on a complete lattice, or more generally on a directed-complete partial order with least element, has a least fixpoint. (For an account of the history of this result, see [LNS82].) Some early uses of this result in the context of formal language theory were [Ard60, GR62]. It had also found applications in recursion theory [Kle52, Pla64]. Its application to the semantics of first-order recursion equations and flowcharts was already well-established among Computer Scientists by the end of the 1960's [dBS69, Bek69, Bek71, Par69]. But Domain Theory proper, at least as we understand the term, began in 1969, and was unambiguously the creation of one man, Dana Scott [Sco69, Sco70, Sco71, Sco72, Sco93]. In particular, the following key insights can be identified in his work:

1. **Domains as types.** The fact that suitable categories of domains are *cartesian closed*, and hence give rise to models of typed λ-calculi. More generally, that domains give mathematical meaning to a broad class of data-structuring mechanisms.

2. **Recursive types.** Scott's key construction was a solution to the "domain equation"
$$D \cong [D \longrightarrow D]$$
thus giving the first mathematical model of the type-free λ-calculus. This led to a general theory of solutions of recursive domain equations. In conjunction with (1), this showed that domains form a suitable universe for the semantics of programming languages. In this way, Scott provided a mathematical foundation for the work of Christopher Strachey on denotational semantics [MS76, Sto77]. This combination of descriptive richness and a powerful and elegant mathematical theory led to denotational semantics becoming a dominant paradigm in Theoretical Computer Science.

3. **Continuity vs. Computability.** *Continuity* is a central pillar of Domain theory. It serves as a qualitative approximation to computability. In other words, for most purposes to detect whether some construction is computationally feasible it is sufficient to check that it is continuous; while continuity is an "algebraic" condition, which is much easier to handle than computability. In order to give this idea of continuity as a smoothed-out version of computability substance, it is not sufficient to work only with a notion of "completeness" or "convergence"; one also needs a notion of *approximation*, which does justice to the idea that infinite objects are given in some coherent way as limits of their finite approximations. This leads to considering, not arbitrary complete partial orders, but the *continuous* ones. Indeed, Scott's early work on Domain Theory was seminal to the subsequent extensive development of the theory of continuous lattices, which also drew heavily on ideas from topology, analysis, topological algebra and category theory [GHK$^+$80].

4. **Partial information.** A natural concomitant of the notion of approximation in domains is that they form the basis of a theory of partial information, which extends the familiar notion of partial function to encompass a whole spectrum of

"degrees of definedness". This has important applications to the semantics of programming languages, where such multiple degrees of definition play a key role in the analysis of computational notions such as lazy vs. eager evaluation, and call-by-name vs. call-by-value parameter-passing mechanisms for procedures.

General considerations from recursion theory dictate that partial functions are unavoidable in any discussion of computability. Domain Theory provides an appropriately abstract, structural setting in which these notions can be lifted to higher types, recursive types, etc.

1.2 Our approach

It is a striking fact that, although Domain Theory has been around for a quarter-century, no book-length treatment of it has yet been published. Quite a number of books on semantics of programming languages, incorporating substantial introductions to domain theory as a necessary tool for denotational semantics, have appeared [Sto77, Sch86, Gun92b, Win93]; but there has been no text devoted to the underlying mathematical theory of domains. To make an analogy, it is as if many Calculus textbooks were available, offering presentations of some basic analysis interleaved with its applications in modelling physical and geometrical problems; but no textbook of Real Analysis. Although this Handbook Chapter cannot offer the comprehensive coverage of a full-length textbook, it is nevertheless written in the spirit of a presentation of Real Analysis. That is, we attempt to give a crisp, efficient presentation of the mathematical theory of domains without excursions into applications. We hope that such an account will be found useful by readers wishing to acquire some familiarity with Domain Theory, including those who seek to apply it. Indeed, we believe that the chances for exciting new applications of Domain Theory will be enhanced if more people become aware of the full richness of the mathematical theory.

1.3 Overview

Domains individually

We begin by developing the basic mathematical language of Domain Theory, and then present the central pillars of the theory: convergence and approximation. We put considerable emphasis on bases of continuous domains, and show how the theory can be developed in terms of these. We also give a first presentation of the topological view of Domain Theory, which will be a recurring theme.

Domains collectively

We study special classes of maps which play a key role in domain theory: retractions, adjunctions, embeddings and projections. We also look at construction on domains such as products, function spaces, sums and lifting; and at bilimits of directed systems of domains and embeddings.

Cartesian closed categories of domains

A particularly important requirement on categories of domains is that they should be cartesian closed (i.e. closed under function spaces). This creates a tension with the requirement for a good theory of approximation for domains, since neither the category **CONT** of all continuous domains, nor the category **ALG** of all algebraic domains is cartesian closed. This leads to a non-trivial analysis of necessary and sufficient conditions on domains to ensure closure under function spaces, and striking results on the classification of the maximal cartesian closed full subcategories of **CONT** and **ALG**. This material is based on [Jun89, Jun90].

Recursive domain equations

The theory of recursive domain equations is presented. Although this material formed the very starting point of Domain Theory, a full clarification of just what canonicity of solutions means, and how it can be translated into proof principles for reasoning about these canonical solutions, has only emerged over the past two or three years, through the work of Peter Freyd and Andrew Pitts [Fre91, Fre92, Pit93b]. We make extensive use of their insights in our presentation.

Equational theories

We present a general theory of the construction of free algebras for inequational theories over continuous domains. These results, and the underlying constructions in terms of bases, appear to be new. We then apply this general theory to powerdomains and give a comprehensive treatment of the Plotkin, Hoare and Smyth powerdomains. In addition to characterizing these as free algebras for certain inequational theories, we also prove representation theorems which characterize a powerdomain over D as a certain space of subsets of D; these results make considerable use of topological methods.

Domains and logic

We develop the logical point of view of Domain Theory, in which domains are characterized in terms of their observable properties, and functions in terms of their actions on these properties. The general framework for this is provided by Stone duality; we develop the rudiments of Stone duality in some generality, and then specialize it to domains. Finally, we present "Domain Theory in Logical Form" [Abr91b], in which a metalanguage of types and terms suitable for denotational semantics is extended with a language of properties, and presented axiomatically as a programming logic in such a way that the lattice of properties over each type is the Stone dual of the domain denoted by that type, and the prime filter of properties which can be proved to hold of a term correspond under Stone duality to the domain element denoted by that term. This yields a systematic way of moving back and forth between the logical and denotational descriptions of some computational situation, each determining the other up to isomorphism.

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Our major intellectual debts, inevitably, are to Dana Scott and Gordon Plotkin. The more we learn about Domain Theory, the more we appreciate the depth of their insights.

2 Domains individually

We will begin by introducing the basic language of Domain Theory. Most topics we deal with in this section are treated more thoroughly and at a more leisurely pace in [DP90].

2.1 Convergence

2.1.1 Posets and preorders

Definition 2.1.1. *A set P with a binary relation $\sqsubseteq$ is called a* partially ordered set *or* poset *if the following holds for all $x, y, z \in P$:*

1. $x \sqsubseteq x$ *(Reflexivity)*
2. $x \sqsubseteq y \wedge y \sqsubseteq z \Longrightarrow x \sqsubseteq z$ *(Transitivity)*
3. $x \sqsubseteq y \wedge y \sqsubseteq x \Longrightarrow x = y$ *(Antisymmetry)*

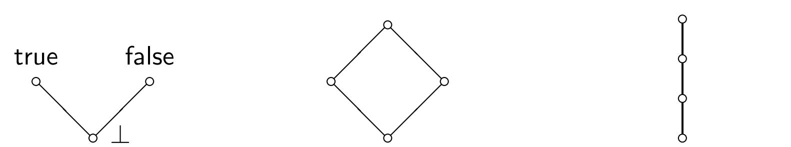
Small finite partially ordered sets can be drawn as line diagrams (Hasse diagrams). Examples are given in Figure 1. We will also allow ourselves to draw infinite posets by showing a finite part which illustrates the building principle. Three examples are given in Figure 2. We prefer the notation $\sqsubseteq$ to the more common $\leq$ because the order on domains we are studying here often coexists with an otherwise unrelated intrinsic order. The flat and lazy natural numbers from Figure 2 illustrate this.

If we drop antisymmetry from our list of requirements then we get what is known as *preorders*. This does not change the theory very much. As is easily seen, the sub-relation $\sqsubseteq \cap \sqsupseteq$ is in any case an equivalence relation and if two elements from two equivalence classes $x \in A, y \in B$ are related by $\sqsubseteq$, then so is any pair of elements from A and B. We can therefore pass from a preorder to a canonical partially ordered set by taking equivalence classes. Pictorially, the situation then looks as in Figure 3.

Many notions from the theory of ordered sets make sense even if reflexivity fails. Hence we may sum up these considerations with the slogan: *Order theory is the study of transitive relations*. A common way to extract the order-theoretic content from a relation R is to pass to the transitive closure of R, defined as $\bigcup_{n \in \mathbb{N} \setminus \{0\}} R^n$.

Ordered sets can be turned upside down:

Proposition 2.1.2. *If $\langle P, \sqsubseteq \rangle$ is an ordered set then so is $P^{op} = \langle P, \sqsupseteq \rangle$.*



The flat booleans | The four-element lattice | The four-element chain

Figure 1: A few posets drawn as line diagrams.

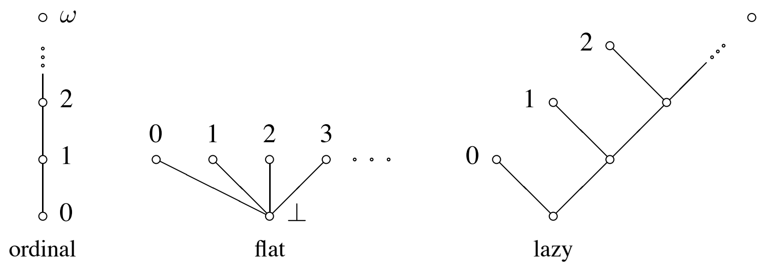


Figure 2: Three versions of the natural numbers.

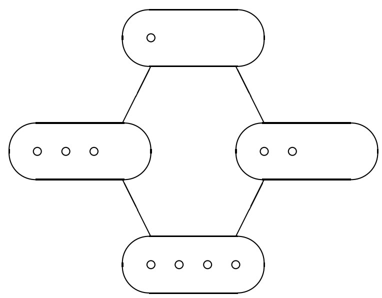


Figure 3: A preorder whose canonical quotient is the four-element lattice.

One consequence of this observation is that each of the concepts introduced below has a dual counterpart.

2.1.2 Notation from order theory

The following concepts form the core language of order theory.

Definition 2.1.3. *Let* $(P, \sqsubseteq)$ *be an ordered set.*

1. *A subset* A *of* P *is an* upper set *if* $x \in A$ *implies* $y \in A$ *for all* $y \sqsupseteq x$*. We denote by* $\uparrow A$ *the set of all elements above some element of* A*. If no confusion is to be feared then we abbreviate* $\uparrow\{x\}$ *as* $\uparrow x$*. The dual notions are* lower set *and* $\downarrow A$*.*

2. *An element* $x \in P$ *is called an* upper bound *for a subset* $A \subseteq P$*, if* x *is above every element of* A*. We often write* $A \sqsubseteq x$ *in this situation. We denote by* $\mathsf{ub}(A)$ *the set of all upper bounds of* A*. Dually,* $\mathsf{lb}(A)$ *denotes the set of lower bounds of* A*.*

3. *An element* $x \in P$ *is* maximal *if there is no other element of* P *above it:* $\uparrow x \cap P = \{x\}$*. Minimal elements are defined dually. For a subset* $A \subseteq P$ *the minimal elements of* $\mathsf{ub}(A)$ *are called* minimal upper bounds of A*. The set of all minimal upper bounds of* A *is denoted by* $\mathsf{mub}(A)$*.*

4. *If all elements of P are below a single element $x \in P$, then x is said to be the* largest element. *The dually defined* least element *of a poset is also called* bottom *and is commonly denoted by $\perp$. In the presence of a least element we speak of a* pointed poset.

5. *If for a subset $A \subseteq P$ the set of upper bounds has a least element x, then x is called the* supremum *or* join. *We write $x = \bigsqcup A$ in this case. In the other direction we speak of* infimum *or* meet *and write $x = \bigsqcap A$.*

6. *A partially ordered set P is a* $\sqcup$-semilattice *(*$\sqcap$-semilattice*) if the supremum (infimum) for each pair of elements exists. If P is both a $\sqcup$- and a $\sqcap$-semilattice then P is called a* lattice. *A lattice is* complete *if suprema and infima exist for all subsets.*

The operations of forming suprema, resp. infima, have a few basic properties which we will use throughout this text without mentioning them further.

Proposition 2.1.4. *Let P be a poset such that the suprema and infima occurring in the following formulae exist. (A, B and all A_i are subsets of P.)*

1. *$A \subseteq B$ implies $\bigsqcup A \sqsubseteq \bigsqcup B$ and $\bigsqcap A \sqsupseteq \bigsqcap B$.*
2. *$\bigsqcup A = \bigsqcup(\downarrow A)$ and $\bigsqcap A = \bigsqcap(\uparrow A)$.*
3. *If $A = \bigcup_{i \in I} A_i$ then $\bigsqcup A = \bigsqcup_{i \in I}(\bigsqcup A_i)$ and similarly for the infimum.*

Proof. We illustrate order theoretic reasoning with suprema by showing (3). The element $\bigsqcup A$ is above each element $\bigsqcup A_i$ by (1), so it is an upper bound of the set $\{\bigsqcup A_i \mid i \in I\}$. Since $\bigsqcup_{i \in I}(\bigsqcup A_i)$ is the least upper bound of this set, we have $\bigsqcup A \sqsupseteq \bigsqcup_{i \in I}(\bigsqcup A_i)$. Conversely, each $a \in A$ is contained in some A_i and therefore below the corresponding $\bigsqcup A_i$ which in turn is below $\bigsqcup_{i \in I}(\bigsqcup A_i)$. Hence the right hand side is an upper bound of A and as $\bigsqcup A$ is the least such, we also have $\bigsqcup A \sqsubseteq \bigsqcup_{i \in I}(\bigsqcup A_i)$. □

Let us conclude this subsection by looking at an important family of examples of complete lattices. Suppose X is a set and $\mathfrak{L}$ is a family of subsets of X. We call $\mathfrak{L}$ a *closure system* if it is closed under the formation of intersections, that is, whenever each member of a family $(A_i)_{i \in I}$ belongs to $\mathfrak{L}$ then so does $\bigcap_{i \in I} A_i$. Because we have allowed the index set to be empty, this implies that X is in $\mathfrak{L}$. We call the members of $\mathfrak{L}$ *hulls* or *closed sets*. Given an arbitrary subset A of X, one can form $\bigcap\{B \in \mathfrak{L} \mid A \subseteq B\}$. This is the least superset of A which belongs to $\mathfrak{L}$ and is called the *hull* or the *closure* of A.

Proposition 2.1.5. *Every closure system is a complete lattice with respect to inclusion.*

Proof. Infima are given by intersections and for the supremum one takes the closure of the union. □

2.1.3 Monotone functions

Definition 2.1.6. *Let P and Q be partially ordered sets. A function $f\colon P \to Q$ is called* monotone *if for all $x, y \in P$ with $x \sqsubseteq y$ we also have $f(x) \sqsubseteq f(y)$ in Q.*

'Monotone' is really an abbreviation for 'monotone order-preserving', but since we have no use for monotone order-reversing maps ($x \sqsubseteq y \Longrightarrow f(x) \sqsupseteq f(y)$), we have opted for the shorter expression. Alternative terminology is *isotone* (vs. *antitone*) or the other half of the full expression: *order-preserving* mapping.

The set $[P \xrightarrow{m} Q]$ of all monotone functions between two posets, when ordered *pointwise* (i.e. $f \sqsubseteq g$ if for all $x \in P$, $f(x) \sqsubseteq g(x)$), gives rise to another partially ordered set, the *monotone function space* between P and Q. The category **POSET** of posets and monotone maps has pleasing properties, see Exercise 2.3.9(9).

Proposition 2.1.7. *If L is a complete lattice then every monotone map from L to L has a fixpoint. The least of these is given by*

$$\bigsqcap \{x \in L \mid f(x) \sqsubseteq x\} ,$$

the largest by

$$\bigsqcup \{x \in L \mid x \sqsubseteq f(x)\} .$$

Proof. Let $A = \{x \in L \mid f(x) \sqsubseteq x\}$ and $a = \bigsqcap A$. For each $x \in A$ we have $a \sqsubseteq x$ and $f(a) \sqsubseteq f(x) \sqsubseteq x$. Taking the infimum we get $f(a) \sqsubseteq \bigsqcap f(A) \sqsubseteq \bigsqcap A = a$ and $a \in A$ follows. On the other hand, $x \in A$ always implies $f(x) \in A$ by monotonicity. Applying this to a yields $f(a) \in A$ and hence $a \sqsubseteq f(a)$. □

For lattices, the converse is also true: The existence of fixpoints for monotone maps implies completeness. But the proof is much harder and relies on the Axiom of Choice, see [Mar76].

2.1.4 Directed sets

Definition 2.1.8. *Let P be a poset. A subset A of P is* directed*, if it is nonempty and each pair of elements of A has an upper bound in A. If a directed set A has a supremum then this is denoted by $\bigsqcup^{\uparrow} A$.*

Directed lower sets are called ideals*. Ideals of the form $\downarrow x$ are called* principal.

The dual notions are filtered set *and (*principal) filter.

Simple examples of directed sets are *chains*. These are non-empty subsets which are totally ordered, i.e. for each pair x, y either $x \sqsubseteq y$ or $y \sqsubseteq x$ holds. The chain of natural numbers with their natural order is particularly simple; subsets of a poset isomorphic to it are usually called ω-chains. Another frequent type of directed set is given by the set of finite subsets of an arbitrary set. Using this and Proposition 2.1.4(3), we get the following useful decomposition of general suprema.

Proposition 2.1.9. *Let A be a non-empty subset of a $\sqcup$-semilattice for which $\bigsqcup A$ exists. Then the join of A can also be written as*

$$\bigsqcup^{\uparrow} \{\bigsqcup M \mid M \subseteq A \textit{ finite and non-empty}\} .$$

General directed sets, on the other hand, may be quite messy and unstructured. Sometimes one can find a well-behaved cofinal subset, such as a chain, where we say that A is *cofinal* in B, if for all $b \in B$ there is an $a \in A$ above it. Such a cofinal subset will have the same supremum (if it exists). But cofinal chains do not always exist, as Exercise 2.3.9(6) shows. Still, every directed set may be thought of as being equipped externally with a nice structure as we will now work out.

Definition 2.1.10. *A* monotone net *in a poset P is a monotone function α from a directed set I into P. The set I is called the* index set *of the net.*

Let $\alpha\colon I \to P$ be a monotone net. If we are given a monotone function $\beta\colon J \to I$, where J is directed and where for all $i \in I$ there is $j \in J$ with $\beta(j) \geq i$, then we call $\alpha \circ \beta\colon J \to P$ a subnet *of α.*

A monotone net $\alpha\colon I \to P$ has a supremum *in P, if the set $\{\alpha(i) \mid i \in I\}$ has a supremum in P.*

Every directed set can be viewed as a monotone net: let the set itself be the index set. On the other hand, the image of a monotone net $\alpha\colon I \to P$ is a directed set in P. So what are nets good for? The answer is given in the following proposition (which seems to have been stated first in [Kra39]).

Lemma 2.1.11. *Let P be a poset and let $\alpha\colon I \to P$ be a monotone net. Then α has a subnet $\alpha \circ \beta\colon J \to P$, whose index set J is a lattice in which every principal ideal is finite.*

Proof. Let J be the set of finite subsets of I. Clearly, J is a lattice in which every principal ideal is finite. We define the mapping $\beta\colon J \to I$ by induction on the cardinality of the elements of J:

$$\begin{array}{rcl} \beta(\phi) & = & \text{any element of } I; \\ \beta(A) & = & \text{any upper bound of the set } A \cup \{\beta(B) \mid B \subset A\}, A \neq \phi. \end{array}$$

It is obvious that β is monotone and defines a subnet. □

This lemma allows us to base an induction proof on an arbitrary directed set. This was recently applied to settle a long-standing conjecture in lattice theory, see [TT93].

Proposition 2.1.12. *Let I be directed and $\alpha\colon I \times I \to P$ be a monotone net. Under the assumption that the indicated directed suprema exist, the following equalities hold:*

$$\bigsqcup_{i,j \in I}^{\uparrow} \alpha(i,j) = \bigsqcup_{i \in I}^{\uparrow}\Big(\bigsqcup_{j \in J}^{\uparrow} \alpha(i,j)\Big) = \bigsqcup_{j \in J}^{\uparrow}\Big(\bigsqcup_{i \in I}^{\uparrow} \alpha(i,j)\Big) = \bigsqcup_{i \in I}^{\uparrow} \alpha(i,i).$$

2.1.5 Directed-complete partial orders

Definition 2.1.13. *A poset D in which every directed subset has a supremum we call a* directed-complete partial order, *or* dcpo *for short.*

Examples 2.1.14.
- *Every complete lattice is also a dcpo. Instances of this are powersets, topologies, subgroup lattices, congruence lattices, and, more generally, closure systems. As Proposition 2.1.9 shows, a lattice which is also a dcpo is almost complete. Only a least element may be missing.*

- *Every finite poset is a dcpo.*
- *The set of natural numbers with the usual order does not form a dcpo; we have to add a top element as done in Figure 2. In general, it is a difficult problem how to add points to a poset so that it becomes a dcpo. Using Proposition 2.1.15 below, Markowsky has defined such a completion via chains in [Mar76]. Luckily, we need not worry about this problem in domain theory because here we are usually interested in algebraic or continuous dcpo's where a completion is easily defined, see Section 2.2.6 below. The correct formulation of what constitutes a completion, of course, takes also morphisms into account. A general framework is described in [Poi92], Sections 3.3 to 3.6.*
- *The points of a locale form a dcpo in the specialization order, see [Vic89, Joh82].*

More examples will follow in the next subsection. There we will also discuss the question of whether directed sets or ω-chains should be used to define dcpo's. Arbitrarily long chains have the full power of directed sets (despite Exercise 2.3.9(6)) as the following proposition shows.

Proposition 2.1.15. *A partially ordered set D is a dcpo if and only if each chain in D has a supremum.*

The proof, which uses the Axiom of Choice, goes back to a lemma of Iwamura [Iwa44] and can be found in [Mar76].

The following, which may also be found in [Mar76], complements Proposition 2.1.7 above.

Proposition 2.1.16. *A pointed poset P is a dcpo if and only if every monotone map on P has a least fixpoint.*

2.1.6 Continuous functions

Definition 2.1.17. *Let D and E be dcpo's. A function $f\colon D \to E$ is (*Scott-*)* continuous *if it is monotone and if for each directed subset A of D we have $f(\bigsqcup^{\uparrow} A) = \bigsqcup^{\uparrow} f(A)$. We denote the set of all continuous functions from D to E, ordered pointwise, by $[D \longrightarrow E]$.*

A function between pointed dcpo's, which preserves the bottom element, is called strict. *We denote the space of all continuous strict functions by $[D \xrightarrow{\perp!} E]$.*

The identity function on a set A is denoted by id_A, the constant function with image $\{x\}$ by c_x.

The preservation of joins of directed sets is actually enough to define continuous maps. In practice, however, one usually needs to show first that $f(A)$ is directed. This is equivalent to monotonicity.

Proposition 2.1.18. *Let D and E be dcpo's. Then $[D \longrightarrow E]$ is again a dcpo. Directed suprema in $[D \longrightarrow E]$ are calculated pointwise.*

Proof. Let F be a directed collection of functions from D to E. Let $g\colon D \to E$ be the function, which is defined by $g(x) = \bigsqcup^{\uparrow}_{f\in F} f(x)$. Let $A \subseteq D$ be directed.

$$\begin{aligned}
g(\bigsqcup^{\uparrow} A) &= \bigsqcup_{f\in F}^{\uparrow} f(\bigsqcup^{\uparrow} A) \\
&= \bigsqcup_{f\in F}^{\uparrow} \bigsqcup_{a\in A}^{\uparrow} f(a) \\
&= \bigsqcup_{a\in A}^{\uparrow} \bigsqcup_{f\in F}^{\uparrow} f(a) \\
&= \bigsqcup_{a\in A}^{\uparrow} g(a).
\end{aligned}$$

This shows that g is continuous. ☐

The class of all dcpo's together with Scott-continuous functions forms a category, which we denote by **DCPO**. It has strong closure properties as we shall see shortly. For the moment we concentrate on that property of continuous maps which is one of the main reasons for the success of domain theory, namely, that fixpoints can be calculated easily and uniformly.

Theorem 2.1.19. *Let D be a pointed dcpo.*

1. *Every continuous function f on D has a least fixpoint. It is given by $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} f^n(\bot)$.*
2. *The assignment $\mathsf{fix}\colon [D \longrightarrow D] \to D$, $f \mapsto \bigsqcup^{\uparrow}_{n\in\mathbb{N}} f^n(\bot)$ is continuous.*

Proof. (1) The set $\{f^n(\bot) \mid n \in \mathbb{N}\}$ is a chain. This follows from $\bot \sqsubseteq f(\bot)$ and the monotonicity of f. Using continuity of f we get $f(\bigsqcup^{\uparrow}_{n\in\mathbb{N}} f^n(\bot)) = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} f^{n+1}(\bot)$ and the latter is clearly equal to $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} f^n(\bot)$.

If x is any other fixpoint of f then from $\bot \sqsubseteq x$ we get $f(\bot) \sqsubseteq f(x) = x$ and so on by induction. Hence x is an upper bound of all $f^n(\bot)$ and that is why it must be above $\mathsf{fix}(f)$.

(2) Let us first look at the n-fold iteration operator $\mathsf{it}_n\colon [D \longrightarrow D] \to D$ which maps f to $f^n(\bot)$. We show its continuity by induction. The 0th iteration operator equals $\mathsf{c}_\bot$ so nothing has to be shown there. For the induction step let F be a directed family of continuous functions on D. We calculate:

$$\begin{array}{rcll}
\mathsf{it}_{n+1}(\bigsqcup^{\uparrow} F) &=& (\bigsqcup^{\uparrow} F)(\mathsf{it}_n(\bigsqcup^{\uparrow} F)) & \text{definition} \\
&=& (\bigsqcup^{\uparrow} F)(\bigsqcup^{\uparrow}_{f\in F} \mathsf{it}_n(f)) & \text{ind. hypothesis} \\
&=& \bigsqcup^{\uparrow}_{g\in F} g(\bigsqcup^{\uparrow}_{f\in F}(\mathsf{it}_n(f))) & \text{Prop. 2.1.18} \\
&=& \bigsqcup^{\uparrow}_{g\in F} \bigsqcup^{\uparrow}_{f\in F} g(\mathsf{it}_n(f)) & \text{continuity of } g \\
&=& \bigsqcup^{\uparrow}_{f\in F} f^{n+1}(\bot) & \text{Prop. 2.1.12}
\end{array}$$

The pointwise supremum of all iteration operators (which form a chain as we have seen in (1)) is precisely fix and so the latter is also continuous. ☐

The least fixpoint operator is the mathematical counterpart of recursive and iterative statements in programming languages. When proving a property of such a statement semantically, one often employs the following proof principle which is known under the name *fixpoint induction* (see [Ten91] or any other book on denotational semantics). Call a predicate on (i.e. a subset of) a dcpo *admissible* if it contains $\perp$ and is closed under suprema of ω-chains. The following is then easily established:

Lemma 2.1.20. *Let D be a dcpo, $P \subseteq D$ an admissible predicate, and $f\colon D \to D$ a Scott-continuous function. If it is true that $f(x)$ satisfies P whenever x satisfies P, then it must be true that* $\mathsf{fix}(f)$ *satisfies P.*

We also note the following invariance property of the least fixpoint operator. In fact, it characterizes fix uniquely among all fixpoint operators (Exercise 2.3.9(16)).

Lemma 2.1.21. *Let D and E be pointed dcpo's and let*

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "f"'] & E \arrow[d, "g"] \\ D \arrow[r, "h"] & E \end{tikzcd}$$

be a commutative diagram of continuous functions where h is strict. Then $\mathsf{fix}(g) = h(\mathsf{fix}(f))$.

Proof. Using continuity of h, commutativity of the diagram, and strictness of h in turn we calculate:

$$\begin{aligned} h(\mathsf{fix}(f)) &= h\Big(\bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} f^n(\perp)\Big) \\ &= \bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} h \circ f^n(\perp) \\ &= \bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} g^n \circ h(\perp) \\ &= \mathsf{fix}(g) \end{aligned}$$

□

2.2 Approximation

In the last subsection we have explained the kind of limits that domain theory deals with, namely, suprema of directed sets. We could have said much more about these “convergence spaces” called dcpo’s. But the topic can easily become esoteric and lose its connection with computing. For example, the cardinality of dcpo’s has not been restricted yet and indeed, we didn’t have the tools to sensibly do so (Exercise 2.3.9(18)). We will in this subsection introduce the idea that elements are composed of (or ‘approximated by’) ‘simple’ pieces. This will enrich our theory immensely and will also give the desired connection to semantics.

2.2.1 The order of approximation

Definition 2.2.1. *Let x and y be elements of a dcpo D. We say that x* approximates *y if for all directed subsets A of D, $y \sqsubseteq \bigsqcup^{\uparrow} A$ implies $x \sqsubseteq a$ for some $a \in A$. We say that x is* compact *if it approximates itself.*

We introduce the following notation for $x, y \in D$ and $A \subseteq D$:

$$\begin{array}{rcl} x \ll y & \Leftrightarrow & x \text{ approximates } y \\ \Downarrow x & = & \{y \in D \mid y \ll x\} \\ \Uparrow x & = & \{y \in D \mid x \ll y\} \\ \Uparrow A & = & \bigcup_{a \in A} \Uparrow a \\ \mathsf{K}(D) & = & \{x \in D \mid x \text{ compact}\} \end{array}$$

The relation $\ll$ is traditionally called 'way-below relation'. M.B. Smyth introduced the expression 'order of definite refinement' in [Smy86]. Throughout this text we will refer to it as the *order of approximation*, even though the relation is not reflexive. Other common terminology for 'compact' is *finite* or *isolated*. The analogy to finite sets is indeed very strong; however one covers a finite set M by a directed collection $(A_i)_{i \in I}$ of sets, M will always be contained in some A_i already.

In general, approximation is not an absolute property of single points. Rather, we could phrase $x \ll y$ as "x is a lot simpler than y", which clearly depends on y as much as it depends on x.

An element which is compact approximates every element above it. More generally, we observe the following basic properties of approximation.

Proposition 2.2.2. *Let D be a dcpo. Then the following is true for all $x, x', y, y' \in D$:*

1. $x \ll y \Longrightarrow x \sqsubseteq y$*;*
2. $x' \sqsubseteq x \ll y \sqsubseteq y' \Longrightarrow x' \ll y'$.

2.2.2 Bases in dcpo's

Definition 2.2.3. *We say that a subset B of a dcpo D is a* basis *for D, if for every element x of D the set $B_x = \Downarrow x \cap B$ contains a directed subset with supremum x. We call elements of B_x* approximants to x relative to B.

We may think of the rational numbers as a basis for the reals (with a top element added, in order to get a dcpo), but other choices are also possible: dyadic numbers, irrational numbers, etc.

Proposition 2.2.4. *Let D be a dcpo with basis B.*

1. *For every $x \in D$ the set B_x is directed and $x = \bigsqcup^{\uparrow} B_x$.*
2. *B contains $\mathsf{K}(D)$.*
3. *Every superset of B is also a basis for D.*

Proof. (1) It is clear that the join of B_x equals x. The point is directedness. From the definition we know there is some directed subset A of B_x with $\bigsqcup^{\uparrow} A = x$. Let now y, y' be elements approximating x. There must be elements a, a' in A above y, y', respectively. These have an upper bound a'' in A, which by definition belongs to B_x.

(2) We have to show that every element c of $\mathsf{K}(D)$ belongs to B. Indeed, since $c = \bigsqcup^{\uparrow} B_c$ there must be an element $b \in B_c$ above c. All of B_c is below c, so b is actually equal to c.

(3) is immediate from the definition. □

Corollary 2.2.5. *Let D be a dcpo with basis B.*

1. *The largest basis for D is D itself.*
2. *B is the smallest basis for D if and only if $B = \mathsf{K}(D)$.*

The 'only if' part of (2) is not a direct consequence of the preceding proposition. We leave its proof as Exercise 2.3.9(26).

2.2.3 Continuous and algebraic domains

Definition 2.2.6. *A dcpo is called* continuous *or a* continuous domain *if it has a basis. It is called* algebraic *or an* algebraic domain *if it has a basis of compact elements. We say D is ω-continuous if there exists a countable basis and we call it ω-algebraic if $\mathsf{K}(D)$ is a countable basis.*

Here we are using the word "domain" for the first time. Indeed, for us a structure only qualifies as a domain if it embodies both a notion of convergence and a notion of approximation.

In the light of Proposition 2.2.4 we can reformulate Definition 2.2.6 as follows, avoiding existential quantification.

Proposition 2.2.7.

1. *A dcpo D is continuous if and only if for all $x \in D$, $x = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\!\downarrow} x$ holds.*
2. *It is algebraic if and only if for all $x \in D$, $x = \bigsqcup^{\uparrow} \mathsf{K}(D)_x$ holds.*

The word 'algebraic' points to algebra. Let us make this connection precise.

Definition 2.2.8. *A closure system $\mathfrak{L}$ (cf. Section 2.1.2) is called* inductive*, if it is closed under directed union.*

Proposition 2.2.9. *Every inductive closure system $\mathfrak{L}$ is an algebraic lattice. The compact elements are precisely the finitely generated hulls.*

Proof. If A is the hull of a finite set M and if $(B_i)_{i \in I}$ is a directed family of hulls such that $\bigsqcup^{\uparrow}_{i \in I} B_i = \bigcup_{i \in I} B_i \supseteq A$, then M is already contained in some B_i. Hence hulls of finite sets are compact elements in the complete lattice $\mathfrak{L}$. On the other hand, every closed set is the directed union of finitely generated hulls, so these form a basis. By Proposition 2.2.4(2), there cannot be any other compact elements. □

Given a group, (or, more generally, an algebra in the sense of universal algebra), then there are two canonical inductive closure systems associated with it, the lattice of subgroups (subalgebras) and the lattice of normal subgroups (congruence relations).

Other standard examples of algebraic domains are:

- Any set with the discrete order is an algebraic domain. In semantics one usually adds a bottom element (standing for divergence) resulting in so-called *flat domains*. (The flat natural numbers are shown in Figure 2.) A basis must in either case contain all elements.
- The set $[X \rightharpoonup Y]$ of partial functions between sets X and Y ordered by graph inclusion. Compact elements are those functions which have a finite carrier. It is naturally isomorphic to $[X \longrightarrow Y_\perp]$ and to $[X_\perp \xrightarrow{\perp !} Y_\perp]$.
- Every finite poset.

Continuous domains:

- Every algebraic dcpo is also continuous. This follows directly from the definition. The order of approximation is characterized by $x \ll y$ if and only if there exists a compact element c between x and y.
- The unit interval is a continuous lattice. It plays a central role in the theory of continuous lattices, see [GHK$^+$80], Chapter IV and in particular Theorem 2.19.

  Another way of modelling the real numbers in domain theory is to take all closed intervals of finite length and to order them by reversed inclusion. Single element intervals are maximal in this domain and provide a faithful representation of the real line. A countable basis is given by the set of intervals with rational endpoints.
- The lattice of open subsets of a sober space X forms a continuous lattice if and only if X is locally compact. Compact Hausdorff spaces are a special case. Here $O \ll U$ holds if and only if there exists a compact set C such that $O \subseteq C \subseteq U$. This meeting point of topology and domain theory is discussed in detail in [Smy92, Vic89, Joh82, GHK$^+$80] and will also be addressed in Chapter 7.

At this point it may be helpful to give an example of a non-continuous dcpo. The easiest to explain is depicted in Figure 4 (labelled D). We show that the order of approximation on D is empty. Pairs (a_i, b_j) and (b_i, a_j) cannot belong to the order of approximation because they are not related in the order. Two points $a_i \sqsubseteq a_j$ in the same 'leg' are still not approximating because $(b_n)_{n \in \mathbb{N}}$ is a directed set with supremum above a_j but containing no element above a_i.

A non-continuous distributive complete lattice is much harder to visualize by a line diagram. From what we have said we know that the topology of a sober space which is not locally compact is such a lattice. Exercise 2.3.9(21) discusses this in detail.

If D is pointed then the order of approximation is non-empty because a bottom element approximates every other element.

A basis not only gives approximations for elements, it also approximates the order relation:

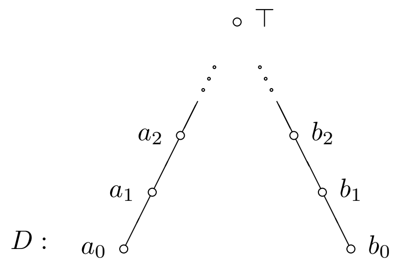


Figure 4: A continuous (E) and a non-continuous (D) dcpo.

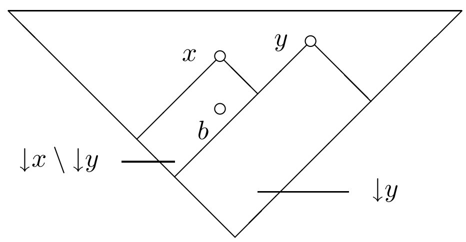


Figure 5: Basis element b witnesses that x is not below y.

Proposition 2.2.10. *Let D be a continuous domain with basis B and let x and y be elements of D. Then $x \sqsubseteq y$, $B_x \subseteq B_y$ and $B_x \subseteq \downarrow y$ are all equivalent.*

The form in which we will usually apply this proposition is: $x \not\sqsubseteq y$ implies there exists $b \in B_x$ with $b \not\sqsubseteq y$. A picture of this situation is given in Figure 5.

In the light of Proposition 2.2.10 we can now also give a more intuitive reason why the dcpo D in Figure 4 is not continuous. A natural candidate for a basis in D is the collection of all a_i's and b_i's (certainly, $\top$ doesn't approximate anything). Proposition 2.2.10 expresses the idea that in a continuous domain all information about how elements are related is contained in the basis already. And the fact that $\bigsqcup^{\uparrow}_{n \in \mathbb{N}} a_n = \bigsqcup^{\uparrow}_{n \in \mathbb{N}} b_n = \top$ holds in D is precisely what is not visible in the would-be basis. Thus, the dcpo should look rather like E in the same figure (which indeed is an algebraic domain).

Bases allow us to express the continuity of functions in a form reminiscent of the ϵ-δ definition for real-valued functions.

Proposition 2.2.11. *A map f between continuous domains D and E with bases B and C, respectively, is continuous if and only if for each $x \in D$ and $e \in C_{f(x)}$ there exists $d \in B_x$ with $f(\uparrow d) \subseteq \uparrow e$.*

Proof. By continuity we have $f(x) = f(\bigsqcup^{\uparrow} B_x) = \bigsqcup^{\uparrow}_{d \in B_x} f(d)$. Since e approximates $f(x)$, there exists $d \in B_x$ with $f(d) \sqsupseteq e$. Monotonicity of f then implies $f(\uparrow d) \subseteq \uparrow e$.

For the converse we first show monotonicity. Suppose $x \sqsubseteq y$ holds but $f(x)$ is not below $f(y)$. By Proposition 2.2.10 there is $e \in C_{f(x)} \setminus \downarrow f(y)$ and from our assumption we get $d \in B_x$ such that $f(\uparrow d) \subseteq \uparrow e$. Since y belongs to $\uparrow d$ this is a contradiction. Now let A be a directed subset of D with x as its join. Monotonicity implies $\bigsqcup^{\uparrow} f(A) \sqsubseteq f(\bigsqcup^{\uparrow} A) = f(x)$. If the converse relation does not hold then we can again choose $e \in C_{f(x)}$ with $e \not\sqsubseteq \bigsqcup^{\uparrow} f(A)$ and for some $d \in B_x$ we have $f(\uparrow d) \subseteq \uparrow e$. Since d approximates x, some $a \in A$ is above d and we get $\bigsqcup^{\uparrow} f(A) \sqsupseteq f(a) \sqsupseteq f(d) \sqsupseteq e$ contradicting our choice of e. □

Finally, we cite a result which reduces the calculation of least fixpoints to a basis. The point here is that a continuous function need not preserve compactness nor the order of approximation and so the sequence $\perp, f(\perp), f(f(\perp)), \ldots$ need not consist of basis elements.

Proposition 2.2.12. *If D is a pointed ω-continuous domain with basis B and if $f\colon D \to D$ is a continuous map, then there exists an ω-chain $b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \ldots$ of basis elements such that the following conditions are satisfied:*

1. $b_0 = \perp$,
2. $\forall n \in \mathbb{N}.\, b_{n+1} \sqsubseteq f(b_n)$,
3. $\bigsqcup^{\uparrow}_{n \in \mathbb{N}} b_n = \mathsf{fix}(f) \quad (= \bigsqcup^{\uparrow}_{n \in \mathbb{N}} f^n(\perp))$.

A proof may be found in [Abr90b].

2.2.4 Comments on possible variations

directed sets vs. ω-chains Let us start with the following observation.

Proposition 2.2.13. *If a dcpo D has a countable basis then every directed subset of D contains an ω-chain with the same supremum.*

This raises the question whether one shouldn't build up the whole theory using ω-chains. The basic definitions then read: An ω-ccpo is a poset in which every ω-chain has a supremum. A function is ω-continuous if it preserves joins of ω-chains. An element x is ω-approximating y if $\bigsqcup^{\uparrow}_{n \in \mathbb{N}} a_n \sqsupseteq y$ implies $a_n \sqsupseteq x$ for some $n \in \mathbb{N}$. An ω-ccpo is continuous if there is a countable subset B such that every element is the join of an ω-chain of elements from B ω-approximating it. Similarly for algebraicity. (This is the approach adopted in [Plo81], for example.) The main point about these definitions is the countability of the basis. It ensures that they are in complete harmony with our set-up, because we can show:

Proposition 2.2.14. *1. Every continuous ω-ccpo is a continuous dcpo.*

2. Every algebraic ω-ccpo is an algebraic dcpo.

3. Every ω-continuous map between continuous ω-ccpo's is continuous.

Proof. (1) Let $(b_n)_{n\in\mathbb{N}}$ be an enumeration of a basis B for D. We first show that the continuous ω-ccpo D is directed-complete, so let A be a directed subset of D. Let B' be the set of basis elements which are below some element of A and, for simplicity, assume that $B = B'$. We construct an ω-chain in A as follows: let a_0 be an element of A which is above b_0. Then let b_{n_1} be the first basis element not below a_0. It must be below some $a_1' \in A$ and we set a_1 to be an upper bound of a_0 and a_1' in A. We proceed by induction. It does not follow that the resulting chain $(a_n)_{n\in\mathbb{N}}$ is cofinal in A but it is true that its supremum is also the supremum of A, because both subsets of D dominate the same set of basis elements.

This construction also shows that ω-approximation is the same as approximation in a continuous ω-ccpo. The same basis B may then be used to show that D is a continuous domain. (The directedness of the sets B_x follows as in Proposition 2.2.4(1).)

(2) follows from the proof of (1), so it remains to show (3). Monotonicity of the function f is implied in the definition of ω-continuity. Therefore a directed set $A \subseteq D$ is mapped onto a directed set in E and also $f(\bigsqcup^{\uparrow} A) \sqsupseteq \bigsqcup^{\uparrow} f(A)$ holds. Let $(a_n)_{n\in\mathbb{N}}$ be an ω-chain in A with $\bigsqcup^{\uparrow} A = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} a_n$, as constructed in the proof of (1). Then we have $f(\bigsqcup^{\uparrow} A) = f(\bigsqcup^{\uparrow}_{n\in\mathbb{N}} a_n) = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} f(a_n) \sqsubseteq \bigsqcup^{\uparrow} f(A)$. $\Box$

If we drop the crucial assumption about the countability of the basis then the two theories bifurcate and, in our opinion, the theory based on ω-chains becomes rather bizarre. To give just one illustration, observe that simple objects, such as powersets, may fail to be algebraic domains. There remains the question, however, whether in the realm of a mathematical theory of computation one should start with ω-chains. Arguments in favor of this approach point to pedagogy and foundations. The pedagogical aspect is somewhat weakened by the fact that even in a continuous ω-ccpo the sets $\mathop{\downarrow\!\!\!\downarrow} x$ *happen to be* directed. Glossing over this fact would tend to mislead the student. In our eyes, the right middle ground for a *course* on domain theory, then, would be to start with ω-chains and motivations from semantics and then at some point (probably where the ideal completion of a poset is discussed) to switch to directed sets as the more general concept. This suggestion is hardly original. It is in direct analogy with the way students are introduced to topological concepts.

Turning to foundations, we feel that the necessity to *choose* chains where directed subsets are naturally available (such as in function spaces) and thus to rely on the Axiom of Choice without need, is a serious stain on this approach. To take foundational questions seriously implies a much deeper re-working of the theory: some pointers to the literature will be found in Section 8.

We do not feel the need to say much about the use of chains of arbitrary cardinality. This adds nothing in strength (because of Proposition 2.1.15) but has all the disadvantages pointed out for ω-chains already.

bases vs. intrinsic descriptions. The definition of a continuous domain given here differs from, and is in fact more complicated than the standard one (which we presented as Proposition 2.2.7(1)). We nevertheless preferred this approach to the concept of approximation for three reasons. Firstly, the standard definition does not allow the restriction of the size of continuous domains. In this respect not the cardinality of a domain but the minimal cardinality of a basis is of interest. Secondly, we wanted to point out the strong analogy between algebraic and continuous domains. And, indeed, the proofs we have given so far for continuous domains specialize directly to the algebraic case if one replaces 'B' by '$\mathsf{K}(D)$' throughout. Thus far at least, proofs for algebraic domains alone would not be any shorter. And, thirdly, we wanted to stress the idea of approximation by elements which are (for whatever reason) simpler than others. Such a notion of simplicity does often exist for continuous domains (such as rational vs. real numbers), even though its justification is not purely order-theoretical (see 8.1.1).

algebraic vs. continuous. This brings up the question of why one bothers with continuous domains at all. There are two important reasons but they depend on definitions introduced later in this text. The first is the simplification of the mathematical theory of domains stemming from the possibility of freely using retracts (see Theorem 3.1.4 below). The second is the observation that in algebraic domains two fundamental concepts of domain theory essentially coincide, namely, that of a Scott-open set and that of a compact saturated set. We find it pedagogically advantageous to be able to distinguish between the two.

continuous dcpo vs. continuous domain. It is presently common practice to start a paper in semantics or domain theory by defining the subclass of dcpo's of interest and then assigning the name 'domain' to these structures. We fully agree with this custom of using 'domain' as a generic name. In this article, however, we will study a full range of possible definitions, the most general of which is that of a dcpo. We have nevertheless avoided calling these domains. For us, 'domain' refers to both ideas essential to the theory, namely, the idea of convergence and the idea of approximation.

2.2.5 Useful properties

Let us start right away with the single most important feature of the order of approximation, the *interpolation property*.

Lemma 2.2.15. *Let D be a continuous domain and let $M \subseteq D$ be a finite set each of whose elements approximates y. Then there exists $y' \in D$ such that $M \ll y' \ll y$ holds. If B is a basis for D then y' may be chosen from B. (We say, y'* interpolates *between M and y.)*

Proof. Given $M \ll y$ in D we define the set

$$A = \{a \in D \mid \exists a' \in D : a \ll a' \ll y\}.$$

It is clearly non-empty. It is directed because if $a \ll a' \ll y$ and $b \ll b' \ll y$ then by the directedness of $\mathord{\downarrow\!\!\!\downarrow} y$ there is $c' \in D$ such that $a' \sqsubseteq c' \ll y$ and $b' \sqsubseteq c' \ll y$ and again by the directedness of $\mathord{\downarrow\!\!\!\downarrow} c'$ there is $c \in D$ with $a \sqsubseteq c \ll c'$ and $b \sqsubseteq c \ll c'$. We calculate the supremum of A: let y' be any element approximating y. Since $\mathord{\downarrow\!\!\!\downarrow} y' \subseteq A$ we have that $\bigsqcup^{\uparrow} A \sqsupseteq \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\!\downarrow} y' = y'$. This holds for all $y' \ll y$ so by continuity $y = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\!\downarrow} y \sqsubseteq \bigsqcup^{\uparrow} A$. All elements of A are less than y, so in fact equality holds: $\bigsqcup^{\uparrow} \mathord{\downarrow\!\!\!\downarrow} y = \bigsqcup^{\uparrow} A$. Remember that we started out with a set M whose elements approximate y. By definition there is $a_m \in A$ with $m \sqsubseteq a_m$ for each $m \in M$. Let a be an upper bound of the a_m in A. By definition, for some a', $a \ll a' \ll y$, and we can take a' as an interpolating element between M and y. The proof remains the same if we allow only basis elements to enter A. □

Corollary 2.2.16. *Let D be a continuous domain with a basis B and let A be a directed subset of D. If c is an element approximating $\bigsqcup^{\uparrow} A$ then c already approximates some $a \in A$. As a formula:*

$$\mathop{\downarrow\!\!\downarrow} \bigsqcup^{\uparrow} A = \bigcup_{a \in A} \mathop{\downarrow\!\!\downarrow} a.$$

Intersecting with the basis on both sides gives

$$B_{\bigsqcup^{\uparrow} A} = \bigcup_{a \in A} B_a.$$

Next we will illustrate how in a domain we can restrict attention to principal ideals.

Proposition 2.2.17. *1. If D is a continuous domain and if x, y are elements in D, then x approximates y if and only if for all directed sets A with $\bigsqcup^{\uparrow} A = y$ there is an $a \in A$ such that $a \sqsupseteq x$.*

2. The order of approximation on a continuous domain is the union of the orders of approximation on all principal ideals.

3. A dcpo is continuous if and only if each principal ideal is continuous.

4. For a continuous domain D we have $\mathsf{K}(D) = \bigcup_{x \in D} \mathsf{K}(\downarrow x)$.

5. A dcpo is algebraic if and only if each principal ideal is algebraic.

Proposition 2.2.18. *1. In a continuous domain minimal upper bounds of finite sets of compact elements are again compact.*

2. In a complete lattice the sets $\mathop{\downarrow\!\!\downarrow} x$ are $\sqcup$-sub-semilattices.

3. In a complete lattice the join of finitely many compact elements is again compact.

Corollary 2.2.19. *A complete lattice is algebraic if and only if each element is the join of compact elements.*

The infimum of compact elements need not be compact again, even in an algebraic lattice. An example is given in Figure 6.

2.2.6 Bases as objects

In Section 2.2.2 we have seen how we can use bases in order to express properties of the ambient domain. We will now study the question of how far we can reduce domain theory to a theory of (abstract) bases. The resulting techniques will prove useful in later chapters but we hope that they will also deepen the reader's understanding of the nature of domains.

We start with the question of what additional information is necessary in order to reconstruct a domain from one of its bases. Somewhat surprisingly, it is just the order of approximation. Thus we define:

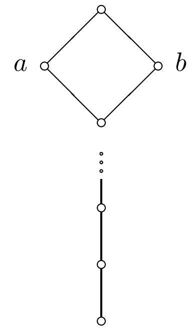


Figure 6: The meet of the compact elements a and b is not compact.

Definition 2.2.20. *An (*abstract*)* basis *is given by a set B together with a transitive relation $\prec$ on B, such that*

$$\text{(INT)} \qquad M \prec x \Longrightarrow \exists y \in B.\ M \prec y \prec x$$

holds for all elements x and finite subsets M of B.

Abstract bases were introduced in [Smy77] where they are called "R-structures". Examples of abstract bases are concrete bases of continuous domains, of course, where the relation $\prec$ is the restriction of the order of approximation. Axiom (INT) is satisfied because of Lemma 2.2.15 and because we have required bases in domains to have directed sets of approximants for each element.

Other examples are partially ordered sets, where (INT) is satisfied because of reflexivity. We will shortly identify posets as being exactly the bases of compact elements of algebraic domains.

In what follows we will use the terminology developed at the beginning of this chapter, even though the relation $\prec$ on an abstract basis need neither be reflexive nor antisymmetric. This is convenient but in some instances looks more innocent than it is. An ideal A in a basis, for example, has the property (following from directedness) that for every $x \in A$ there is another element $y \in A$ with $x \prec y$. In posets this doesn't mean anything but here it becomes an important feature. Sometimes this is stressed by using the expression 'A is a *round* ideal'. Note that a set of the form $\downarrow x$ is always an ideal because of (INT) but that it need not contain x itself. We will refrain from calling $\downarrow x$ 'principal' in these circumstances.

Definition 2.2.21. *For a basis $\langle B, \prec\rangle$ let $\mathsf{Idl}(B)$ be the set of all ideals ordered by inclusion. It is called the* ideal completion *of B. Furthermore, let $i\colon B \to \mathsf{Idl}(B)$ denote the function which maps $x \in B$ to $\downarrow x$. If we want to stress the relation with which B is equipped then we write $\mathsf{Idl}(B, \prec)$ for the ideal completion.*

Proposition 2.2.22. *Let $\langle B, \prec\rangle$ be an abstract basis.*

1. *The ideal completion of* B *is a dcpo.*

2. $A \ll A'$ *holds in* $\mathsf{Idl}(B)$ *if and only if there are* $x \prec y$ *in* B *such that* $A \subseteq i(x) \subseteq i(y) \subseteq A'$.

3. $\mathsf{Idl}(B)$ *is a continuous domain and a basis of* $\mathsf{Idl}(B)$ *is given by* $i(B)$.

4. *If* $\prec$ *is reflexive then* $\mathsf{Idl}(B)$ *is algebraic.*

5. *If* $\langle B, \prec\rangle$ *is a poset then* B, $\mathsf{K}(\mathsf{Idl}(B))$, *and* $i(B)$ *are all isomorphic.*

Proof. (1) holds because clearly the directed union of ideals is an ideal. Roundness implies that every $A \in \mathsf{Idl}(B)$ can be written as $\bigcup_{x\in A} \downarrow\!x$. This union is directed because A is directed. This proves (2) and also (3). The fourth claim follows from the characterization of the order of approximation. The last clause holds because there is only one basis of compact elements for an algebraic domain. $\square$

Defining the product of two abstract bases as one does for partially ordered sets, we have the following:

Proposition 2.2.23. $\mathsf{Idl}(B \times B') \cong \mathsf{Idl}(B) \times \mathsf{Idl}(B')$

Our 'completion' has a weak universal property:

Proposition 2.2.24. *Let* $\langle B, \prec\rangle$ *be an abstract basis and let* D *be a dcpo. For every monotone function* $f\colon B \to D$ *there is a largest continuous function* $\hat{f}\colon \mathsf{Idl}(B) \to D$ *such that* $\hat{f} \circ i$ *is below* f. *It is given by* $\hat{f}(A) = \bigsqcup^{\uparrow} f(A)$.

$$\begin{tikzcd} B \arrow[d, "i"'] \arrow[dr, "f"] & \\ \mathsf{Idl}(B) \arrow[r, dashed, "\hat{f}"] & D \end{tikzcd}$$

The assignment $f \mapsto \hat{f}$ *is a Scott-continuous map from* $[B \xrightarrow{m} D]$ *to* $[\mathsf{Idl}(B) \longrightarrow D]$.

If the relation $\prec$ *is reflexive then* $\hat{f} \circ i$ *equals* f.

Proof. Let us first check continuity of $\hat{f}$. To this end let $(A_i)_{i\in I}$ be a directed collection of ideals. Using general associativity (Proposition 2.1.4(3)) we can calculate: $\hat{f}(\bigsqcup^{\uparrow}_{i\in I} A_i) = \hat{f}(\bigcup_{i\in I} A_i) = \bigsqcup^{\uparrow}\{f(x) \mid x \in \bigcup_{i\in I} A_i\} = \bigsqcup^{\uparrow}_{i\in I} \bigsqcup^{\uparrow}\{f(x) \mid x \in A_i\} = \bigsqcup^{\uparrow}_{i\in I} \hat{f}(A_i)$.

Since f is assumed to be monotone, $f(x)$ is an upper bound for $f(\downarrow\!x)$. This proves that $\hat{f} \circ i$ is below f. If, on the other hand, $g\colon \mathsf{Idl}(B) \to D$ is another continuous function with this property then we have $g(A) = g(\bigcup_{x\in A} \downarrow\!x) = \bigsqcup^{\uparrow}_{x\in A} g(\downarrow\!x) = \bigsqcup^{\uparrow}_{x\in A} g(i(x)) \sqsubseteq \bigsqcup^{\uparrow}_{x\in A} f(x) = \hat{f}(A)$.

The claim about the continuity of the assignment $f \mapsto \hat{f}$ is shown by the usual switch of directed suprema.

If $\prec$ is a preorder then we can show that $\hat{f} \circ i = f$: $\hat{f}(i(x)) = \hat{f}(\downarrow\!x) = \bigsqcup^{\uparrow} f(\downarrow\!x) = f(x)$. $\square$

A particular instance of this proposition is the case that B and B' are two abstract bases and $f\colon B \to B'$ is monotone. By the extension of f to $\mathsf{Idl}(B)$ we mean the map $\widehat{i' \circ f}\colon \mathsf{Idl}(B) \to \mathsf{Idl}(B')$. It maps an ideal $A \subseteq B$ to the ideal $\downarrow\! f(A)$.

Proposition 2.2.25. *Let D be a continuous domain with basis B. Viewing $\langle B, \ll\rangle$ as an abstract basis, we have the following:*

1. *$\mathsf{Idl}(B)$ is isomorphic to D. The isomorphism $\sigma\colon \mathsf{Idl}(B) \to D$ is the extension $\hat{e}$ of the embedding of B into D. Its inverse β maps elements $x \in D$ to B_x.*
2. *For every dcpo E and continuous function $f\colon D \to E$ we have $f = \hat{g} \circ \beta$ where g is the restriction of f to B.*

Proof. In a continuous domain we have $x = \bigsqcup^{\uparrow} B_x$ for all elements, so $\sigma \circ \beta = \mathsf{id}_D$. Composing the maps the other way round we need to see that every $c \in B$ which approximates $\bigsqcup^{\uparrow} A$, where A is an ideal in $\langle B, \ll\rangle$, actually belongs to A. We interpolate: $c \ll d \ll \bigsqcup^{\uparrow} A$ and using the defining property of the order of approximation, we find $a \in A$ above d. Therefore c approximates a and belongs to A.

The calculation for (2) is straightforward: $f(x) = f(\bigsqcup^{\uparrow} B_x) = \bigsqcup^{\uparrow} f(B_x) = \hat{g}(B_x) = \hat{g}(\beta(x))$. ☐

Corollary 2.2.26. *A continuous function from a continuous domain D to a dcpo E is completely determined by its behavior on a basis of D.*

As we now know how to reconstruct a continuous domain from its basis and how to recover a continuous function from its restriction to the basis, we may wonder whether it is possible to work with bases alone. There is one further problem to overcome, namely, the fact that continuous functions do not preserve the order of approximation. The only way out is to switch from functions to relations, where we relate a basis element c to all basis elements approximating $f(c)$. This can be axiomatized as follows.

Definition 2.2.27. *A relation R between abstract bases B and C is called* approximable *if the following conditions are satisfied:*

1. *$\forall x \in B\ \forall y, y' \in C.\ (xRy \succ y' \Longrightarrow xRy')$;*
2. *$\forall x \in B\ \forall M \subseteq_{\mathit{fin}} C.\ (\forall y \in M.\ xRy \Longrightarrow (\exists z \in C.\ xRz$ and $z \succ M))$;*
3. *$\forall x, x' \in B\ \forall y \in C.\ (x' \succ xRy \Longrightarrow x'Ry)$;*
4. *$\forall x \in B\ \forall y \in C.\ (xRy \Longrightarrow (\exists z \in B.\ x \succ zRy))$.*

The following is then proved without difficulties.

Theorem 2.2.28. *The category of abstract bases and approximable relations is equivalent to* **CONT**, *the category of continuous dcpo's and continuous maps.*

The formulations we have chosen in this section allow us immediately to read off the corresponding results in the special case of algebraic domains. In particular:

Theorem 2.2.29. *The category of preorders and approximable relations is equivalent to* **ALG**, *the category of algebraic dcpo's and continuous maps.*

2.3 Topology

By a *topology* on a space X we understand a system of subsets of X (called the *open sets*), which is closed under finite intersections and infinite unions. It is an amazing fact that by a suitable choice of a topology we can encode all information about convergence, approximation, continuity of functions, and even points of X themselves. To a student of Mathematics this appears to be an immense abstraction from the intuitive beginnings of analysis. In domain theory we are in the lucky situation that we can tie up open sets with the concrete idea of observable properties. This has been done in detail earlier in this handbook, [Smy92], and we may therefore proceed swiftly to the mathematical side of the subject.

2.3.1 The Scott-topology on a dcpo

Definition 2.3.1. *Let D be a dcpo. A subset A is called* (Scott-)closed *if it is a lower set and is closed under suprema of directed subsets. Complements of closed sets are called* (Scott-)open*; they are the elements of σ_D, the* Scott-topology *on D.*

We shall use the notation $\mathsf{Cl}(A)$ for the smallest closed set containing A. Similarly, $\mathsf{Int}(A)$ will stand for the open kernel of A.

A Scott-open set O is necessarily an upper set. By contraposition it is characterized by the property that every directed set whose supremum lies in O has a non-empty intersection with O.

Basic examples of closed sets are principal ideals. This knowledge is enough to show the following:

Proposition 2.3.2. *Let D be a dcpo.*

1. *For elements $x, y \in D$ the following are equivalent:*
   - *(a)* $x \sqsubseteq y$,
   - *(b) Every Scott-open set which contains x also contains y,*
   - *(c)* $x \in \mathsf{Cl}(\{y\})$.
2. *The Scott-topology satisfies the T_0 separation axiom.*
3. *$\langle D, \sigma_D\rangle$ is a Hausdorff ($= \mathrm{T}_2$) topological space if and only if the order on D is trivial.*

Thus we can reconstruct the order between elements of a dcpo from the Scott-topology. The same is true for limits of directed sets.

Proposition 2.3.3. *Let A be a directed set in a dcpo D. Then $x \in D$ is the supremum of A if and only if it is an upper bound for A and every Scott-neighborhood of x contains an element of A.*

Proof. Indeed, the closed set $\downarrow \bigsqcup^{\uparrow} A$ separates the supremum from all other upper bounds of A. □

Proposition 2.3.4. *For dcpo's D and E, a function f from D to E is Scott-continuous if and only if it is topologically continuous with respect to the Scott-topologies on D and E.*

Proof. Let f be a continuous function from D to E and let O be an open subset of E. It is clear that $f^{-1}(O)$ is an upper set because continuous functions are monotone. If f maps the element $x = \bigsqcup^{\uparrow}_{i\in I} x_i \in D$ into O then we have $f(x) = f(\bigsqcup^{\uparrow}_{i\in I} x_i) = \bigsqcup^{\uparrow}_{i\in I} f(x_i) \in O$ and by definition there must be some x_i which is mapped into O. Hence $f^{-1}(O)$ is open in D.

For the converse assume that f is topologically continuous. We first show that f must be monotone: Let $x \sqsubseteq x'$ be elements of D. The inverse image of the Scott-closed set $\downarrow f(x')$ contains x'. Hence it also contains x. Now let $A \subseteq D$ be directed. Look at the inverse image of the Scott-closed set $\downarrow(\bigsqcup^{\uparrow}_{a\in A} f(a))$. It contains A and is Scott-closed, too. So it must also contain $\bigsqcup^{\uparrow} A$. Since by monotonicity $f(\bigsqcup^{\uparrow} A)$ is an upper bound of $f(A)$, it follows that $f(\bigsqcup^{\uparrow} A)$ is the supremum of $f(A)$. □

So much for the theme of convergence. Let us now proceed to see in how far approximation is reflected in the Scott-topology.

2.3.2 The Scott-topology on domains

In this subsection we work with the second-most primitive form of open sets, namely those which can be written as $\twoheaduparrow x$. We start by characterizing the order of approximation.

Proposition 2.3.5. *Let D be a continuous domain. Then the following are equivalent for all pairs $x, y \in D$:*

1. $x \ll y$,
2. $y \in \mathsf{Int}(\uparrow x)$,
3. $y \in \twoheaduparrow x$.

Comment: Of course, (1) is equivalent to (3) in *all* dcpos.

Proposition 2.3.6. *Let D be a continuous domain with basis B. Then openness of a subset O of D can be characterized in the following two ways:*

1. $O = \bigcup_{x\in O} \twoheaduparrow x$,
2. $O = \bigcup_{x\in O\cap B} \twoheaduparrow x$.

This can be read as saying that every open set is supported by its members from the basis. We may therefore ask how the Scott-topology is derived from an abstract basis.

Proposition 2.3.7. *Let $(B, \prec)$ be an abstract basis and let M be any subset of B. Then the set $\{A \in \mathsf{Idl}(B) \mid M \cap A \neq \emptyset\}$ is Scott-open in $\mathsf{Idl}(B)$ and all open sets on $\mathsf{Idl}(B)$ are of this form.*

This, finally, nicely connects the theory up with the idea of an observable property. If we assume that the elements of an abstract basis are finitely describable and finitely recognisable (and we strongly approve of this intuition) then it is clear how to observe a property in the completion: we have to wait until we see an element from a given set of basis elements.

We also have the following sharpening of Proposition 2.3.6:

Lemma 2.3.8. *Every Scott-open set in a continuous domain is a union of Scott-open filters.*

Proof. Let x be an element in the open set O. By Proposition 2.3.6 there is an element $y \in O$ which approximates x. We repeatedly interpolate between y and x. This gives us a sequence $y \ll \ldots \ll y_n \ll \ldots \ll y_1 \ll x$. The union of all $\uparrow y_n$ is a Scott-open filter containing x and contained in O. $\square$

In this subsection we have laid the groundwork for a formulation of Domain Theory purely in terms of the lattice of Scott-open sets. Since we construe open sets as properties we have also brought logic into the picture. This relationship will be looked at more closely in Chapter 7. There and in Section 4.2.3 we will also exhibit more properties of the Scott-topology on domains.

Exercises 2.3.9.

1. *Formalize the passage from preorders to their quotient posets.*
2. *Draw line diagrams of the powersets of a one, two, three, and four element set.*
3. *Show that a poset which has all suprema also has all infima, and vice versa.*
4. *Refine Proposition 2.1.7 by showing that the fixpoints of a monotone function on a complete lattice form a complete lattice. Is it a sublattice?*
5. *Show that finite directed sets have a largest element. Characterize the class of posets in which this is true for every directed set.*
6. *Show that the directed set of finite subsets of real numbers does not contain a cofinal chain.*
7. *Which of the following are dcpo's:* $\mathbb{R}$, $[0,1]$ *(unit interval),* $\mathbb{Q}$, $\mathbb{Z}^-$ *(negative integers)?*
8. *Let* f *be a monotone map between complete lattices* L *and* M *and let* A *be a subset of* L*. Prove:* $f(\bigsqcup A) \sqsupseteq \bigsqcup f(A)$.
9. *Show that the category of posets and monotone functions forms a cartesian closed category.*
10. *Draw the line diagram for the function space of the flat booleans (see Figure 1).*
11. *Show that an ideal in a (binary) product of posets can always be seen as the product of two ideals from the individual posets.*

12. *Show that a map f between two dcpo's D and E is continuous if and only if for all directed sets A in D, $f(\bigsqcup^{\uparrow} A) = \bigsqcup f(A)$ holds (i.e., monotonicity does not need to be required explicitly).*

13. *Give an example of a monotone map f on a pointed dcpo D for which $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} f^n(\bot)$ is not a fixpoint. (Some fixpoint must exist by Proposition 2.1.16.)*

14. *Use fixpoint induction to prove the following. Let $f, g\colon D \to D$ be continuous functions on a pointed dcpo D with $f(\bot) = g(\bot)$, and $f \circ g = g \circ f$. Then $\mathsf{fix}(f) = \mathsf{fix}(g)$.*

15. *(Dinaturality of fixpoints) Let D, E be pointed dcpo's and let $f\colon D \to E, g\colon E \to D$ be continuous functions. Prove*

$$\mathsf{fix}(g \circ f) = g(\mathsf{fix}(f \circ g)) \ .$$

16. *Show that Lemma 2.1.21 uniquely characterizes* fix *among all fixpoint operators.*

17. *Prove: Given pointed dcpo's D and E and a continuous function $f\colon D \times E \to E$ there is a continuous function $Y(f)\colon D \to E$ such that $Y(f) = f \circ \langle \mathsf{id}_D, Y(f)\rangle$ holds. (This is the general definition of a category having fixpoints.) How does Theorem 2.1.19 follow from this?*

18. *Show that each version of the natural numbers as shown in Figure 2 is an example of a countable dcpo whose function space is uncountable.*

19. *Characterize the order of approximation on the unit interval. What are the compact elements?*

20. *Show that in finite posets every element is compact.*

21. *Let L be the lattice of open sets of $\mathbb{Q}$, where $\mathbb{Q}$ is equipped with the ordinary metric topology. Show that no two non-empty open sets approximate each other. Conclude that L is not continuous.*

22. *Prove Proposition 2.2.10.*

23. *Extend Proposition 2.2.10 in the following way: For every finite subset M of a continuous dcpo D with basis B there exists $M' \subseteq B$, such that $x \mapsto x'$ is an order-isomorphism between M and M' and such that for all $x \in M$, the element x' belongs to B_x.*

24. *Prove Proposition 2.2.17.*

25. *Show that elements of an abstract basis, which approximate no other element, may be deleted without changing the ideal completion.*

26. *Show that if x is a non-compact element of a basis B for a continuous domain D then $B \setminus \{x\}$ is still a basis. (Hint: Use the interpolation property.)*

27. *The preceding exercise shows that different bases can generate the same domain. Show that for a fixed basis different orders of approximation may also yield the same domain. Show that this will definitely be the case if the two orders* $\prec_1$ *and* $\prec_2$ *satisfy the equations* $\prec_1 \circ \prec_2 = \prec_1$ *and* $\prec_2 \circ \prec_1 = \prec_2$.

28. *Consider Proposition 2.2.22(2). Give an example of an abstract basis* B *which shows that* $i(x) \ll i(y)$ *in* $\mathsf{Idl}(B)$ *does not entail* $x \prec y$.

29. *What is the ideal completion of* $\langle \mathbb{Q}, < \rangle$*?*

30. *Let* $\prec$ *be a relation on a set* B *such that* $\prec \circ \prec \, = \, \prec$ *holds. Give an example showing that Axiom (INT) (Definition 2.2.20) need not be satisfied. Nevertheless,* $\mathsf{Idl}(B, \prec)$ *is a continuous domain. What is the advantage of our axiomatization over this simpler concept?*

31. *Spell out the proof of Theorem 2.2.28.*

32. *Prove that in a dcpo every upper set is the intersection of its Scott-neighborhoods.*

33. *Show that in order to construct the Scott-closure of a lower set* A *of a continuous domain it is sufficient to add all suprema of directed subsets to* $\downarrow A$*. Give an example of a non-continuous dcpo where this fails.*

34. *Given a subset* X *in a dcpo* D *let* $\bar{X}$ *be the smallest superset of* X *which is closed against the formation of suprema of directed subsets. Show that the cardinality of* $\bar{X}$ *can be no greater than* $2^{|X|}$*. (Hint: Construct a directed suprema closed superset of* X *by adding* all *existing suprema to* X*.)*

3 Domains collectively

3.1 Comparing domains

3.1.1 Retractions

A reader with some background in universal algebra may already have missed a discussion of sub-dcpo's and quotient-dcpo's. The reason for this omission is quite simple: there is no fully satisfactory notion of sub-object or quotient in domain theory based on general Scott-continuous functions. And this is because the formation of directed suprema is a partial operation of unbounded arity. We therefore cannot hope to be able to employ the tools of universal algebra. But if we *combine* the ideas of sub-object and quotient then the picture looks quite nice.

Definition 3.1.1. *Let P and Q be posets. A pair $s\colon P \to Q$, $r\colon Q \to P$ of monotone functions is called a* monotone section retraction pair *if $r \circ s$ is the identity on P. In this situation we will call P a* monotone retract *of Q.*

If P and Q are dcpo's and if both functions are continuous then we speak of a continuous section retraction pair.

We will omit the qualifying adjective 'monotone', respectively 'continuous', if the properties of the functions are clear from the context. We will also use *s-r-pair* as a shorthand.

One sees immediately that in an s-r-pair the retraction is surjective and the section is injective, so our intuition about P being both a sub-object and a quotient of Q is justified. In such a situation P inherits many properties from Q:

Proposition 3.1.2. *Let P and Q be posets and let $s\colon P \to Q$, $r\colon Q \to P$ be a monotone section retraction pair.*

1. *Let A be any subset of P. If $s(A)$ has a supremum in Q then A has a supremum in P. It is given by $r(\bigsqcup s(A))$. Similarly for the infimum.*

2. *If Q is a (pointed) dcpo, a semilattice, a lattice or a complete lattice then so is P.*

Proof. Because of $r \circ s = \mathsf{id}_P$ and the monotonicity of r it is clear that $r(\bigsqcup s(A))$ is an upper bound for A. Let x be another such. Then by the monotonicity of s we have that $s(x)$ is an upper bound of $s(A)$ and hence it is above $\bigsqcup s(A)$. So we get $x = r(s(x)) \sqsupseteq r(\bigsqcup s(A))$.

The property of being a (pointed) dcpo, semilattice, etc., is defined through the existence of suprema or infima of certain subsets. The shape of these subsets is preserved by monotone functions and so (2) follows from (1). $\square$

Let us now turn to continuous section retraction pairs.

Lemma 3.1.3. *Let (s, r) be a continuous section retraction pair between dcpo's D and E and let B be a basis for E. Then $r(B)$ is a basis for D.*

Proof. Let $c \in B$ be an approximant to $s(x)$ for $x \in D$. We show that $r(c)$ approximates x. To this end let A be a directed subset of D with $\bigsqcup^{\uparrow} A \sqsupseteq x$. By the continuity of s we have $\bigsqcup^{\uparrow} s(A) = s(\bigsqcup^{\uparrow} A) \sqsupseteq s(x)$ and so for some $a \in A$, $s(a) \sqsupseteq c$ must hold. This implies $a = r(s(a)) \sqsupseteq r(c)$. The continuity of r gives us that x is the supremum of $r(B_{s(x)})$. $\square$

Theorem 3.1.4. *A retract of a continuous domain via a continuous s-r-pair is continuous.*

The analogous statement for algebraic domains does not hold in general. Instead of constructing a particular counterexample, we use our knowledge about the ideal completion to get a general, positive result which implies this negative one.

Theorem 3.1.5. *Every (ω-) continuous domain is the retract of an (ω-) algebraic domain via a continuous s-r-pair.*

In more detail, we have:

Proposition 3.1.6. *Let D be a continuous domain with basis B. Then the maps $s\colon D \to \mathsf{Idl}(B, \sqsubseteq), x \mapsto B_x$ and $r\colon \mathsf{Idl}(B, \sqsubseteq) \to D, A \mapsto \bigsqcup^{\uparrow} A$ constitute a continuous section retraction pair between D and $\mathsf{Idl}(B, \sqsubseteq)$.*

Proof. The continuity of r follows from general associativity, Proposition 2.1.4, and the fact that directed suprema in $\mathsf{Idl}(B)$ are directed unions. For the continuity of s we use the interpolation property in the form of Proposition 2.2.16(2). $\square$

3.1.2 Idempotents

Often the section part of an s-r-pair is really a subset inclusion. In this case we can hide it and work with the map $s \circ r$ on E alone. It is idempotent, because $(s \circ r) \circ (s \circ r) = s \circ (r \circ s) \circ r = s \circ r$.

Proposition 3.1.7.
1. *The image of a continuous idempotent map f on a dcpo D is a dcpo. The suprema of directed subsets of $\mathsf{im}(f)$, calculated in $\mathsf{im}(f)$, coincide with those calculated in D. The inclusion $\mathsf{im}(f) \to D$ is Scott-continuous.*
2. *The set of all continuous idempotent functions on a dcpo is again a dcpo.*

Proof. (1) The first part follows from Proposition 3.1.2 because the inclusion is surely monotone. For the second part let A be a directed set contained in $\mathsf{im}(f)$. We need to see that $\bigsqcup^{\uparrow} A$ belongs to $\mathsf{im}(f)$ again. This holds because f is continuous: $\bigsqcup^{\uparrow} A = \bigsqcup^{\uparrow} f(A) = f(\bigsqcup^{\uparrow} A)$.

(2) Let $(f_i)_{i \in I}$ be a directed family of continuous idempotents. For any $x \in D$ we

can calculate

$$\begin{aligned}
(\bigsqcup_{i\in I}^{\uparrow} f_i)\circ(\bigsqcup_{j\in I}^{\uparrow} f_j)(x) &= \bigsqcup_{i\in I}^{\uparrow} f_i(\bigsqcup_{j\in I}^{\uparrow} f_j(x))\\
&= \bigsqcup_{i\in I}^{\uparrow}\bigsqcup_{j\in I}^{\uparrow} f_i(f_j(x))\\
&= \bigsqcup_{i\in I}^{\uparrow} f_i(f_i(x))\\
&= \bigsqcup_{i\in I}^{\uparrow} f_i(x).
\end{aligned}$$

Hence the supremum of continuous idempotents is again an idempotent function. We have proved in Proposition 2.1.18 that it is also continuous. □

If f is a continuous idempotent map on a continuous domain D then we know that its image is again continuous. But it is *not* true that the order of approximation on $\mathsf{im}(f)$ is the restriction of the order of approximation on D. For example, every constant map is continuous and idempotent. Its image is an algebraic domain with one element, which is therefore compact. But surely not every element of a continuous domain is compact. However, we can say something nice about the Scott-topology on the image:

Proposition 3.1.8. *If f is a continuous idempotent function on a dcpo D then the Scott-topology on $\mathsf{im}(f)$ is the restriction of the Scott-topology on D to $\mathsf{im}(f)$.*

Proof. This follows immediately because a continuous idempotent function f gives rise to a continuous s-r-pair between $\mathsf{im}(f)$ and D. □

Useful examples of idempotent self-maps are retractions ret_x onto principal ideals. They are given by

$$\mathsf{ret}_x(y) = \begin{cases} y, & \text{if } y \sqsubseteq x;\\ x, & \text{otherwise.}\end{cases}$$

Their continuity follows from the fact that $\downarrow x$ is always Scott-closed. Dually, we can define a retraction onto a principal filter $\uparrow c$. It is Scott-continuous if (but not only if) its generator c is compact.

3.1.3 Adjunctions

An easy way to avoid writing this subsection would be to refer to category theory and to translate the general theory of adjoint functors into the poset setting. However, we feel that the right way to get used to the idea of adjointness is to start out with a relatively simple situation such as is presented by domain theory. (In fact, we will use adjoint functors later on, but really in a descriptive fashion only.)

Let us start with the example of a surjective map f from a poset Q onto a poset P. It is natural to ask whether there is a one-sided inverse $e\colon P \to Q$ for f, i.e. a map such that $f \circ e = \mathsf{id}_P$ holds. Figure 7 illustrates this situation. Such a map must

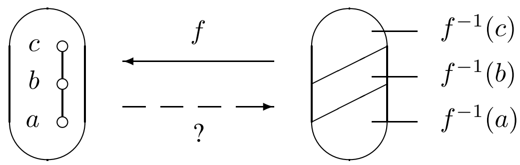


Figure 7: The right inverse problem for a surjective function

pick out a representative from $f^{-1}(x)$ for each $x \in P$. Set-theoretically this can be done, but the point here is that we want e to be monotone. If we succeed then e and f form a (monotone) section retraction pair. Even nicer would it be if we could pick out a canonical representative from $f^{-1}(x)$, which in the realm of order theory means that we want $f^{-1}(x)$ to have a least (or largest) element. If this is the case then how can we ensure that the assignment $e\colon x \mapsto \mathsf{min}(f^{-1}(x))$ is monotone? The solution is suggested by the observation that if e is monotone then $e(x)$ is not only the least element of $f^{-1}(x)$ but also of $f^{-1}(\uparrow x)$. This condition is also sufficient. The switch from $f^{-1}(x)$ to $f^{-1}(\uparrow x)$ (and this is a trick to remember) may allow us to construct a partial right inverse even if f is not surjective. Thus we arrive at a first, tentative definition of an adjunction.

Definition 3.1.9. (preliminary) *Let P and Q be posets and let $l\colon P \to Q$ and $u\colon Q \to P$ be monotone functions. We say that (l, u) is an* adjunction *between P and Q if for every $x \in P$ we have that $l(x)$ is the least element of $u^{-1}(\uparrow x)$.*

This definition is simple and easy to motivate. But it brings out just one aspect of adjoint pairs, namely, that l is uniquely determined by u. There is much more:

Proposition 3.1.10. *Let P and Q be posets and $l\colon P \to Q$ and $u\colon Q \to P$ be monotone functions. Then the following are equivalent:*

1. $\forall x \in P.\ l(x) = \mathsf{min}(u^{-1}(\uparrow x))$,
2. $\forall y \in Q.\ u(y) = \mathsf{max}(l^{-1}(\downarrow y))$,
3. $l \circ u \sqsubseteq \mathsf{id}_Q$ *and* $u \circ l \sqsupseteq \mathsf{id}_P$,
4. $\forall x \in P\ \forall y \in Q.\ (x \sqsubseteq u(y) \Leftrightarrow l(x) \sqsubseteq y)$.

(For (4)$\Longrightarrow$(1) the monotonicity of u and l is not needed.)

Proof. (1)$\Longrightarrow$(2) Pick an element $y \in Q$. Then because $u(y) \sqsubseteq u(y)$ we have from (1) that $l(u(y)) \sqsubseteq y$ holds. So $u(y)$ belongs to $l^{-1}(\downarrow y)$. Now let x' be any element of $l^{-1}(\downarrow y)$, or, equivalently, $l(x') \sqsubseteq y$. Using (1) again, we see that this can only happen if $u(y) \sqsupseteq x'$ holds. So $u(y)$ is indeed the largest element of $l^{-1}(\downarrow y)$. The converse is proved analogously, of course.

(1) and (2) together immediately give both (3) and (4).

From (3) we get (4) by applying the monotone map l to the inequality $x \sqsubseteq u(y)$ and using $l \circ u \sqsubseteq \mathsf{id}_Q$.

Assuming (4) we see immediately that $l(x)$ is a lower bound for $u^{-1}(\uparrow x)$. But because $l(x) \sqsubseteq l(x)$ and hence $x \sqsubseteq u(l(x))$ we have that $l(x)$ also belongs to $u^{-1}(\uparrow x)$. We get the monotonicity of l as follows: If $x \sqsubseteq x'$ holds in P then because $l(x') \sqsubseteq l(x')$ we have $x' \sqsubseteq u(l(x'))$ and by transitivity $x \sqsubseteq u(l(x'))$. Using (4) again, we get $l(x) \sqsubseteq l(x')$. □

We conclude that despite the lopsided definition, the situation described by an adjunction is completely symmetric. And indeed, adjunctions are usually introduced using either (3) or (4).

Definition 3.1.11. (official) *Let P and Q be posets and let $l\colon P \to Q$ and $u\colon Q \to P$ be functions. We say that (l, u) is an* adjunction *between P and Q if for all $x \in P$ and $y \in Q$ we have $x \sqsubseteq u(y) \Leftrightarrow l(x) \sqsubseteq y$. We call l the* lower *and u the* upper adjoint *and write $l\colon P \leftrightarrows Q : u$.*

Proposition 3.1.12. *Let $l\colon P \leftrightarrows Q : u$ be an adjunction between posets.*

1. *$u \circ l \circ u = u$ and $l \circ u \circ l = l$,*

2. *The image of u and the image of l are order-isomorphic. The isomorphisms are given by the restrictions of u and l to $\mathsf{im}(l)$ and $\mathsf{im}(u)$, respectively.*

3. *u is surjective $\Leftrightarrow u \circ l = \mathsf{id}_P \Leftrightarrow l$ is injective,*

4. *l is surjective $\Leftrightarrow l \circ u = \mathsf{id}_Q \Leftrightarrow u$ is injective,*

5. *l preserves existing suprema, u preserves existing infima.*

Proof. (1) We use Proposition 3.1.10(3) twice: $u = \mathsf{id}_P \circ u \sqsubseteq (u \circ l) \circ u = u \circ (l \circ u) \sqsubseteq u \circ \mathsf{id}_Q = u$.

(2) The equations from (1) say precisely that on the images of u and l, $u \circ l$ and $l \circ u$, respectively, act like identity functions.

(3) If u is surjective then we can cancel u on the right in the equation $u \circ l \circ u = u$ and get $u \circ l = \mathsf{id}_P$. From this it follows that l must be injective.

(5) Let $x = \bigsqcup A$ for $A \subseteq P$. By monotonicity, $l(x) \sqsupseteq l(a)$ for each $a \in A$. Conversely, let y be any upper bound of $l(A)$. Then $u(y)$ is an upper bound for each $u(l(a))$ which in turn is above a. So $u(y) \sqsupseteq \bigsqcup A = x$ holds and this is equivalent to $y \sqsupseteq l(x)$. □

The last property in the preceding proposition may be used to define an adjunction in yet another way, the only prerequisite being that there are enough sets with an infimum (or supremum). This is the Adjoint Functor Theorem for posets.

Proposition 3.1.13. *Let $f\colon L \to P$ be a monotone function from a complete lattice to a poset. Then the following are equivalent:*

1. *f preserves all infima,*

2. *f has a lower adjoint.*

And similarly: f preserves all suprema if and only if f has an upper adjoint.

Proof. We already know how to define a candidate for a lower adjoint g; we try $g(x) = \bigsqcap f^{-1}(\uparrow x)$. All that remains, is to show that $g(x)$ belongs to $f^{-1}(\uparrow x)$. This follows because f preserves meets: $f(g(x)) = f(\bigsqcap f^{-1}(\uparrow x)) = \bigsqcap f(f^{-1}(\uparrow x)) \sqsupseteq \bigsqcap \uparrow x = x$. $\square$

This proposition gives us a way of recognizing an adjoint situation in cases where only one function is explicitly given. It is then useful to have a notation for the missing mapping. We write f^* for the upper and f_* for the lower adjoint of f.

Now it is high time to come back to domains and see what all this means in our setting.

Proposition 3.1.14. *Let $l\colon D \leftrightarrows E : u$ be an adjunction between dcpo's.*

1. *l is Scott-continuous.*

2. *If u is Scott-continuous then l preserves the order of approximation.*

3. *If D is continuous then the converse of (2) is also true.*

Proof. Continuity of the lower adjoint follows from Proposition 3.1.12(5). So let $x \ll y$ be elements in D and let A be a directed subset of E such that $l(y) \sqsubseteq \bigsqcup^{\uparrow} A$ holds. This implies $y \sqsubseteq u(\bigsqcup^{\uparrow} A)$ and from the continuity of u we deduce $y \sqsubseteq \bigsqcup^{\uparrow} u(A)$. Hence some $u(a)$ is above x which, going back to E, means $l(x) \sqsubseteq a$.

(3) For the converse let A be any directed subset of E. Monotonicity of u yields $\bigsqcup^{\uparrow} u(A) \sqsubseteq u(\bigsqcup^{\uparrow} A)$. In order to show that the other inequality also holds, we prove that $\bigsqcup^{\uparrow} u(A)$ is above every approximant to $u(\bigsqcup^{\uparrow} A)$. Indeed, if $x \ll u(\bigsqcup^{\uparrow} A)$ we have $l(x) \ll l(u(\bigsqcup^{\uparrow} A)) \sqsubseteq \bigsqcup^{\uparrow} A$ by assumption. So some a is above $l(x)$ and for this a we have $x \sqsubseteq u(a) \sqsubseteq \bigsqcup^{\uparrow} u(A)$. $\square$

3.1.4 Projections and sub-domains

Let us now combine the ideas of Section 3.1.1 and 3.1.3.

Definition 3.1.15. *Let D and E be dcpo's and let $e\colon D \to E$ and $p\colon E \to D$ be continuous functions. We say that (e, p) is a* continuous embedding projection pair *(or* e-p-pair*) if $p \circ e = \mathsf{id}_D$ and $e \circ p \sqsubseteq \mathsf{id}_E$.*

We note that the section retraction pair between a continuous domain and its ideal completion as constructed in Section 3.1.1 is really an embedding projection pair.

From the general theory of adjunctions and retractions we already know quite a bit about e-p-pairs. The embedding is injective, p is surjective, e preserves existing suprema and the order of approximation, p preserves existing infima, D is continuous if E is continuous, and, finally, embeddings and projections uniquely determine each other. Because of this last property the term 'embedding' has a well-defined meaning; it is an injective function which has a Scott-continuous upper adjoint.

An injective lower adjoint also reflects the order of approximation:

Proposition 3.1.16. *Let $e\colon D \leftrightarrows E : p$ be an e-p-pair between dcpo's and let x and y be elements of D. Then $e(x) \ll e(y)$ holds in E if and only if x approximates y in D.*

Let us also look at the associated idempotent $e \circ p$ on E. As it is below the identity, it makes good sense to call such a function a *kernel operator*, but often such maps are just called *projections*. We denote the set of kernel operators on a dcpo D by $[D \xrightarrow{\downarrow} D]$. It is important to note that while a kernel operator preserves infima as a map from D to its image, it does *not* have any preservation properties as a map from D to D besides Scott-continuity. What we can say is summarized in the following proposition.

Proposition 3.1.17. *Let D be a dcpo.*

1. $[D \xrightarrow{\downarrow} D]$ *is a dcpo.*
2. *If p is a kernel operator on D then for all $x \in D$ we have that* $p(x) = \mathsf{max}\{y \in \mathsf{im}(p) \mid y \sqsubseteq x\}$.
3. *The image of a kernel operator is closed under existing suprema.*
4. $\ll_{\mathsf{im}(p)} = (\ll_D) \cap (\mathsf{im}(p) \times \mathsf{im}(p))$.
5. *For kernel operators p, p' on D we have $p \sqsubseteq p'$ if and only if* $\mathsf{im}(p) \subseteq \mathsf{im}(p')$.

Proof. (1) is proved as Proposition 3.1.7 and (2) follows because p together with the inclusion of $\mathsf{im}(p)$ into D form an adjunction. This also shows (4). Finally, (3) and (5) are direct consequences of (2). ☐

In the introduction we explained the idea that the order on a semantic domain models the relation of some elements being *better* than others, where—at least in semantics—'better' may be replaced more precisely by 'better termination'. Thus we view elements at the bottom of a domain as being less desirable than those higher up; they are 'proto-elements' from which fully developed elements evolve as we go up in the order. Now, the embedding part of an e-p-pair $e\colon D \leftrightharpoons E : p$ places D at the bottom of E. Following the above line of thought, we may think of D as being a collection of proto-elements from which the elements of E evolve. Because there is the projection part as well, every element of E exists in some primitive form in D already. Also, D contains some information about the order and the order of approximation on E. We may therefore think of D as a preliminary version of E, as an *approximation* to E on the domain level. This thought is made fruitful in Sections 4.2 and 5. Although the word does not convey the whole fabric of ideas, we name D a *sub-domain* of E, just in case there is an e-p-pair $e\colon D \leftrightharpoons E : p$.

3.1.5 Closures and quotient domains

The sub-domain relation is preeminent in domain theory but, of course, we can also combine retractions and adjunctions the other way around. Thus we arrive at *continuous insertion closure pairs* (*i-c-pairs*). Because adjunctions are not symmetric as far as the order of approximation is concerned, Proposition 3.1.14, the situation is not just the order dual of that of the previous subsection. We know that the insertion preserves existing infima and so on, but in addition we now have that the surjective part preserves the order of approximation and therefore, D is algebraic if E is.

The associated idempotent is called a *closure operator*. For closure operators the same caveat applies as for kernel operators; they need not preserve suprema. Worse, such functions do no longer automatically have a Scott-continuous (upper) adjoint. This is the price we have to pay for the algebraicity of the image. Let us formulate this precisely.

Proposition 3.1.18. *Let D be an algebraic domain and let $c\colon D \to D$ be a monotone idempotent function above* id_D*. Then* $\mathsf{im}(c)$ *is again an algebraic domain if and only if it is closed under directed suprema.*

The reader will no doubt recognize this statement as being a reformulation and generalization of our example of inductive closure systems from Chapter 2, Proposition 2.2.9. It is only consequent to call D a *quotient domain* of the continuous domain E if there exists an i-c-pair $e\colon D \rightleftharpoons E\colon c$.

3.2 Finitary constructions

In this section we will present a few basic ways of putting domains together so as to build up complicated structures from simple ones. There are three aspects of these constructions which we are interested in. The first one is simply the order-theoretic definition and the proof that we stay within dcpo's and Scott-continuous functions. The second one is the question how the construction can be described in terms of bases and whether the principle of approximation can be retained. The third one, finally, is the question of what universal property the construction has. This is the categorical viewpoint. Since this Handbook contains a chapter on category theory, [Poi92] (in particular, Chapter 2), we need not repeat here the arguments for why this is a fruitful and enlightening way of looking at these type constructors.

There are, however, several categories that we are interested in and a construction may play different roles in different settings. Let us therefore list the categories that, at this point, seem suitable as a *universe of discourse*. There is, first of all, **DCPO**, the category of dcpo's and Scott-continuous functions as introduced in Section 2.1. We can restrict the objects by taking only continuous or, more special, algebraic domains. Thus we arrive at the full subcategories **CONT** and **ALG** of **DCPO**. Each of these may be further restricted by requiring the objects to have a bottom element (and Theorem 2.1.19 tells us why one would be interested in doing so) resulting in the categories $\mathbf{DCPO}_\perp$, $\mathbf{CONT}_\perp$, and $\mathbf{ALG}_\perp$. The presence of a distinguished point in each object suggests that morphisms should preserve them. But this is not really appropriate in semantics; strict functions are tied to a particular evaluation strategy. For us this means that there is yet another cascade of categories, $\mathbf{DCPO}_{\perp!}$, $\mathbf{CONT}_{\perp!}$, and $\mathbf{ALG}_{\perp!}$, where objects have bottom elements and morphisms are strict and Scott-continuous. Finally, we may bound the size of (minimal) bases for continuous and algebraic domains to be countable. We indicate this by the prefix 'ω-'.

3.2.1 Cartesian product

Definition 3.2.1. *The* cartesian product *of two dcpo's D and E is given by the following data:*

$$D \times E = \{\langle x, y\rangle \mid x \in D, y \in E\},$$
$$\langle x, y\rangle \sqsubseteq \langle x', y'\rangle \textit{ if and only if } x \sqsubseteq x' \textit{ and } y \sqsubseteq y'.$$

This is just the usual product of sets, augmented by the coordinatewise order. Through induction, we can define the cartesian product for finite non-empty collections of dcpo's. For the product over the empty index set we define the result to be a fixed one-element dcpo $\mathbb{I}$.

Proposition 3.2.2. *The cartesian product of dcpo's is a dcpo. Suprema and infima are calculated coordinatewise.*

With each product $D \times E$ there are associated two projections:

$$\pi_1 \colon D \times E \to D \text{ and } \pi_2 \colon D \times E \to E.$$

These projections are always surjective but they are upper adjoints only if D and E are pointed. So there is a slight mismatch with Section 3.1.4 here. Given a dcpo F and continuous functions $f \colon F \to D$ and $g \colon F \to E$, we denote the mediating morphism from F to $D \times E$ by $\langle f, g\rangle$. It maps $x \in F$ to $\langle f(x), g(x)\rangle$.

Proposition 3.2.3. *Projections and mediating morphisms are continuous.*

If $f \colon D \to D'$ and $g \colon E \to E'$ are Scott-continuous, then so is the mediating map $\langle f \circ \pi_1, g \circ \pi_2\rangle \colon D \times E \to D' \times E'$. The common notation for it is $f \times g$. Since our construction is completely explicit, we have thus defined a functor in two variables on **DCPO**.

Proposition 3.2.4. *Let D and E be dcpo's.*

1. *A tuple $\langle x, y\rangle$ approximates a tuple $\langle x', y'\rangle$ in $D \times E$ if and only if x approximates x' in D and y approximates y' in E.*
2. *If B and B' are bases for D and E, respectively, then $B \times B'$ is a basis for $D \times E$.*
3. *$D \times E$ is continuous if and only if D and E are.*
4. *$\mathsf{K}(D \times E) = \mathsf{K}(D) \times \mathsf{K}(E)$.*

The categorical aspect of the cartesian product is quite pleasing; it is a categorical product in each case. But we can say even more.

Lemma 3.2.5. *Let* **C** *be a full subcategory of* **DCPO** *or* $\mathbf{DCPO}_{\perp!}$ *which has finite products. Then these are isomorphic to the cartesian product.*

In a restricted setting this was first observed in [Smy83a]. The general proof may be found in [Jun89].

A useful property which does not follow from general categorical or topological considerations, is the following.

Lemma 3.2.6. *A function $f\colon D \times E \to F$ is continuous if and only if it is continuous in each variable separately.*

Proof. Assume $f\colon D \times E \to F$ is separately continuous. Then f is monotone, because given $(x, y) \sqsubseteq (x', y')$ we can fill in (x, y') and use coordinatewise monotonicity twice. The same works for continuity: if $A \subseteq D \times E$ is directed then

$$\begin{aligned}
\bigsqcup^{\uparrow}_{(x,y)\in A} f(x,y) &= \bigsqcup^{\uparrow}_{x\in\pi_1(A)} \bigsqcup^{\uparrow}_{y\in\pi_2(A)} f(x,y) \\
&= \bigsqcup^{\uparrow}_{x\in\pi_1(A)} f(x, \bigsqcup^{\uparrow}_{y\in\pi_2(A)} y) \\
&= f(\bigsqcup^{\uparrow}_{x\in\pi_1(A)} x, \bigsqcup^{\uparrow}_{y\in\pi_2(A)} y) \\
&= f(\bigsqcup^{\uparrow} A).
\end{aligned}$$

This proves the interesting direction. $\square$

3.2.2 Function space

We have introduced the function space in Section 2.1.6 already. It consists of all continuous functions between two dcpo's ordered pointwise. We know that this is again a dcpo. The first morphism which is connected with this construction is $\mathsf{apply}\colon [D \longrightarrow E] \times D \to E$, $\langle f, x\rangle \mapsto f(x)$. It is continuous because it is continuous in each argument separately: in the first because directed suprema of functions are calculated pointwise, in the second, because $[D \longrightarrow E]$ contains only continuous functions.

The second standard morphism is the operation which rearranges a function of two arguments into a combination of two unary functions. That is, if f maps $D \times E$ to F, then $\mathsf{Curry}(f)\colon D \to [E \longrightarrow F]$ is the mapping which assigns to $d \in D$ the function which assigns to $e \in E$ the element $f(d, e)$. $\mathsf{Curry}(f)$ is a continuous function because of Lemma 3.2.6. And for completely general reasons we have that Curry itself is a continuous operation from $[D \times E \longrightarrow F]$ to $[D \longrightarrow [E \longrightarrow F]]$. Another derived operation is composition which is a continuous operation from $[D \longrightarrow E] \times [E \longrightarrow F]$ to $[D \longrightarrow F]$.

All this shows that the continuous function space is the exponential in **DCPO**. Taking cartesian products and function spaces together we have shown that **DCPO** is cartesian closed.

We turn the function space construction into a functor from $\mathbf{DCPO}^{op} \times \mathbf{DCPO}$ to **DCPO** by setting $[\cdot \longrightarrow \cdot](f, g)(h) = g \circ h \circ f$, where $f\colon D' \to D$, $g\colon E \to E'$ and h is an element of $[D \longrightarrow E]$.

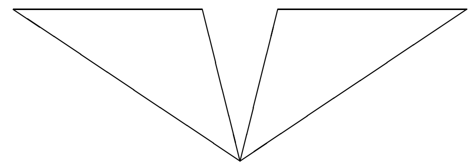


Figure 8: The coalesced sum of two pointed dcpo's.

As for the product we can show that the choice of the exponential object is more or less forced upon us. This again was first noticed by Smyth in the above mentioned reference.

Lemma 3.2.7. *Let* **C** *be a cartesian closed full subcategory of* **DCPO**. *The exponential of two objects D and E of* **C** *is isomorphic to $[D \longrightarrow E]$.*

Let us now turn to the theme of approximation in function spaces. The reader should brace himself for a profound disappointment: Even for algebraic domains it may be the case that the order of approximation on the function space is empty! (Exercise 3.3.12(11) discusses an example.) This fact together with Lemmas 3.2.5 and 3.2.7 implies that neither **CONT** nor **ALG** are cartesian closed. The only way to remedy this situation is to move to more restricted classes of domains. This will be the topic of Chapter 4.

3.2.3 Coalesced sum

In the category of sets the coproduct is given by disjoint union. This works equally well for dcpo's and there isn't really anything interesting to prove about it. We denote it by $D \dot{\cup} E$.

Disjoint union, however, destroys the property of having a least element and this in turn is indispensable in proving that every function has a fixpoint, Theorem 2.1.19. One therefore looks for substitutes for disjoint union which retain pointedness, but, of course, one cannot expect a clean categorical characterization such as for cartesian product or function space. (See also Exercise 3.3.12(12).) In fact, it has been shown in [HP90] that we cannot have cartesian closure, the fixpoint property and coproducts in a non-degenerate category.

So let us now restrict attention to pointed dcpo's. One way of putting a family of them together is to identify their bottom elements. This is called the *coalesced sum* and denoted $D \oplus E$. Figure 8 illustrates this operation. Elements from $D \oplus E$ different from $\bot_{D \oplus E}$ carry a label which indicates where they came from. We write them in the form $(x\colon i), i \in \{1, 2\}$.

Proposition 3.2.8. *The coalesced sum of pointed dcpo's is a pointed dcpo.*

The individual dcpo's may be injected into the sum in the obvious way:

$$\mathsf{inl}(x) = \begin{cases} (x\colon 1), & x \neq \bot_D; \\ \bot_{D\oplus E}, & x = \bot_D \end{cases} ;$$

and

$$\mathsf{inr}(x) = \begin{cases} (x\colon 2), & x \neq \bot_E; \\ \bot_{D\oplus E}, & x = \bot_E \end{cases} .$$

A universal property for the sum holds only in the realm of strict functions:

Proposition 3.2.9. *The coalesced sum of pointed dcpo's is the coproduct in* $\mathbf{DCPO}_{\bot!}$, $\mathbf{CONT}_{\bot!}$*, and* $\mathbf{ALG}_{\bot!}$.

Once we accept the restriction to bottom preserving functions it is clear how to turn the coalesced sum into a functor.

3.2.4 Smash product and strict function space

It is clear that inside $\mathbf{DCPO}_{\bot!}$ a candidate for the exponential is not the full function space but rather the set $[D \xrightarrow{\bot!} E]$ of strict continuous functions from D to E. However, it does not harmonize with the product in $\mathbf{DCPO}_{\bot!}$, which, as we have seen, must be the cartesian product. We do get a match if we consider the so-called *smash product*. It is defined like the cartesian product but all tuples which contain at least one bottom element are identified. Common notation is $D \otimes E$.

We leave it to the reader to check that smash product and strict function space turn $\mathbf{DCPO}_{\bot!}$ into a monoidal closed category.

3.2.5 Lifting

Set-theoretically, *lifting* is the simple operation of adding a new bottom element to a dcpo. Applied to D, the resulting structure is denoted by $D_\bot$. Clearly, continuity or algebraicity don't suffer any harm from this.

Associated with it is the map $\mathsf{up}\colon D \to D_\bot$ which maps each $x \in D$ to its namesake in $D_\bot$.

The categorical significance of lifting stems from the fact that it is left adjoint to the inclusion functor from $\mathbf{DCPO}_{\bot!}$ into $\mathbf{DCPO}$. (Where a morphism $f\colon D \to E$ is lifted by mapping the new bottom element of $D_\bot$ to the new bottom element of $E_\bot$.)

3.2.6 Summary

For quick reference let us compile a table of the constructions looked at so far. A '✓' indicates that the category is closed under the respective construction, a '+' says that, in addition, the construction plays the expected categorical role as a product, exponential or coproduct, respectively. Observe that for the constructions considered in this section it makes no difference if we restrict the size of a (minimal) basis.

	DCPO	$\text{DCPO}_\perp$	$\text{DCPO}_{\perp!}$	CONT ALG	$\text{CONT}_\perp$ $\text{ALG}_\perp$	$\text{CONT}_{\perp!}$ $\text{ALG}_{\perp!}$
$D \times E$	$+$	$+$	$+$	$+$	$+$	$+$
$[D \longrightarrow E]$	$+$	$+$	✓			
$D \dot{\cup} E$	$+$			$+$		
$D \otimes E$		✓	✓		✓	✓
$[D \stackrel{\perp!}{\longrightarrow} E]$		✓	$+$			
$D \oplus E$		✓	$+$		✓	$+$
$D_\perp$	✓	✓	✓	✓	✓	✓

3.3 Infinitary constructions

The product and sum constructions from the previous section have infinitary counterparts. Generally, these work nicely as long as we are only concerned with questions of convergence, but they cause problems with respect to the order of approximation. This is exemplified by the fact that an infinite power of a finite poset may fail to be algebraic. In any case, there is not much use of these operations in semantics. Much more interesting is the idea of incrementally building up a domain in a limit process. This is the topic of this section.

3.3.1 Limits and colimits

Our limit constructions are to be understood categorically and hence we refer once more to [Poi92] for motivation and general results. Here are the basic definitions. A *diagram* in a category **C** is given by a functor from a small category **I** to **C**. We can describe, somewhat sloppily but more concretely, a diagram by a pair $\langle (D_i)_{i\in O}, (f_j\colon D_{d(j)} \to D_{c(j)})_{j\in M}\rangle$ of a family of objects and a family of *connecting morphisms*. The shape of the diagram is thus encoded in the index sets O (which correspond to the objects of **I**) and M (which correspond to the morphisms of **I**) and in the maps $c, d\colon M \to O$ which corresponds to the dom and codom map on **I**. What is lost is the information about composition in **I**. In the cases which interest us, this is not a problem. A *cone* over such a diagram is given by an object D and a family $(f_i\colon D \to D_i)_{i\in O}$ of morphisms such that for all $j \in M$ we have $f_j \circ f_{d(j)} = f_{c(j)}$. A cone is *limiting* if for every other cone $\langle E, (g_i)_{i\in O}\rangle$ there is exactly one morphism $f\colon E \to D$ such that for all $i \in O$, $g_i = f_i \circ f$. If $\langle D, (f_i)_{i\in O}\rangle$ is a limiting cone, then D is called *limit* and the f_i are called *limiting morphisms*. The dual notions are *cocone*, *colimit*, and *colimiting morphism*.

Theorem 3.3.1. **DCPO** *has limits of arbitrary diagrams.*

Proof. The proof follows general category theoretic principles. We describe the limit of the diagram $\langle (D_i)_{i\in O}, (f_j\colon D_{d(j)} \to D_{c(j)})_{j\in M}\rangle$ as a set of particular elements of the product of all D_i's, the so-called *commuting tuples*.

$$D = \{\langle x_i : i \in O\rangle \in \prod_{i\in O} D_i \mid \forall j \in M.\ x_{c(j)} = f_j(x_{d(j)})\}$$

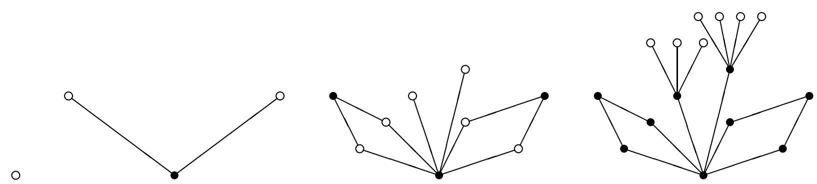


Figure 9: An expanding sequence of finite domains.

The order on the limit object is inherited from the product, that is, tuples are ordered coordinatewise. It is again a dcpo because the coordinatewise supremum of commuting tuples is commuting as all f_j are Scott-continuous. This also proves that the projections $\pi_j\colon \prod_{i\in O} D_i \to D_j$ restricted to D are continuous. They give us the maps needed to complement D to a cone.

Given any other cone $\langle E, (g_i\colon E \to D_i)_{i\in O}\rangle$, we define the mediating morphism $h\colon E \to D$ by $h(x) = \langle g_i(x) : i \in O\rangle$. Again, it is obvious that this is well-defined and continuous, and that it is the only possible choice. □

We also have the dual:

Theorem 3.3.2. **DCPO** *has colimits of arbitrary diagrams.*

This was first noted in [Mar77] and, for a somewhat different setting, in [Mes77]. The simplest way to prove it is by reducing it to completeness à la Theorem 23.14 of [HS73]. This appears in [LS81]. A more detailed analysis of colimits appears in [Fie92]. There the problem of retaining algebraicity is also addressed.

Theorem 3.3.3. **DCPO** *is cartesian closed, complete and cocomplete.*

Theorem 3.3.4. $\mathbf{DCPO}_{\perp!}$ *is monoidal closed, complete and cocomplete.*

How about $\mathbf{DCPO}_{\perp}$, where objects have least elements but morphisms need not preserve them? The truth is that both completeness and cocompleteness fail for this category. On the other hand, it is the right setting for denotational semantics in most cases. As a result of this mismatch, we quite often must resort to detailed proofs on the element level and cannot simply apply general category theoretic principles. Let us nevertheless write down the one good property of $\mathbf{DCPO}_{\perp}$:

Theorem 3.3.5. $\mathbf{DCPO}_{\perp}$ *is cartesian closed.*

3.3.2 The limit-colimit coincidence

The theorems of the previous subsection fall apart completely if we pass to domains, that is, to **CONT** or **ALG**. To get better results for limits and colimits we must restrict both the shape of the diagrams and the connecting morphisms used.

For motivation let us look at a chain $D_1, D_2, \ldots$ of domains where each D_n is a sub-domain of D_{n+1} in the sense of Section 3.1.4. Taking up again the animated language from that section we may think of the points of D_{n+1} as growing out of points of D_n, the latter being the buds which contain the leaves and flowers to be seen at later stages. Figure 9 shows a, hopefully inspiring, example. Intuition suggests that in such a situation a well-structured limit can be found by adding limit points to the union of the D_n, and that it will be algebraic if the D_n are.

Definition 3.3.6. *A diagram $\langle (D_n)_{n\in\mathbb{N}}, (e_{mn}\colon D_n \to D_m)_{n\leq m\in\mathbb{N}}\rangle$ in the category* **DCPO** *is called an* expanding sequence*, if the following conditions are satisfied:*

1. *Each $e_{mn}\colon D_n \to D_m$ is an embedding. (The associated projection e^*_{mn} we denote by p_{nm}.)*
2. *$\forall n \in \mathbb{N}.\ e_{nn} = \mathsf{id}_{D_n}$.*
3. *$\forall n \leq m \leq k \in \mathbb{N}.\ e_{kn} = e_{km} \circ e_{mn}$.*

Note that because lower adjoints determine upper adjoints and vice versa, we have $p_{nk} = p_{nm} \circ p_{mk}$ whenever $n \leq m \leq k \in \mathbb{N}$.

It turns out that, in contrast to the general situation, the colimit of an expanding sequence can be calculated easily via the associated projections.

Theorem 3.3.7. *Let $\langle (D_n)_{n\in\mathbb{N}}, (e_{mn}\colon D_n \to D_m)_{n\leq m\in\mathbb{N}}\rangle$ be an expanding sequence in* **DCPO***. Define*

$$D = \{\langle x_n : n \in \mathbb{N}\rangle \in \textstyle\prod_{n\in\mathbb{N}} D_n \mid \forall n \leq m \in \mathbb{N}.\ x_n = p_{nm}(x_m)\},$$
$$p_m\colon D \to D_m, \langle x_n : n \in \mathbb{N}\rangle \mapsto x_m, m \in \mathbb{N},$$
$$e_m\colon D_m \to D, x \mapsto \langle \textstyle\bigsqcup^{\uparrow}_{k\sqsupseteq n,m} p_{nk} \circ e_{km}(x) : n \in \mathbb{N}\rangle, m \in \mathbb{N}\,.$$[3]

Then

1. *The maps (e_m, p_m), $m \in \mathbb{N}$, form embedding projection pairs and $\bigsqcup^{\uparrow}_{m\in\mathbb{N}} e_m \circ p_m = \mathsf{id}_D$ holds.*
2. *$\langle D, (p_n)_{n\in\mathbb{N}}\rangle$ is a limit of the diagram $\langle (D_n)_{n\in\mathbb{N}}, (p_{nm})_{n\leq m\in\mathbb{N}}\rangle$. If $\langle C, (g_n)_{n\in\mathbb{N}}\rangle$ is another cone, then the mediating map from C to D is given by $g(x) = \langle g_n(x) : n \in \mathbb{N}\rangle$ or $g = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} e_n \circ g_n$.*
3. *$\langle D, (e_n)_{n\in\mathbb{N}}\rangle$ is a colimit of the diagram $\langle (D_n)_{n\in\mathbb{N}}, (e_{mn})_{n\leq m\in\mathbb{N}}\rangle$. If $\langle E, (f_n)_{n\in\mathbb{N}}\rangle$ is another cocone, then the mediating map from D to E is given by $f(\langle x_n : n \in \mathbb{N}\rangle) = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} f_n(x_n)$ or $f = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} f_n \circ p_n$.*

Proof. We have already shown in Theorem 3.3.1 that a limit of the diagram $\langle (D_n), (p_{nm})\rangle$ is given by $\langle D, (p_n)\rangle$ and that the mediating morphism has the (first) postulated form.

[3]he directed supremum $\bigsqcup^{\uparrow}_{k\sqsupseteq n,m} p_{nk} \circ e_{km}(x)$ in the definition of e_m could be replaced by $p_{nk} \circ e_{km}(x)$ for *any* upper bound k of $\{n, m\}$ in $\mathbb{N}$. However, this would actually make the proofs more cumbersome to write down.

For the rest, let us start by showing that the functions e_m are well-defined, i.e. that $y = e_m(x)$ is a commuting tuple. Assume $n \leq n'$. Then we have $p_{nn'}(y_{n'}) = p_{nn'}(\bigsqcup^{\uparrow}_{k \sqsupseteq n',m} p_{n'k} \circ e_{km}(x)) = \bigsqcup^{\uparrow}_{k \sqsupseteq n',m} p_{nn'} \circ p_{n'k} \circ e_{km}(x) = \bigsqcup^{\uparrow}_{k \sqsupseteq n',m} p_{nk} \circ e_{km}(x) = y_n$. The assignment $x \mapsto e_m(x)$ is Scott-continuous because of general associativity and because only Scott-continuous maps are involved in the definition.

Next, let us now check that e_m and p_m form an e-p-pair.

$$\begin{array}{rcl} e_m \circ p_m(\langle x_n : n \in \mathbb{N} \rangle) & = & e_m(x_m) \\ & = & \langle \bigsqcup^{\uparrow}_{k \sqsupseteq n,m} p_{nk} \circ e_{km}(x_m) : n \in \mathbb{N} \rangle \\ & = & \langle \bigsqcup^{\uparrow}_{k \sqsupseteq n,m} p_{nk} \circ e_{km} \circ p_{mk}(x_k) : n \in \mathbb{N} \rangle \\ & \sqsubseteq & \langle \bigsqcup^{\uparrow}_{k \sqsupseteq n,m} p_{nk}(x_k) : n \in \mathbb{N} \rangle \\ & = & \langle x_n : n \in \mathbb{N} \rangle \end{array}$$

and $p_m \circ e_m(x) = p_m(\langle \bigsqcup^{\uparrow}_{k \sqsupseteq n,m} p_{nk} \circ e_{km}(x) : n \in \mathbb{N} \rangle) = \bigsqcup^{\uparrow}_{k \sqsupseteq m} p_{mk} \circ e_{km}(x) = x$.

A closer analysis reveals that $e_m \circ p_m$ will leave all those elements of the tuple $\langle x_n : n \in \mathbb{N} \rangle$ unchanged for which $n \leq m$:

$$\begin{array}{rcl} p_n(e_m \circ p_m(\langle x_n : n \in \mathbb{N} \rangle)) & = & \ldots = \displaystyle\bigsqcup^{\uparrow}_{k \geq n,m} p_{nk} \circ e_{km} \circ p_{mk}(x_k) \\ & = & \displaystyle\bigsqcup^{\uparrow}_{k \geq n,m} p_{nm} \circ p_{mk} \circ e_{km} \circ p_{mk}(x_k) \\ & = & \displaystyle\bigsqcup^{\uparrow}_{k \geq n,m} p_{nm} \circ p_{mk}(x_k) = \bigsqcup^{\uparrow}_{k \geq n,m} x_n = x_n \end{array}$$

This proves that the $e_m \circ p_m, m \in \mathbb{N}$, add up to the identity, as stated in (1). Putting this to use, we easily get the second representation for the mediating map into D viewed as a limit: $g = \mathsf{id} \circ g = \bigsqcup^{\uparrow}_{m \in \mathbb{N}} e_m \circ p_m \circ g = \bigsqcup^{\uparrow}_{m \in \mathbb{N}} e_m \circ g_m$.

It remains to prove the universal property of D as a colimit. To this end let $\langle E, (f_n)_{n \in \mathbb{N}} \rangle$ be a cocone over the expanding sequence. We have to check that $f = \bigsqcup^{\uparrow}_{n \in \mathbb{N}} f_n \circ p_n$ is well-defined in the sense that the supremum is over a directed set. So let $n \leq m$. We get $f_n \circ p_n = f_m \circ e_{mn} \circ p_{nm} \circ p_m \sqsubseteq f_m \circ p_m$. It commutes with the colimiting maps because

$$\begin{array}{rcl} f \circ e_m & = & \displaystyle\bigsqcup^{\uparrow}_{n \geq m} f_n \circ p_n \circ e_m \\ & = & \displaystyle\bigsqcup^{\uparrow}_{n \geq m} f_n \circ p_n \circ e_n \circ e_{nm} \\ & = & \displaystyle\bigsqcup^{\uparrow}_{n \geq m} f_n \circ e_{nm} = \bigsqcup^{\uparrow}_{n \geq m} f_m = f_m \end{array}$$

We also have to show that there is no other choice for f. Again the equation in (1) comes in handy: Let f' be any mediating morphism. It must satisfy $f' \circ e_m = f_m$ and so $f' \circ e_m \circ p_m = f_m \circ p_m$. Forming the supremum on both sides gives $f' = \bigsqcup^{\uparrow}_{m \in \mathbb{N}} f_m \circ p_m$ which is the definition of f. $\square$

This fact, that the colimit of an expanding sequence is canonically isomorphic to the limit of the associated dual diagram, is called the *limit-colimit coincidence*. It is one of the fundamental tools of domain theory and plays its most prominent role in the solution of recursive domain equations, Chapter 5. Because of this coincidence we will henceforth also speak of the *bilimit* of an expanding sequence and denote it by $\mathsf{bilim}\langle(D_n),(e_{mn})\rangle$.

We can generalize Theorem 3.3.7 in two ways; we can replace $\mathbb{N}$ by an arbitrary directed set (in which case we will speak of an *expanding system*) and we can use general Scott-continuous adjunctions instead of e-p-pairs. The first generalization is harmless and does not need any serious adjustments in the proofs. We will freely use it from now on. The second, on the other hand, is quite interesting. By the passage from embeddings to, no longer injective, lower adjoints, we allow domains not only to grow but also to shrink as we move on in the index set. Thus points, which at some stage looked different, may at a later stage be recognised to be the same. The interested reader will find an outline of the mathematical theory of this in the exercises. For the main text, we must remind ourselves that this generalization has so far not found any application in semantics.

Part (1) of the preceding theorem gives a characterization of bilimits:

Lemma 3.3.8. *Let* $\langle E,(f_n)_{n\in\mathbb{N}}\rangle$ *be a cocone for the expanding sequence* $\langle(D_n)_{n\in\mathbb{N}},(e_{mn}\colon D_n\to D_m)_{n\le m\in\mathbb{N}}\rangle$. *It is colimiting if and only if, firstly, there are Scott-continuous functions* $g_n\colon E\to D_n$ *such that each* (f_n,g_n) *is an e-p-pair and, secondly,* $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} f_n\circ g_n = id_E$ *holds.*

Proof. Necessity is Part (1) of Theorem 3.3.7. For sufficiency we show that the bilimit D as constructed there, is isomorphic to E. We already have maps $f\colon D\to E$ and $g\colon E\to D$ because D is the bilimit. These commute with the limiting and the colimiting morphisms, respectively. So let us check that they compose to identities:

$$\begin{aligned} f\circ g(x) &= f(\langle g_n(x) : n\in\mathbb{N}\rangle)\\ &= \bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} f_n\circ g_n(x)\\ &= x \end{aligned}$$

and

$$\begin{aligned} g\circ f &= (\bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} e_n\circ g_n)\circ(\bigsqcup_{m\in\mathbb{N}}{}^{\uparrow} f_m\circ p_m)\\ &= \bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} e_n\circ g_n\circ f_n\circ p_n\\ &= \bigsqcup_{n\in\mathbb{N}}{}^{\uparrow} e_n\circ p_n = \mathsf{id}_D. \end{aligned}$$

$\square$

We note that in the proof we have used the condition $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} f_n\circ g_n = \mathsf{id}_E$ only for the first calculation. Without it, we still get that f and g form an e-p-pair. Thus we have:

Proposition 3.3.9. *Let $\langle E, (f_n)_{n\in\mathbb{N}}\rangle$ be a cocone over the expanding sequence $\langle (D_n)_{n\in\mathbb{N}}, (e_{mn}\colon D_n \to D_m)_{n\leq m\in\mathbb{N}}\rangle$ where the f_n are embeddings. Then the bilimit of the sequence is a sub-domain of E.*

In other words:

Corollary 3.3.10. *The bilimit of an expanding sequence is also the colimit (limit) in the restricted category of dcpo's with embeddings (projections) as morphisms.*

3.3.3 Bilimits of domains

Theorem 3.3.11. *Let $\langle (D_n)_{n\in\mathbb{N}}, (e_{mn}\colon D_n \to D_m)_{n\leq m\in\mathbb{N}}\rangle$ be an expanding sequence and $\langle D, (e_n)_{n\in\mathbb{N}}\rangle$ its bilimit.*

1. *If all D_n are (ω-)continuous then so is D. If we are given bases $B^n, n \in \mathbb{N}$ for each D_n then a basis for D is given by $\bigcup_{n\in\mathbb{N}} e_n(B^n)$.*

2. *If all D_n are (ω-)algebraic then so is D and $\mathsf{K}(D) = \bigcup_{n\in\mathbb{N}} e_n(\mathsf{K}(D_n))$.*

Proof. Given an element $x \in D$ we first show that $\bigcup_{n\in\mathbb{N}} e_n(B^n_{p_n(x)})$ is directed. To this end it is sufficient to show that for all $n \leq m \in \mathbb{N}$ and for each $y \in B^n_{p_n(x)}$ there is $y' \in B^m_{p_m(x)}$ with $e_n(y) \sqsubseteq e_m(y')$. Well, because y approximates $p_n(x)$ and because embeddings preserve the order of approximation, we have $e_{mn}(y) \ll e_{mn}(p_n(x)) = e_{mn}(p_{nm} \circ p_m(x)) \sqsubseteq p_m(x)$. Since $p_m(x) = \bigsqcup^{\uparrow} B^m_{p_m(x)}$, some $y' \ll p_m(x)$ is above $e_{mn}(y)$. This implies $e_n(y) = e_m(e_{mn}(y)) \sqsubseteq e_m(y')$.

The set $\bigcup_{n\in\mathbb{N}} e_n(B^n_{p_n(x)})$ gives back x because $x = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} e_n \circ p_n(x) = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} e_n(\bigsqcup^{\uparrow} B^n_{p_n(x)}) = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} \bigsqcup^{\uparrow} e_n(B^n_{p_n(x)}) = \bigsqcup^{\uparrow} \bigcup e_n(B^n_{p_n(x)})$. It consists solely of approximants to x because the e_n are lower adjoints. $\square$

Exercises 3.3.12.

1. *Let D be a continuous domain and let $f\colon D \to D$ be an idempotent Scott-continuous function. Show that $f(x) \ll f(y)$ holds in the image of f if and only if there exists $z \ll f(y)$ in D such that $f(x) \sqsubseteq f(z) \sqsubseteq f(y)$. In the case that D is algebraic conclude that an element x of $\mathsf{im}(f)$ is compact if and only if there is $c \in \mathsf{K}(D)_{f(x)}$ with $f(c) = f(x)$.*

2. *Let p be a kernel operator with finite image. Show that $\mathsf{im}(p)$ is contained in $\mathsf{K}(D)$ and that p itself is compact in $[D \longrightarrow D]$.*

3. *[Hut92] A chain C is called* order dense *if it has more than one element and for each pair $x \sqsubset y$ there exists $z \in C$ such that $x \sqsubset z \sqsubset y$.*

   (a) *Let C be an order dense chain of compact elements in an algebraic domain D with least element. Consider the function $g(x) = \bigsqcup\{c \in C \mid c \sqsubset x\}$. Show that this is continuous and below the identity. Give an example to demonstrate that g need not be idempotent. Show that $h = g \circ g$* is *idempotent and hence a kernel operator. Finally, show that the image of h is not algebraic (it must be continuous by Theorem 3.1.4).*

(b) *Let, conversely, f be a continuous and idempotent function on an algebraic dcpo D such that its image is not algebraic. Show that $\mathsf{K}(D)$ contains an order dense chain.*

(c) *An algebraic domain is called* projection stable *if every projection on D has an algebraic image. Conclude that an algebraic domain with bottom is projection stable if and only if $\mathsf{K}(D)$ does not contain an order dense chain.*

4. *Let $e\colon D \leftrightharpoons E : p$ be an embedding projection pair between $\sqcap$-semilattices. Show that $\mathsf{im}(e)$ is a lower set in E if and only if for all $x \sqsubseteq y$ in E we have $e(p(x)) = e(p(y)) \sqcap x$.*

5. *Formulate and prove a generalization of Proposition 3.1.13 for arbitrary posets.*

6. *Formulate an analogue of Proposition 3.2.4 for infinite products. Proceed as follows: First restrict to pointed dcpo's. Next find an example of a (non-pointed) finite poset which has a non-algebraic infinite power. This should give you enough intuition to try the general case.*

7. *A dcpo may be seen as a topological space with respect to the Scott-topology. Given two dcpo's we can form their product in* **DCPO**. *Show that the Scott-topology on the product need not be the product topology but that the two topologies coincide if one of the factors is a continuous domain.*

8. *Construct an example which shows that Lemma 3.2.6 does not hold for infinite products.*

9. *Derive* Curry *and composition as maps in an arbitrary cartesian closed category.*

10. *Let* **C** *be a cartesian closed full subcategory of* **DCPO**. *Let* **R-C** *be the full subcategory of* **DCPO** *whose objects are the retracts of objects of* **C**. *Show that* **R-C** *is cartesian closed.*

11. *Let $\mathbb{Z}^-$ be the negative integers with the usual ordering. Show that the order of approximation on $[\mathbb{Z}^- \longrightarrow \mathbb{Z}^-]$ is empty. Find a pointed algebraic dcpo in which a similar effect takes place.*

12. *Show that* $\mathbf{DCPO}_\perp$ *does not have coproducts.*

13. *Show that* **CONT** *does not have equalizers for all pairs of morphisms. (Hint: First convince yourself that limits in* **CONT**, *if they exist, have the same underlying dcpo as when they are calculated in* **DCPO**.*)*

14. *Complement the table in Section 3.2.6 with the infinitary counterparts of cartesian product, disjoint union, smash product and sum. Observe that for these the cardinality of the basis does play a role, so you have to add columns for* ω-**CONT** *etc.*

15. *Show that the embeddings into the bilimit of an expanding sequence are given more concretely by* $e_m(x) = \langle x_n : n \in \mathbb{N}\rangle$ *with*

$$x_n = \begin{cases} p_{nm}(x), & n < m; \\ e_{nm}(x), & n \geq m. \end{cases}$$

Find a similar description for expanding systems.

16. *Redo Section 3.3.2 for directed index sets and Scott-continuous adjunctions. The following are the interesting points:*

    (a) *The limit-colimit coincidence, Theorem 3.3.7, holds verbatim.*

    (b) *The characterization of bilimits given in Lemma 3.3.8 does not suffice. It states that* E *must not contain superfluous elements. Now we also need to say that* E *does not identify too many elements.*

    (c) *Given an expanding system* $\langle (D_i), (l_{ji})\rangle$ *with adjunctions, we can pass to quotient domains* D_i' *by setting* $D_i' = \mathsf{im}(\bigsqcup^{\uparrow}_{k \sqsupseteq i} u_{ik} \circ l_{ki})$*. Show that the original adjunctions when restricted and corestricted to the* D_i' *become e-p-pairs and that these define the same bilimit.*

17. *Let* RD *be the space of Scott-continuous idempotents on a dcpo* D*. Apply the previous exercise to show that* $\bigsqcup^{\uparrow}_{i \in I} r_i = r$ *in* RD *implies* $\mathsf{bilim}(\mathsf{im}(r_i)) \cong \mathsf{im}(r)$ *(where the connecting adjunctions are given by restricting the retractions to the respective image).*

18. *Prove that the Scott-topology on a bilimit of continuous domains is the restriction of the product topology on the product of the individual domains.*

4 Cartesian closed categories of domains

In the last chapter we have seen that our big ambient categories **DCPO** and $\mathbf{DCPO}_\perp$ are, among other things, cartesian closed and we have already pointed out that for the natural classes of *domains*, **CONT** and **ALG**, this is no longer true. The problematic construction is the exponential, which as we know by Lemma 3.2.7, must be the set of Scott-continuous functions ordered pointwise. If, on the other hand, we find a full subcategory of **CONT** which is closed under terminal object, cartesian product and function space, then it is also cartesian closed, because the necessary universal properties are inherited from **DCPO**.

Let us study more closely why function spaces may fail to be domains. The fact that the order of approximation may be empty tells us that there may be no natural candidates for basis elements in a function space. This we can better somewhat by requiring the image domain to contain a bottom element.

Definition 4.0.1. *For D and E dcpo's where E has a least element and $d \in D, e \in E$, we define the* step function $(d \searrow e)\colon D \to E$ *by*

$$(d \searrow e)(x) = \begin{cases} e, & \text{if } x \in \mathsf{Int}(\uparrow d); \\ \perp_E, & \text{otherwise.} \end{cases}$$

More generally, we will use $(O \searrow e)$ for the function which maps the Scott-open set O to e and everything else to $\perp$.

Proposition 4.0.2. *1. Step functions are Scott-continuous.*

2. *Let D and E be dcpo's where E is pointed and let $f\colon D \to E$ be continuous. If e approximates $f(d)$ then $(d \searrow e)$ approximates f.*

3. *If, in addition, D and E are continuous then f is a supremum of step functions.*

Proof. (1) Continuity follows from the openness of $\mathsf{Int}(\uparrow d)$, respectively O.

(2) Let G be a directed family of functions with $\bigsqcup^\uparrow G \sqsupseteq f$. Suprema in $[D \longrightarrow E]$ are calculated pointwise so we also have $\bigsqcup^\uparrow_{g \in G} g(d) \sqsupseteq f(d)$. This implies that for some $g \in G$, $g(d) \sqsupseteq e$ holds. A simple case distinction then shows that g must be above $(d \searrow e)$ everywhere.

(3) We show that for each $d \in D$ and each $e \ll f(d)$ in E there is a step function approximating f which maps d to e. Indeed, from $d = \bigsqcup^\uparrow \mathord{\downarrow\!\!\downarrow} d$ we get $f(d) = f(\bigsqcup^\uparrow_{y \ll d} y) = \bigsqcup^\uparrow_{y \ll d} f(y)$ and so for some $y \ll d$ we have $f(y) \sqsupseteq e$. The desired step function is therefore given by $(y \searrow e)$. Continuity of E implies that we can get arbitrarily close to $f(d)$ this way. □

Note that the supremum in (3) need not be directed, so we have *not* shown that $[D \longrightarrow E]$ is again continuous. Was it a mistake to require directedness for the set of approximants? The answer is no, because without it we could not have proved (3) in the first place.

The problem of joining finitely many step functions together, so as to build *directed* collections of approximants, comes up already in the case of two step functions $(d_1 \searrow$

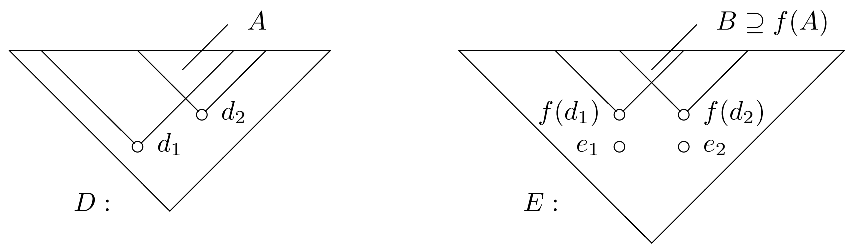


Figure 10: Finding an upper bound for two step functions.

e_1) and $(d_2 \searrow e_2)$ which approximate a given continuous function f. The situation is illustrated in Figure 10. The problem is where to map the (Scott-open but otherwise unstructured) set $A = \Uparrow d_1 \cap \Uparrow d_2$. It has to be done in such a way that the resulting function still approximates f. As it will turn out, it suffices to make special assumptions about *either* the image domain E—the topic of Section 4.1— *or* about the pre-image domain D – the topic of Section 4.2. In both cases we restrict our attention to pointed domains, and we work with step functions and joins of these. From these we can pass to more general domains in again two ways. This will be outlined briefly in Section 4.3.2. The question then arises whether we have not missed out on some alternative way of building a cartesian closed category. This is not the case as we will see in Section 4.3. The basic tool for this fundamental result, Lemma 4.3.1, will nicely connect up with the dichotomy distinguishing 4.1 and 4.2.

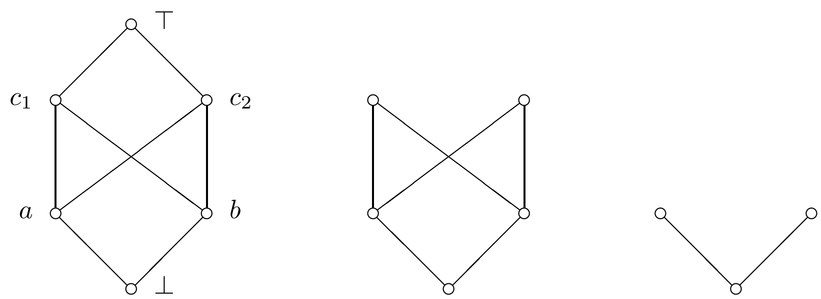
4.1 Local uniqueness: Lattice-like domains

The idea for adjusting the image domain is simple; we assume that e_1 and e_2 have a least upper bound e (if bounded at all). Mapping the intersection A to e (and $\Uparrow d_1 \setminus A$ to e_1 and $\Uparrow d_2 \setminus A$ to e_2) results in a continuous function h which is above $(d_1 \searrow e_1)$ and $(d_2 \searrow e_2)$ and still approximates f. This is seen as follows: Suppose G is a directed collection of functions with supremum above f. Some $g_1 \in G$ must be above $(d_1 \searrow e_1)$ and some $g_2 \in G$ must be above $(d_2 \searrow e_2)$. Then by construction every upper bound of $\{g_1, g_2\}$ in G is above h.

In fact, we do not need that the join of e_1 and e_2 exists globally in E. It suffices to form the join for every $a \in A$ inside $\downarrow f(a)$, because we have seen in Proposition 2.2.17 that all considerations about the order of approximation can be performed inside principal ideals. We have the following list of definitions.

Definition 4.1.1. *Let E be a pointed continuous domain. We say that E is*

1. *an* L-domain, *if each pair $e_1, e_2 \in E$ bounded by $e \in E$ has a supremum in $\downarrow e$;*
2. *a* bounded-complete domain *(or* bc-domain*), if each bounded pair $e_1, e_2 \in E$ has a supremum;*



$X \in \mathbf{ALG} \setminus \mathbf{L}$ $\qquad$ $C \in \mathbf{L} \setminus \mathbf{BC}$ $\qquad$ $V \in \mathbf{BC} \setminus \mathbf{LAT}$

Figure 11: Separating examples for the categories of lattice-like domains.

3. *(repeated for comparison) a* continuous lattice*, if each pair* $e_1, e_2 \in E$ *has a supremum.*

We denote the full subcategories of $\mathbf{CONT}_\bot$ *corresponding to these definitions by* **L***,* **BC***, and* **LAT***. For the algebraic counterparts we use* **aL***,* **aBC***, and* **aLAT***.*

All this still makes sense if we forget about approximation but, surely, at this point the reader does not suffer from a lack of variety as far as categories are concerned. We would like to point out that continuous lattices are the main objects of study in [GHK$^+$80], a mathematically oriented text, whereas the objects of ω-**aBC** are often the domains of choice in semantics, where they appear under the name *Scott-domain*. Typical examples are depicted in Figure 11. They even characterize the corresponding categories, see Exercise 4.3.11(3).

Since domains have directed joins anyway, we see that in L-domains every subset of a principal ideal has a supremum in that ideal. We also know that complete lattices can alternatively be characterized by infima. The same game can be played for the other two definitions:

Proposition 4.1.2. *Let D be a pointed continuous domain. Then D is an L-domain, a bc-domain, or a continuous lattice if and only if it has infima for bounded non-empty, non-empty, or arbitrary subsets, respectively.*

The consideration of infima may seem a side issue in the light of the problem of turning function spaces into domains. Its relevance becomes clear when we remember that upper adjoints preserve infima. The second half of the following is therefore a simple observation. The first half follows from Proposition 3.1.2 and Theorem 3.1.4.

Proposition 4.1.3. *Retracts and bilimits of L-domains (bc-domains, continuous lattices) are again L-domains (bc-domains, continuous lattices).*

We can treat continuous and algebraic lattice-like domains nicely in parallel because the ideal completion respects these definitions:

Proposition 4.1.4. *Let D be an L-domain (bc-domain, continuous lattice). Then* $\mathsf{Idl}(D, \sqsubseteq)$ *is an algebraic L-domain (bc-domain, lattice).*

Thus **L**, **BC**, and **LAT** contain precisely the retracts of objects of **aL**, **aBC**, and **aLAT**, respectively. We conclude this section by stating the desired closure property of lattice-like domains.

Proposition 4.1.5. *Let D be a continuous domain and E an L-domain (bc-domain, continuous lattice). Then $[D \longrightarrow E]$ is again an L-domain (bc-domain, continuous lattice).*

Corollary 4.1.6. *The categories* **L***,* **BC***,* **LAT***, and their algebraic counterparts are cartesian closed.*

4.2 Finite choice: Compact domains

Let us now turn our attention to the first argument of the function space construction, which means by the general considerations from the beginning of this chapter, the study of open sets and their finite intersections. Step functions are defined using basic open sets of the form $\twoheaduparrow d$, and the fact that there is a single generator d was crucial in the proof that $(d \searrow e)$ approximates f whenever e approximates $f(d)$. Arbitrary open sets are unions of such basic opens (Proposition 2.3.6) but in general this is an infinite union and so the proof of Proposition 4.0.2 will no longer work. For the first time we have now reached a point in our exposition where the theory of algebraic domains is definitely simpler and better understood than that of continuous domains. Let us therefore treat this case first.

4.2.1 Bifinite domains

Step functions $(d \searrow e)$ may in the algebraic case be defined using compact elements only, where the characteristic pre-image $\twoheaduparrow d$ is actually equal to $\uparrow d$. Taking up our line of thought from above, we want for the algebraicity of the function space that the intersection $A = \uparrow d_1 \cap \uparrow d_2$ is itself generated by finitely many compact points: $A = \uparrow c_1 \cup \ldots \cup \uparrow c_n$. Note that the c_i must be minimal upper bounds of $\{d_1, d_2\}$. For each c_i we choose a compact element below $f(c_i)$ and above e_1, e_2. New intersections then come up, this time between the different $\uparrow c_i$'s. Let us therefore further assume that after finitely many iterations this process stops. It is an easy exercise to show that the function constructed in this way is a compact element below f and above $(d_1 \searrow e_1)$ and $(d_2 \searrow e_2)$. We hope that this provides sufficient motivation for the following list of definitions.

Definition 4.2.1. *Let P be a poset. (Think of P as the basis of an algebraic domain.)*

1. *We say that P is* mub-complete *(or: has* property m*) if for every upper bound x of a finite subset M of P there is a minimal upper bound of M below x. Written as a formula: $\forall M \subseteq_{\mathit{fin}} P.\ \bigcap_{m \in M} \uparrow m = \uparrow \mathsf{mub}(M)$.*
2. *For a subset A of P let its* mub-closure $\mathsf{mc}(A)$ *be the smallest superset of A which for every finite $M \subseteq \mathsf{mc}(A)$ also contains* $\mathsf{mub}(M)$.

3. *We say that P has the* finite mub property *if it is mub-complete and if every finite subset has a finite mub-closure. If, in addition, P has a least element, then we call it a* Plotkin-order.

4. *An algebraic domain whose basis of compact elements is a Plotkin-order is called a* bifinite domain. *The full subcategory of* $\mathbf{ALG}_\perp$ *of bifinite domains we denote by* $\mathbf{B}$.

With this terminology we can formulate precisely how finitely many step functions combine to determine a compact element in the function space [Abr91b].

Definition 4.2.2. *Let D be a bifinite domain and let E be pointed and algebraic. A finite subset F of $\mathsf{K}(D) \times \mathsf{K}(E)$ is called* joinable *if*

$$\forall G \subseteq F\ \exists H \subseteq F.\ (\pi_1(H) = \mathsf{mub}(\pi_1(G))\ \wedge\ \forall c \in \pi_2(G), d \in \pi_2(H).\ c \sqsubseteq d).$$

The function which we associate with a joinable family F is

$$x \mapsto \bigsqcup\{e \mid \exists d \in \mathsf{K}(D).\ d \sqsubseteq x\ \wedge\ (d, e) \in F\}.$$

Lemma 4.2.3. *If D is a bifinite domain and E is pointed and algebraic, then every joinable subset of $\mathsf{K}(D) \times \mathsf{K}(E)$ gives rise to a compact element of $[D \longrightarrow E]$.*

If F and G are joinable families then the corresponding functions are related if and only if

$$\forall (d, e) \in G\ \exists (d', e') \in F.\ d' \sqsubseteq d \text{ and } e \sqsubseteq e'.$$

The expected result, dual to Proposition 4.1.5 above, then is:

Proposition 4.2.4. *If D is a bifinite domain and E is pointed and algebraic, then $[D \longrightarrow E]$ is algebraic. All compact elements of $[D \longrightarrow E]$ arise from joinable families.*

Comment: Proof sketch: Let f be a continuous function from D to E, and M be a finite mub-closed set of compact elements of D. Let $(e_m)_{m \in M}$ be a collection of compact elements of E such that for all $m \in M$, $e_m \leq f(m)$. Then there exists a collection $(\hat{e}_m)_{m \in M}$ of compact elements of E such that the assignment $m \mapsto \hat{e}_m$ is order-preserving. The new elements can be found by repeatedly considering a minimal element m of M for which $\hat{e}_m$ has not yet been chosen, and by picking an upper bound for $\{e_m\} \cup \{\hat{e}_{m'} \mid m' < m\}$. With this construction one finds a directed collection of compact elements of $[D \longrightarrow E]$ arbitrarily close to f.

Note that this is strictly weaker than Proposition 4.1.5 and we do not immediately get that $\mathbf{B}$ is cartesian closed. For this we have to find alternative descriptions. The fact that we can get an algebraic function space by making special assumptions about *either* the argument domain *or* the target domain was noted in a very restricted form in [Mar81].

The concept of finite mub closure is best explained by illustrating what can go wrong. In Figure 12 we have the three classical examples of algebraic domains which are not bifinite; in the first one the basis is not mub-complete, in the second one there is an infinite mub-set for two compact elements, and in the third one, although all mub-sets are finite, there occurs an infinite mub-closure. On a more positive note, it is clear

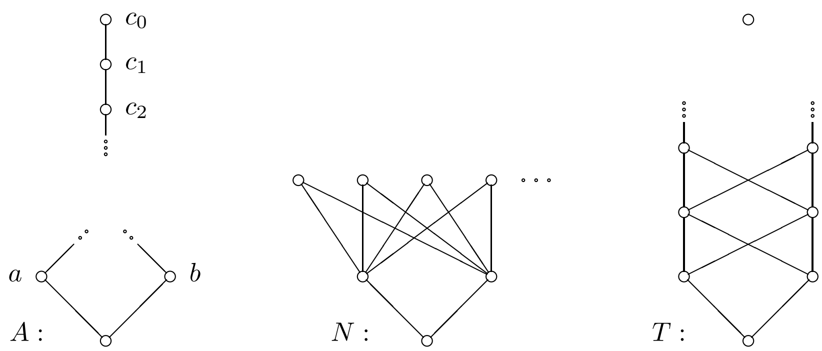


Figure 12: Typical non-bifinite domains.

that every finite and pointed poset is a Plotkin-order and hence bifinite. This trivial example contains the key to a true understanding of bifiniteness; we will now prove that bifinite domains are precisely the bilimits of finite pointed posets.

Proposition 4.2.5. *Let D be an algebraic domain with mub-complete basis $\mathsf{K}(D)$ and let A be a set of compact elements. Then there is a least kernel operator p_A on D which keeps A fixed. It is given by $p_A(x) = \bigsqcup^{\uparrow}\{c \in \mathsf{mc}(A) \mid c \sqsubseteq x\}$.*

Proof. First note that p_A is well-defined because the supremum is indeed over a directed set. This follows from mub-completeness. Continuity follows from Corollary 2.2.16. On the other hand, it is clear that a kernel operator which fixes A must also fix each element of the mub-closure $\mathsf{mc}(A)$, and so p_A is clearly the least monotone function with the desired property. $\square$

In a bifinite domain finite sets of compact elements have finite mub-closures. By the preceding proposition this implies that there are many kernel operators on such a domain which have a finite image. In fact, we get a directed family of them, because the order on kernel operators is completely determined by their images, Proposition 3.1.17. For the sake of brevity, let us call a kernel operator with finite image an *idempotent deflation*.

Theorem 4.2.6. *Let D be a pointed dcpo D. The following are equivalent*

1. *D is a bifinite domain.*
2. *There exists a directed collection $(f_i)_{i \in I}$ of idempotent deflations of D whose supremum equals id_D.*
3. *The set of all idempotent deflations is directed and yields id_D as its join.*

Proof. What we have not yet said is how algebraicity of D follows from the existence of idempotent deflations. For this observe that the inclusion of the image of a kernel operator is a lower adjoint and as such preserves compactness. For the implication '$2 \Longrightarrow 3$' we use the fact that idempotent deflations are in any case compact elements of the function space. □

It is now only a little step to the promised categorical characterization.

Theorem 4.2.7. *A dcpo is bifinite if and only if it is a bilimit of an expanding system of finite pointed posets.*

Proof. Let D be bifinite and let $(f_i)_{i \in I}$ be a family of idempotent deflations generating the identity. Construct an expanding system by taking as objects the images of the deflations and as connecting embeddings the inclusion of images. The associated upper adjoint is given by f_i restricted to $\mathsf{im}(f_j)$. D is the bilimit of this system by Lemma 3.3.8.

If, conversely, $\langle D, (f_i)_{i \in I} \rangle$ is a bilimit of finite posets then clearly the compositions $f_i \circ g_i$, where g_i is the upper adjoint of f_i, satisfy the requirements of Theorem 4.2.6. □

So we have three characterizations of bifiniteness, the original one, which may be called an internal description, a functional description by Theorem 4.2.6, and a categorical one by Theorem 4.2.7. Often, the functional characterization is the most handy one in proofs. We should also mention that bifinite domains were first defined by Gordon Plotkin in [Plo76] using expanding sequences. (In our taxonomy these are precisely the countably based bifinite domains.) The acronym he used for them, SFP, continues to be quite popular.

Theorem 4.2.8. *The category* **B** *of bifinite domains is closed under cartesian product, function space, coalesced sum, and bilimits. In particular,* **B** *is cartesian closed.*

Proof. Only function space and bilimit are non-trivial. We leave the latter as an exercise. For the function space let D and E be bifinite with families of idempotent deflations $(f_i)_{i \in I}$ and $(g_j)_{j \in J}$. A directed family of idempotent deflations on $[D \longrightarrow E]$ is given by the maps $F_{ij} \colon h \mapsto g_j \circ h \circ f_i$, $\langle i, j \rangle \in I \times J$. □

4.2.2 FS-domains

Let us now look at continuous domains. The reasoning about what the structure of D should be in order to ensure that $[D \longrightarrow E]$ is continuous is pretty much the same as for algebraic domains. But at the point where we there introduced the mub-closure of a finite set of compact elements, we must now postulate the *existence* of some finite and finitely supported partitioning of D. This is clearly an increase in the logical complexity of our definition and also of doubtful practical use. It is more satisfactory to generalise the functional characterization.

Definition 4.2.9. *Let D be a dcpo and $f \colon D \to D$ be a Scott-continuous function. We say that f is* finitely separated *from the identity on D, if there exists a finite set M*

such that for any $x \in D$ there is $m \in M$ with $f(x) \sqsubseteq m \sqsubseteq x$. We speak of strong separation *if for each x there are elements $m, m' \in M$ with $f(x) \sqsubseteq m \ll m' \sqsubseteq x$.*

A pointed dcpo D is called an FS-domain *if there is a directed collection $(f_i)_{i \in I}$ of continuous functions on D, each finitely separated from id_D, with the identity map as their supremum.*

It is relatively easy to see that FS-domains are indeed continuous. Thus it makes sense to speak of **FS** as the full subcategory of **CONT** where the objects are the FS-domains.

We have exact parallels to the properties of bifinite domains, but often the proofs are trickier.

Proposition 4.2.10. *If D is an FS-domain and E is pointed and continuous then $[D \longrightarrow E]$ is continuous.*

> **Comment:** Unfortunately, the proof of this is not only "trickier" but as yet unknown. What is true, is that when *both* D and E are FS-domains, then $[D \longrightarrow E]$ is also an FS-domains. This was shown in [Jun90]. The following theorem is therefore still valid.

Theorem 4.2.11. *The category* **FS** *is closed under the formation of products, function spaces, coalesced sums, and bilimits. It is cartesian closed.*

What we do not have are a categorical characterization or a description of FS-domains as retracts of bifinite domains. All we can say is the following.

Proposition 4.2.12.

1. *Every bifinite domain is an FS-domain.*
2. *A retract of an FS-domain is an FS-domain.*
3. *An algebraic FS-domain is bifinite.*

To fully expose our ignorance, we conclude this subsection with an example of a well-structured FS-domain of which we do not know whether it is a retract of a bifinite domain.

Example. Let Disc be the collection of all closed discs in the plane plus the plane itself, ordered by reversed inclusion. One checks that the filtered intersection of discs is again a disc, so Disc is a pointed dcpo. A disc d_1 approximates a disc d_2 if and only if d_1 is a neighborhood of d_2. This proves that Disc is continuous. For every $\epsilon > 0$ we define a map f_ϵ on Disc as follows. All discs inside the open disc with radius $\frac{1}{\epsilon}$ are mapped to their closed ϵ-neighborhood, all other discs are mapped to the plane which is the bottom element of Disc. Because the closed discs contained in some compact set form a compact space under the Hausdorff subspace topology, these functions are finitely separated from the identity map. This proves that Disc is a countably based FS-domain.

4.2.3 Coherence

This is a good opportunity to continue our exposition of the topological side of domain theory, which we began in Section 2.3. We need a second tool complementing the lattice σ_D of Scott-open sets, namely, the compact saturated sets. Here 'compact' is to be understood in the classical topological sense of the word, i.e. a set A of a topological space is *compact* if every covering of A by open sets contains a finite subcovering. *Saturated* are those sets which are intersections of their neighborhoods. In dcpo's equipped with the Scott-topology these are precisely the upper sets, as is easily seen using opens of the form $D \setminus \downarrow x$.

Theorem 4.2.13. *Let D be a continuous domain. The sets of open neighborhoods of compact saturated sets are precisely the Scott-open filters in σ_D.*

By Proposition 7.2.27 this is a special case of the Hofmann-Mislove Theorem 7.2.9.

Let us denote the set of compact saturated sets of a dcpo D, ordered by reverse inclusion, by κ_D. We will refer to families in κ_D which are directed with respect to reverse inclusion, more concretely as filtered families. The following, then, is only a re-formulation of Corollary 7.2.11.

Proposition 4.2.14. *Let D be a continuous domain.*

1. *κ_D is a dcpo. Directed suprema are given by intersection.*
2. *If the intersection of a filtered family of compact saturated sets is contained in a Scott-open set O then some element of it belongs to O already.*
3. *$\kappa_D \setminus \{\emptyset\}$ is a dcpo.*

Proposition 4.2.15. *Let D be a continuous domain.*

1. *κ_D is a continuous domain.*
2. *$A \ll B$ holds in κ_D if and only if there is a Scott-open set O with $B \subseteq O \subseteq A$.*
3. *$O \ll U$ holds in σ_D if and only if there is a compact saturated set A with $O \subseteq A \subseteq U$.*

Proof. All three claims are shown easily using upper sets generated by finitely many points: If O is an open neighborhood of a compact saturated set A then there exists a finite set M of points of O with $A \subseteq \Uparrow M \subseteq \uparrow M \subseteq O$. $\square$

The interesting point about FS-domains then is, that their space of compact saturated sets is actually a continuous lattice. We already have directed suprema (in the form of filtered intersections) and continuity, so this boils down to the property that the intersection of two compact saturated sets is again compact. Let us call domains for which this is true, *coherent domains*. Given the intimate connection between σ_D and κ_D, it is no surprise that we can read off coherence from the lattice of open sets.

Proposition 4.2.16. *A continuous domain D is coherent if and only if for all $O, U_1, U_2 \in \sigma_D$ with $O \ll U_1$ and $O \ll U_2$ we also have $O \ll U_1 \cap U_2$.*

(In Figure 6 we gave an example showing that the condition is not true in arbitrary continuous lattices.)

This result specializes for algebraic domains as follows:

Proposition 4.2.17. *An algebraic domain D is coherent if and only if $\mathsf{K}(D)$ is mub-complete and finite sets of $\mathsf{K}(D)$ have finite sets of minimal upper bounds.*

This proposition was named '2/3-SFP Theorem' in [Plo81] because coherence rules out precisely the first two non-examples of Plotkin-orders, Figure 12, but not the third. The only topological characterization of bifinite domains we have at the moment, makes use of the continuous function space, see Lemma 4.3.2.

We observe that for algebraic coherent domains, σ_D and κ_D have a common sublattice, namely that of compact-open sets. These are precisely the sets of the form $\uparrow c_1 \cup \ldots \cup \uparrow c_n$ with the c_i compact elements. This lattice generates both σ_D and κ_D when we form arbitrary suprema. This pleasant coincidence features prominently in Chapter 7.

Theorem 4.2.18. *FS-domains (bifinite domains) are coherent.*

Let us reformulate the idea of coherence in yet another way.

Definition 4.2.19. *The* Lawson-topology *on a dcpo D is the smallest topology containing all Scott-open sets and all sets of the form $D \setminus \uparrow x$. It is denoted by λ_D.*

Proposition 4.2.20. *Let D be a continuous domain.*

1. *The Lawson-topology on D is Hausdorff. Every Lawson-open set has the form $O \setminus A$ where O is Scott-open and A is Scott-compact saturated.*

2. *The Lawson-topology on D is compact if and only if D is coherent.*

3. *A Scott-continuous retract of a Lawson-compact continuous domain is Lawson-compact and continuous.*

So we see that FS-domains and bifinite domains carry a natural compact Hausdorff topology. We will make use of this in Chapter 6.

4.3 The hierarchy of categories of domains

The purpose of this section is to show that there are no other ways of constructing a cartesian closed full subcategory of **CONT** or **ALG** than those exhibited in the previous two sections. The idea that such a result could hold originated with Gordon Plotkin, [Plo81]. For the particular class ω-$\mathbf{ALG}_\perp$ it was verified by Mike Smyth in [Smy83a], for the other classes by Achim Jung in [Jun88, Jun89, Jun90]. All these classification results depend on the Axiom of Choice.

4.3.1 Domains with least element

Let us start right away with the crucial bifurcation lemma on which everything else in this section is based.

Lemma 4.3.1. *Let D and E be continuous domains, where E is pointed, such that $[D \longrightarrow E]$ is continuous. Then D is coherent or E is an L-domain.*

Proof. By contradiction. Assume D is not coherent and E is not an L-domain. By Proposition 4.2.16 there exist open sets O, U_1, and U_2 in D such that $O \ll U_1$ and $O \ll U_2$ hold but not $O \ll U_1 \cap U_2$. Therefore there is a directed collection $(V_i)_{i \in I}$ of open sets covering $U_1 \cap U_2$, none of which covers O. We shall also need interpolating sets U_1' and U_2', that is, $O \ll U_1' \ll U_1$ and $O \ll U_2' \ll U_2$.

The assumption about E not being an L-domain can be transformed into two special cases. Either E contains the algebraic domain A from Figure 12 (where the descending chain in A may generally be an ordinal) or X from Figure 11 as a retract. We have left the proof of this as Exercise 4.3.11(3). Note that if E' is a retract of E then $[D \longrightarrow E']$ is a retract of $[D \longrightarrow E]$ and hence the former is continuous if the latter is. Let us now prove for both cases that $[D \longrightarrow E]$ is not continuous.

Case 1: $E = A$. Consider the step functions $f_1 = (U_1' \searrow a)$ and $f_2 = (U_2' \searrow b)$. They clearly approximate f, which is defined by

$$f(x) = \begin{cases} c_0, & \text{if } x \in U_1 \cap U_2; \\ a, & \text{if } x \in U_1 \setminus U_2; \\ b, & \text{if } x \in U_2 \setminus U_1; \\ \perp, & \text{otherwise.} \end{cases}$$

Since approximating sets are directed we ought to find an upper bound g for f_1 and f_2 approximating f. But this impossible: Given an upper bound of $\{f_1, f_2\}$ below f we have the directed collection $(h_i)_{i \in I}$ defined by

$$h_i(x) = \begin{cases} c_0, & \text{if } x \in V_i; \\ c_{n+1}, & \text{if } x \in (U_1 \cap U_2) \setminus V_i \text{ and } g(x) = c_n; \\ g(x), & \text{otherwise.} \end{cases}$$

No h_i is above g because $(U_1 \cap U_2) \setminus V_i$ must contain a non-empty piece of O and there h_i is strictly below g. The supremum of the h_i, however, equals f. Contradiction.

Case 2: $E = X$. We choose open sets in D as in the previous case. The various functions, giving the contradiction, are now defined by $f_1 = (U_1' \searrow a)$, $f_2 = (U_2' \searrow b)$,

$$f(x) = \begin{cases} c_1, & \text{if } x \in U_1 \cap U_2; \\ a, & \text{if } x \in U_1 \setminus U_2; \\ b, & \text{if } x \in U_2 \setminus U_1; \\ \perp, & \text{otherwise.} \end{cases}$$

$$h_i(x) = \begin{cases} \top, & \text{if } x \in V_i; \\ c_2, & \text{if } x \in (U_1 \cap U_2) \setminus V_i; \\ g(x), & \text{otherwise.} \end{cases}$$

□

The remaining problem is that coherence does not imply that D is an FS-domain (nor, in the algebraic case, that it is bifinite). It is taken care of by passing to higher-order function spaces:

Lemma 4.3.2. *Let D be a continuous domain with bottom element. Then D is an FS-domain if and only if both D and $[D \longrightarrow D]$ are coherent.*

(The proof may be found in [Jun90].)

Combining the preceding two lemmas with Lemmas 3.2.5 and 3.2.7 we get the promised classification result.

Theorem 4.3.3. *Every cartesian closed full subcategory of* $\mathbf{CONT}_\perp$ *is contained in* **FS** *or* **L**.

Adding Proposition 4.2.12 we get the analogue for algebraic domains:

Theorem 4.3.4. *Every cartesian closed full subcategory of* $\mathbf{ALG}_\perp$ *is contained in* **B** *or* **aL**.

Forming the function space of an L-domain may in general increase the cardinality of the basis (Exercise 4.3.11(17)). If we restrict the cardinality, this case is ruled out:

Theorem 4.3.5. *Every cartesian closed full subcategory of* ω-$\mathbf{CONT}_\perp$ *(*ω-$\mathbf{ALG}_\perp$*) is contained in* ω-**FS** *(*ω-**B***).*

4.3.2 Domains without least element

The classification of pointed domains, as we have just seen, is governed by the dichotomy between coherent and lattice-like structures. Expressed at the element level, and at least for algebraic domains we have given the necessary information, it is the distinction between finite mub-closures and locally unique suprema of finite sets. It turns out that passing to domains which do not necessarily have bottom elements implies that we also have to study the mub-closure of the empty set. We get again the same dichotomy. Coherence in this case means that D itself, that is, the largest element of σ_D, is a compact element. This is just the compactness of D as a topological space. And the property that E is lattice-like boils down to the requirement that each element of E is above a unique minimal element, so E is really the disjoint union of pointed components.

Lemma 4.3.6. *Let* D *and* E *be continuous domains such that* $[D \longrightarrow E]$ *is continuous. Then* D *is compact or* E *is a disjoint union of pointed domains.*

The proof is a cut-down version of that of Lemma 4.3.1 above. The surprising fact is that this choice can be made *independently* from the choice between coherent domains and L-domains. Before we state the classification, which because of this independence, will now involve $2 \times 2 = 4$ cases, we have to refine the notion of compactness, because just like coherence it is not the full condition necessary for cartesian closure.

Definition 4.3.7. *A dcpo* D *is a* finite amalgam *if it is the union of finitely many pointed dcpo's* $D_1, \ldots, D_n$ *such that every intersection of* D_i*'s is also a union of* D_i*'s. (Compare the definition of mub-complete.)*

For categories whose objects are finite amalgams of objects from another category **C** *we use the notation* **F-C**. *Similarly, we write* **U-C** *if the objects are disjoint unions of objects of* **C**.

Proposition 4.3.8. *A mub-complete dcpo is a finite amalgam if and only if the mub-closure of the empty set is finite.*

Lemma 4.3.9. *If both D and $[D \longrightarrow D]$ are compact and continuous then D is a finite amalgam.*

Theorem 4.3.10.
1. *The maximal cartesian closed full subcategories of* **CONT** *are* **F-FS**, **U-FS**, **F-L**, *and* **U-L**.
2. *The maximal cartesian closed full subcategories of* **ALG** *are* **F-B**, **U-B**, **F-aL**, *and* **U-aL**.

At this point we can answer a question that may have occurred to the diligent reader some time ago, namely, why we have defined bifinite domains in terms of *pointed* finite posets, where clearly we never needed the bottom element in the characterizations of them. The answer is that we wanted to emphasize the uniform way of passing from pointed to general domains. The fact that the objects of **F-B** can be represented as bilimits of finite posets is then just a pleasant coincidence.

Exercises 4.3.11.
1. *[Jun89] Show that a dcpo D is continuous if the function space $[D \longrightarrow D]$ is continuous.*
2. *Let D be a bounded-complete domain. Show that '$\sqcap$' is a Scott-continuous function from $D \times D$ to D.*
3. *Characterize the lattice-like (pointed) domains by forbidden substructures:*
   (a) *E is ω-continuous but not mub-complete if and only if domain A in Figure 12 is a retract of E.*
   (b) *E is mub-complete but not an L-domain if and only if domain X in Figure 11 is a retract of E.*
   (c) *E is an L-domain but not bounded-complete if and only if domain C in Figure 11 is a retract of E.*
   (d) *E is a bounded-complete domain but not a lattice if and only if domain V in Figure 11 is a retract of E.*
4. *Find a poset in which all pairs have finite mub-closures but in which a triple of points exists with infinite mub-closure.*
5. *Show that if for an algebraic domain D the basis is mub-complete then D itself is not necessarily mub-complete.*
6. *Show that in a bifinite domain finite sets of non-compact elements may have infinitely many minimal upper bounds and, even if these are all finite, may have infinite mub-closures.*
7. *Show that if A is a two-element subset of an L-domain then $A \cup \mathsf{mub}(A)$ is mub-closed.*
8. *Prove that bilimits of bifinite domains are bifinite.*

9. *Prove the following statements about retracts of bifinite domains.*

   (a) *A pointed dcpo D is a retract of a bifinite domain if and only if there is a directed family $(f_i)_{i \in I}$ of functions on D such that each f_i has a finite image and such that $\bigsqcup^{\uparrow}_{i \in I} f_i = \mathsf{id}_D$. (You may want to do this for countably based domains first.)*

   (b) *The ideal completion of a retract of a bifinite domain need not be bifinite.*

   (c) *If D is a countably based retract of a bifinite domain then it is also the image of a projection from a bifinite domain. (Without countability this is an open problem.)*

   (d) *The category of retracts of bifinite domains is cartesian closed and closed under bilimits.*

10. *Prove that FS-domains have infima for downward directed sets. As a consequence, an FS-domain which has binary infima, is a bc-domain.*

11. *Show that in a continuous domain the Lawson-closed upper sets are precisely the Scott-compact saturated sets.*

12. *Characterize Lawson-continuous maps between bifinite domains.*

13. *We have seen that every bifinite domain is the bilimit of finite posets. As such, it can be thought of as a subset of the product of all these finite posets. Prove that the Lawson-topology on the bifinite domain is the restriction of the product topology if each finite poset is equipped with the discrete topology.*

14. *Prove that a coherent L-domain is an FS-domain.*

15. *Characterize those domains which are both L-domains and FS-domains.*

16. *Characterize Scott-topology and Lawson-topology on both L-domains and FS-domains by the ideal of functions approximating the identity.*

17. *[Jun89] Let E be an L-domain such that $[E \longrightarrow E]$ is countably based. Show that E is an FS-domain.*

5 Recursive domain equations

The study of recursive domain equations is not easily motivated by reference to other mathematical structure theories. So we shall allow ourselves to deviate from our general philosophy and spend some time on examples. Beyond motivation, our examples represent three different (and almost disjoint) areas in which recursive domain equations arise, in which they serve a particular role, and in which particular aspects about solutions become prominent. It is an astonishing fact that within domain theory all these aspects are dealt with in a unified and indeed very satisfactory manner. This richness and interconnectedness of the theory of recursive domain equations, beautiful as it is, may nevertheless appear quite confusing on a first encounter. As a general guideline we offer the following: Recursive domain equations and the domain theory for solving them comprise a *technique* that is worth *learning*. But in order to *understand* the *meaning* of a particular recursive domain equation, you have to know the context in which it came up.

5.1 Examples

5.1.1 Genuine equations

The prime example here is $X \cong [X \longrightarrow X]$. Solving this equation in a cartesian closed category gives a model for the untyped λ-calculus [Sco80, Bar84], in which, as we know, no type distinction is made between functions and arguments. When setting up an interpretation of λ-terms with values in D, where D solves this equation, we need the isomorphisms $\phi\colon D \to [D \longrightarrow D]$ and $\psi\colon [D \longrightarrow D] \to D$ explicitly. We conclude that even in the case of a genuine equation we are looking not only for an object but an object *plus* an isomorphism. This is a first hint that we shall need to treat recursive domain equations in a categorical setting. However, the function space operator is contravariant in its first and covariant in its second argument and so there is definitely an obstacle to overcome. A second problem that this example illustrates is that there may be many solutions to choose from. How do we recognize a canonical one? This will be the topic of Section 5.3.

Besides this classical example, genuine equations are rare. They come up in semantics when one is confronted with the ability of computers to treat information both as program text and as data.

5.1.2 Recursive definitions

In semantics we sometimes need to make recursive definitions, for very much the same reasons that we need recursive function calls, namely, we sometimes do not know how often the body of a definition (resp. function) needs to be repeated. To give an example, take the following definition of a space of so-called 'resumptions':

$$R \cong [S \longrightarrow (S \oplus S \times R)].$$

We read it as follows: A resumption is a map which assigns to a state either a final state or an intermediary state together with another resumption representing the remaining computation. Such a recursive definition is therefore nothing but a shorthand for an infinite (but regular) expression. Likewise, a while loop could be replaced by an infinite repetition of its body. This analogy suggests that the way to give meaning to a recursive definition is to seek a limit of the repeated unwinding of the body of the definition starting from a trivial domain. No doubt this is in accordance with our intuition, and indeed this is how we shall solve equations in general. But again, before we can do this, we need to be able to turn the right hand side of the specification into a functor.

5.1.3 Data types

Data types are algebras, i.e. sets together with operations. The study of this notion is known as 'Algebraic Specification' [EM85] or 'Initial Algebra Semantics' [GTW78]. We choose a formulation which fits nicely into our general framework.

Definition 5.1.1. *Let F be a functor on a category* $\mathbf{C}$. *An F-*algebra *is given by an object A and a map $f\colon F(A) \to A$. A homomorphism between algebras $f\colon F(A) \to A$ and $f'\colon F(A') \to A'$ is a map $g\colon A \to A'$ such that the following diagram commutes:*

$$\begin{tikzcd} F(A) \arrow[r, "F(g)"] \arrow[d, "f"'] & F(A') \arrow[d, "f'"] \\ A \arrow[r, "g"] & A' \end{tikzcd}$$

For example, if we let F be the functor over **Set** which assigns $\mathbb{I} \mathbin{\dot\cup} A \times A$ to A (where $\mathbb{I}$ is the one-point dcpo as discussed in Section 3.2.1), then F-algebras are precisely the algebras with one nullary and one binary operation in the sense of universal algebra. Lehmann and Smyth [LS81] discuss many examples. Many of the data types which programming languages deal with are furthermore totally free algebras, or term algebras on no generators. These are distinguished by the fact that there is precisely one homomorphism from them into any other algebra of the same signature. In our categorical language we express this by initiality. Term algebras (alias initial F-algebras) are connected with the topic of this chapter because of the following observation:

Lemma 5.1.2. *If $i\colon F(A) \to A$ is an initial F-algebra then i is an isomorphism.*

Proof. Consider the following composition of homomorphisms:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "i"'] & F^2(A) \arrow[r, "F(i)"] \arrow[d, "F(i)"'] & F(A) \arrow[d, "i"] \\ A \arrow[r, "f"] & F(A) \arrow[r, "i"] & A \end{tikzcd}$$

where f is the unique homomorphism from $i\colon F(A) \to A$ to $F(i)\colon F^2(A) \to F(A)$ guaranteed by initiality. Again by initiality, $i \circ f$ must be id_A. And from the first quadrangle we get $f \circ i = F(i) \circ F(f) = F(\mathsf{id}_A) = \mathsf{id}_{F(A)}$. So f and i are inverses of each other. $\square$

So in order to find an initial F-algebra, we need to solve the equation $X \cong F(X)$. But once we get a solution, we still have to check initiality, that is, we must validate that the isomorphism from $F(X)$ to X is the right structure map.

In category theory we habitually dualize all definitions. In this case we get (final) co-algebras. Luckily, this concept is equally meaningful. Where the map $f\colon F(A) \to A$ describes the way how new objects of type A are *constructed* from old ones, a map $g\colon A \to F(A)$ stands for the opposite process, the *decomposition* of an object into its constituents. Naturally, we want the two operations to be inverses of each other. In other words, if $i\colon F(A) \to A$ is an initial F-algebra, then we require $i^{-1}\colon A \to F(A)$ to be the final co-algebra.

Peter Freyd [Fre91] makes this reasoning the basis of an axiomatic treatment of domain theory. Beyond and above axiomatizing known results, he treats contravariant and mixed variant functors and offers a universal property encompassing both initiality and finality. This will allow us to judge the solution of general recursive domain equations with respect to canonicity.

5.2 Construction of solutions

Suppose we are given a recursive domain equation $X \cong F(X)$ where the right hand side defines a functor on a suitable category of domains. As suggested by the example in Section 5.1.2, we want to repeat the trick which gave us fixpoints for Scott-continuous functions, namely, to take a (bi-)limit of the sequence $\mathbb{I}, F(\mathbb{I}), F(F(\mathbb{I})), \ldots$. Remember that bilimits are defined in terms of e-p-pairs. This makes it necessary that we, at least temporarily, switch to a different category. The convention that we adopt for this chapter is to let **D** stand for any category of *pointed* domains, closed under bilimits. All the cartesian closed categories of pointed domains mentioned in Chapter 4 qualify. We denote the corresponding subcategory where the morphisms are embeddings by $\mathbf{D}^e$. Some results will only hold for strict functions. Recall that our notation for these were $f\colon D \stackrel{\perp!}{\longrightarrow} E$ and $\mathbf{D}_{\perp!}$ for categories. Despite this unhappy (but unavoidable) proliferation of categories, recall that the central limit-colimit Theorem 3.3.7 and Corollary 3.3.10 state a close connection: Colimits of expanding sequences in $\mathbf{D}^e$ are also colimits in **D** and, furthermore, if the embeddings defining the sequence are replaced by their upper adjoints, the colimit coincides with the corresponding limit. This will bear fruit when we analyze the solutions we get in $\mathbf{D}^e$ from various angles as suggested by the examples in the last subsection.

Let us now start by just assuming that our functor restricts to $\mathbf{D}^e$.

5.2.1 Continuous functors

Definition 5.2.1. *A functor $F\colon \mathbf{D}^e \to \mathbf{D}^e$ is called* continuous*, if for every expanding sequence $\langle (D_n)_{n\in\mathbb{N}}, (e_{mn}\colon D_n \to D_m)_{n\sqsubseteq m\in\mathbb{N}}\rangle$ with colimit $\langle D, (e_n)_{n\in\mathbb{N}}\rangle$ we have that $\langle F(D), (F(e_n))_{n\in\mathbb{N}}\rangle$ is a colimit of the sequence $\langle (F(D_n))_{n\in\mathbb{N}}, (F(e_{mn})\colon F(D_n) \to F(D_m))_{n\sqsubseteq m\in\mathbb{N}}\rangle$.*

This, obviously, is Scott-continuity expressed for functors. Whether we formulate it in terms of expanding sequences or expanding systems is immaterial. The question is not, what is allowed to enter the model, but rather, how much do I have to check before I can apply the theorems in this chapter. And sequences are all that is needed.

This, then, is the central lemma on which our domain theoretic technique for solving recursive domain equations is based (recall that f^* is our notation for the upper adjoint of f):

Lemma 5.2.2. *Let F be a continuous functor on a category $\mathbf{D}^e$ of domains. For each embedding $e\colon A \to F(A)$ consider the colimit $\langle D, (e_n)_{n\in\mathbb{N}}\rangle$ of the expanding sequence $A \xrightarrow{e} F(A) \xrightarrow{F(e)} F(F(A)) \xrightarrow{F(F(e))} \cdots$. Then D is isomorphic to $F(D)$ via the maps*

$$\begin{array}{rcl} \mathsf{fold} &=& \bigsqcup^{\uparrow}_{n\in\mathbb{N}} e_{n+1}\circ F(e_n)^* \quad : \quad F(D)\to D, \textit{ and} \\ \mathsf{unfold} &=& \bigsqcup^{\uparrow}_{n\in\mathbb{N}} F(e_n)\circ e^*_{n+1} \quad : \quad D\to F(D). \end{array}$$

For each $n\in\mathbb{N}$ they satisfy the equations

$$\begin{array}{rcl} F(e_n) &=& \mathsf{unfold}\circ e_{n+1} \\ F(e_n)^* &=& e^*_{n+1}\circ \mathsf{fold}\,. \end{array}$$

Proof. We know that $\langle D, (e_n)_{n\in\mathbb{N}\setminus\{0\}}\rangle$ is a colimit over the diagram

$$F(A) \xrightarrow{F(e)} F(F(A)) \xrightarrow{F(F(e))} \cdots$$

(clipping off the first approximation makes no difference), where there is also the cocone $\langle F(D), (F(e_n))_{n\in\mathbb{N}}\rangle$. The latter is also colimiting by the continuity of F. In this situation Theorem 3.3.7 provides us with unique mediating morphisms which are precisely the stated fold and unfold. They are inverses of each other because both cocones are colimiting. The equations follow from the explicit description of mediating morphisms in Theorem 3.3.7. $\square$

Note that since we have restricted attention to pointed domains, we always have the initial embedding $e\colon \mathbb{I}\to F(\mathbb{I})$. The solution to $X\cong F(X)$ based on this embedding we call *canonical* and denote it by $\mathsf{FIX}(F)$.

5.2.2 Local continuity

Continuity of a functor is a hard condition to verify. Luckily there is a property which is stronger but nevertheless much easier to check. It will also prove useful in the next section.

Definition 5.2.3. *A functor F from* **D** *to* **E**, *where* **D** *and* **E** *are categories of domains, is called* locally continuous, *if the maps $\mathsf{Hom}(D,D')\longrightarrow \mathsf{Hom}(F(D),F(D'))$, $f\mapsto F(f)$, are continuous for all objects D and D' from* **D**.

Proposition 5.2.4. *A locally continuous functor $F\colon \mathbf{D}\to\mathbf{E}$ restricts to a continuous functor from $\mathbf{D}^e$ to $\mathbf{E}^e$.*

We will soon generalize this, so there is no need for a proof at this point.

Typically, recursive domain equations are built from the basic constructions listed in Section 3.2. The strategy is to check local continuity for each of these individually and then rely on the fact that composition of continuous functors yields a continuous functor. However, we must realize that the function space construction is contravariant in its first and covariant in its second variable, and so the technique from the preceding paragraph does not immediately apply. Luckily, it can be strengthened to cover this case as well.

Definition 5.2.5. *A functor $F\colon \mathbf{D}^{op} \times \mathbf{D'} \to \mathbf{E}$, contravariant in its first, covariant in its second variable, is called* locally continuous, *if for directed sets $A \subseteq \mathsf{Hom}(D_2, D_1)$ and $A' \subseteq \mathsf{Hom}(D'_1, D'_2)$ (where D_1, D_2 are objects in $\mathbf{D}$ and D'_1, D'_2 are objects in $\mathbf{D'}$) we have*

$$F(\bigsqcup\nolimits^{\uparrow} A, \bigsqcup\nolimits^{\uparrow} A') = \bigsqcup_{f\in A, f'\in A'}^{\uparrow} F(f, f')$$

in $\mathsf{Hom}(F(D_1, D'_1), F(D_2, D'_2))$.

Proposition 5.2.6. *If $F\colon \mathbf{D}^{op} \times \mathbf{D'} \to \mathbf{E}$ is a mixed variant, locally continuous functor, then it defines a continuous covariant functor $\hat{F}$ from $\mathbf{D}^e \times \mathbf{D'}^e$ to $\mathbf{E}^e$ as follows:*

$$\begin{aligned} \hat{F}(D, D') &= F(D, D') \text{ for objects, and} \\ \hat{F}(e, e') &= F(e^*, e') \text{ for embeddings.} \end{aligned}$$

The upper adjoint to $\hat{F}(e, e')$ is given by $F(e, {e'}^*)$.

Proof. Let (e, e^*) and $(e', {e'}^*)$ be e-p-pairs in $\mathbf{D}$ and $\mathbf{D'}$, respectively. We calculate $F(e, {e'}^*) \circ \hat{F}(e, e') = F(e, {e'}^*) \circ F(e^*, e') = F(e^* \circ e, {e'}^* \circ e') = F(\mathsf{id}, \mathsf{id}) = \mathsf{id}$ and $\hat{F}(e, e') \circ F(e, {e'}^*) = F(e^*, e') \circ F(e, {e'}^*) = F(e \circ e^*, e' \circ {e'}^*) \sqsubseteq F(\mathsf{id}, \mathsf{id}) = \mathsf{id}$, so $\hat{F}$ maps indeed pairs of embeddings to embeddings.

For continuity, let $\langle (D_n), (e_{mn})\rangle$ and $\langle (D'_n), (e'_{mn})\rangle$ be expanding sequences in $\mathbf{D}$ and $\mathbf{D'}$ with colimits $\langle D, (e_n)\rangle$ and $\langle D', (e'_n)\rangle$, respectively. By Lemma 3.3.8 this implies $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} e_n \circ e_n^* = \mathsf{id}_D$ and $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} e'_n \circ {e'}^*_n = \mathsf{id}_{D'}$. By local continuity we have $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} \hat{F}(e_n, e'_n) \circ \hat{F}(e_n, e'_n)^* = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} F(e_n^*, e'_n) \circ F(e_n, {e'}^*_n) = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} F(e_n \circ e_n^*, e'_n \circ {e'}^*_n) = F(\bigsqcup^{\uparrow}_{n\in\mathbb{N}} e_n \circ e_n^*, \bigsqcup^{\uparrow}_{n\in\mathbb{N}} e'_n \circ {e'}^*_n) = F(\mathsf{id}_D, \mathsf{id}_{D'}) = \mathsf{id}_{F(D,D')}$ and so $\langle \hat{F}(D, D'), (\hat{F}(e_n, e'_n))_{n\in\mathbb{N}}\rangle$ is a colimit of $\langle (\hat{F}(D_n, D'_n))_{n\in\mathbb{N}}, (\hat{F}(e_{mn}, e'_{mn}))_{n\sqsubseteq m\in\mathbb{N}}\rangle$. □

While it may seem harmless to restrict a covariant functor to embeddings in order to solve a recursive domain equation, it is nevertheless not clear what the philosophical justification for this step is. For mixed variant functors this question becomes even more pressing since we explicitly *change* the functor. As already mentioned, a satisfactory answer has only recently been found, [Fre91, Pit93b]. We present Peter Freyd's solution in the next section.

Let us take stock of what we have achieved so far. Building blocks for recursive domain equations are the constructors of Section 3.2, $\times, \oplus, \to$, etc., each of which is readily seen to define a locally continuous functor. Translating them to embeddings via the preceding proposition, we get continuous functors of one or two variables. We further need the diagonal $\Delta\colon \mathbf{D}^e \to \mathbf{D}^e \times \mathbf{D}^e$ to deal with multiple occurrences of X in the body of the equation. Then we note that colimits in a finite power of $\mathbf{D}^e$ are calculated coordinatewise and hence the diagonal and the tupling of continuous functors are continuous. Finally, we include constant functors to allow for constants to occur in an equation. Two more operators will be added below: the bilimit in the next section and various powerdomain constructions in Chapter 6.

5.2.3 Parameterized equations

Suppose that we are given a locally continuous functor F in two variables. Given any domain D we can solve the equation $X \cong F(D,X)$ using the techniques of the preceding sections. Remember that by default we mean the solution according to Lemma 5.2.2 based on $e\colon \mathbb{I} \to F(D,\mathbb{I})$, so there is no ambiguity. Also, we have given a concrete representation for bilimits in Theorem 3.3.7, so $\mathsf{FIX}(F(D,\cdot))$ is also well-defined in this respect. We want to show that it extends to a functor.

Notation is a bit of a problem. Let $F\colon \mathbf{D}_{\perp!} \times \mathbf{E}_{\perp!} \to \mathbf{E}_{\perp!}$ be a functor in two variables. We set F_D for the functor on $\mathbf{E}_{\perp!}$ which maps E to $F(D,E)$ for objects and $g\colon E \xrightarrow{\perp!} E'$ to $F(\mathsf{id}_D, g)$ for morphisms. Similarly for $F_{D'}$. The embeddings into the canonical fixpoint of F_D, resp. $F_{D'}$, we denote by $e_0, e_1, \ldots$ and $e'_0, e'_1, \ldots$, and we use e and e' for the unique strict function from $\mathbb{I}$ into D and D', respectively.

Proposition 5.2.7. *Let $F\colon \mathbf{D}_{\perp!} \times \mathbf{E}_{\perp!} \to \mathbf{E}_{\perp!}$ be a locally continuous functor. Then the following defines a locally continuous functor from $\mathbf{D}_{\perp!}$ to $\mathbf{E}_{\perp!}$:*

$$\begin{array}{rcl} \textit{On objects} & : & D \mapsto \mathsf{FIX}(F_D), \\ \textit{on morphisms} & : & (f\colon D \to D') \mapsto \bigsqcup_{n\in\mathbb{N}}^{\uparrow} e'_n \circ f_n \circ e_n^* \end{array}$$

where the sequence $(f_n)_{n\in\mathbb{N}}$ is defined recursively by $f_0 = \mathsf{id}_{\mathbb{I}}, f_{n+1} = F(f, f_n)$.

Proof. Let D and D' be objects of $\mathbf{D}_{\perp!}$ and let $f\colon D \xrightarrow{\perp!} D'$ be a strict function. The solution to $X \cong F(D,X)$ is given by the bilimit

$$\begin{tikzcd} & \mathsf{FIX}(F_D) & & \\ \mathbb{I} \arrow[ur, "e_0"] \arrow[r, "e"] & F_D(\mathbb{I}) \arrow[u, "e_1"] \arrow[r, "F_D(e)"] & F_D^2(\mathbb{I}) \arrow[ul, "e_2"'] \arrow[r] & \cdots \end{tikzcd} \quad \ldots$$

and similarly for D'. Corresponding objects of the two expanding sequences are connected by $f_n\colon F_D^n(\mathbb{I}) \xrightarrow{\perp!} F_{D'}^n(\mathbb{I})$. They commute with the embeddings of the expanding sequences: For $n = 0$ we have $F_{D'}^0(e') \circ f_0 = e' \circ \mathsf{id}_{\mathbb{I}} = e' = f_1 \circ e = f_1 \circ F_D^0(e)$ because there is only one strict map from $\mathbb{I}$ to $F^1(D')$. Higher indices follow by induction:

$$\begin{aligned}
F^{n+1}_{D'}(e') \circ f_{n+1} &= F(\mathsf{id}_{D'}, F^n_{D'}(e')) \circ F(f, f_n) \\
&= F(f, F^n_{D'}(e') \circ f_n) \\
&= F(f, f_{n+1} \circ F^n_D(e)) \\
&= F(f, f_{n+1}) \circ F(\mathsf{id}_D, F^n_D(e)) \\
&= f_{n+2} \circ F^{n+1}_D(e).
\end{aligned}$$

So we have a second cocone over the sequence defining $\mathsf{FIX}(F_D)$ and using the fact that colimits in $\mathbf{E_{\perp!}}^e$ are also colimits in $\mathbf{E_{\perp!}}$, we get a (unique) mediating morphism from $\mathsf{FIX}(F_D)$ to $\mathsf{FIX}(F_{D'})$. By Theorem 3.3.7 it has the postulated representation.

Functoriality comes for free from the uniqueness of mediating morphisms. It remains to check local continuity. So let A be a directed set of maps from D to D'. We easily get $(\bigsqcup^{\uparrow} A)_n = \bigsqcup^{\uparrow}_{f \in A} f_n$ by induction and the local continuity of F. The supremum can be brought to the very front by the continuity of composition and general associativity. □

Note that this proof works just as well for mixed variant functors. As an application, suppose we are given a system of simultaneous equations

$$\begin{aligned}
X_1 &\cong F_1(X_1, \ldots, X_n) \\
&\vdots \\
X_n &\cong F_n(X_1, \ldots, X_n).
\end{aligned}$$

We can solve these one after the other, viewing $X_2, \ldots, X_n$ as parameters for the first equation, substituting the result for X_1 in the second equation and so on. It is more direct to pass from $\mathbf{D}$ to $\mathbf{D}^n$, for which Theorem 3.3.7 and the results of this chapter remain true, and then solve these equations simultaneously with the tupling of the F_i. The fact that these two methods yield isomorphic results is known as *Bekič's rule* [Bek69].

5.3 Canonicity

We have seen in the first section of this chapter that recursive domain equations arise in various contexts. After having demonstrated a technique for solving them, we must now check whether the solutions match the particular requirements of these applications.

5.3.1 Invariance and minimality

Let us begin with a technique of internalizing the expanding sequence $\mathbb{I} \to F(\mathbb{I}) \to F(F(\mathbb{I})) \to \cdots$ into the canonical solution. This will allow us to do proofs about $\mathsf{FIX}(F)$ without (explicit) use of the defining expanding sequence.

Lemma 5.3.1. *Let F be a locally continuous functor on a category of domains* $\mathbf{D}$ *and let $i\colon F(A) \to A$ be an isomorphism. Then there exists a least homomorphism $h_{C,A}$*

from A to every other F-algebra $f\colon F(C) \to C$. It equals the least fixpoint of the functional $\phi_{C,A}$ on $[A \longrightarrow C]$ which is defined by

$$\phi_{C,A}(g) = f \circ F(g) \circ i^{-1} .$$

Least homomorphisms compose: If $j\colon F(B) \to B$ is also an isomorphism, then $h_{C,A} = h_{C,B} \circ h_{B,A}$.

Proof. The functional $\phi = \phi_{C,A}$ is clearly continuous because F is locally continuous and composition is a continuous operation. Since we have globally assumed least elements, the function space $[A \longrightarrow C]$ contains $\mathsf{c}_\perp$ as a least element. So the least fixpoint $h_{C,A}$ of $\phi_{C,A}$ calculated as the supremum of the chain $\mathsf{c}_\perp \sqsubseteq \phi(\mathsf{c}_\perp) \sqsubseteq \cdots$ exists. We show by induction that it is below every homomorphism h. For $\mathsf{c}_\perp$ this is obvious. For the induction step assume $g \sqsubseteq h$. We calculate: $\phi(g) = f \circ F(g) \circ i^{-1} \sqsubseteq f \circ F(h) \circ i^{-1} = h$. It follows that $\mathsf{fix}(\phi) = h_{C,A} \sqsubseteq h$ holds. On the other hand, every fixpoint of ϕ is a homomorphism: $h \circ i = \phi(h) \circ i = f \circ F(h) \circ i^{-1} \circ i = f \circ F(h)$.

The claim about composition of least homomorphisms can also be shown by induction. But it is somewhat more elegant to use the invariance of least fixpoints, Lemma 2.1.21. Consider the diagram

$$\begin{tikzcd}
{[B \longrightarrow C]} \arrow[r, "H"] \arrow[d, "\phi_{C,B}"'] & {[A \longrightarrow C]} \arrow[d, "\phi_{C,A}"] \\
{[B \longrightarrow C]} \arrow[r, "H"] & {[A \longrightarrow C]}
\end{tikzcd}$$

where H is the strict operation which assigns $g \circ h_{B,A}$ to $g \in [B \longrightarrow C]$. The diagram commutes, because $H \circ \phi_{C,B}(g) = f \circ F(g) \circ j^{-1} \circ h_{B,A} = f \circ F(g \circ h_{B,A}) \circ i^{-1}$ (because $h_{B,A}$ is an homomorphism) $= \phi_{C,A}(H(g))$. Lemma 2.1.21 then gives us the desired equality: $h_{C,A} = \mathsf{fix}(\phi_{C,A}) = H(\mathsf{fix}(\phi_{C,B})) = \mathsf{fix}(\phi_{C,B}) \circ h_{B,A} = h_{C,B} \circ h_{B,A}$. ☐

Specializing the second algebra in this lemma to be $i\colon F(A) \to A$ itself, we deduce that on every fixpoint of a locally continuous functor there exists a least endomorphism $h_{A,A}$. Since the identity is always an endomorphism, the least endomorphism must be below the identity and idempotent, i.e. a kernel operator and in particular strict. This we will use frequently below.

Theorem 5.3.2. (Invariance, Part 1) *Let F be a locally continuous functor on a category of domains* $\mathbf{D}$ *and let $i\colon F(A) \to A$ be an isomorphism. Then the following are equivalent:*

1. *A is isomorphic to the canonical fixpoint $\mathsf{FIX}(F)$;*
2. *id_A is the least endomorphism of A;*
3. *$\mathsf{id}_A = \mathsf{fix}(\phi_{A,A})$ where $\phi_{A,A}\colon [A \longrightarrow A] \to [A \longrightarrow A]$ is defined by $\phi_{A,A}(g) = i \circ F(g) \circ i^{-1}$;*

4. id_A *is the only strict endomorphism of* A.

Proof. ($1 \Longrightarrow 2$) The least endomorphism on $D = \mathsf{FIX}(F)$ is calculated as the least fixpoint of $\phi_{D,D}: g \mapsto \mathsf{fold} \circ F(g) \circ \mathsf{unfold}$. With the usual notation for the embeddings of $F^n(\mathbb{I})$ into D we get (by induction): $\mathsf{c}_\perp = e_0 \circ e_0^*$ and $\phi^n(\mathsf{c}_\perp) = \phi(\phi^{n-1}(\mathsf{c}_\perp)) = \phi(e_{n-1} \circ e_{n-1}^*) = \mathsf{fold} \circ F(e_{n-1}) \circ F(e_{n-1}^*) \circ \mathsf{unfold} = e_n \circ e_n^*$, where the last equality follows because fold and unfold are mediating morphisms. Lemma 3.3.8 entails that the supremum of the $\phi^n(\mathsf{c}_\perp)$ is the identity.

The equivalence of (2) and (3) is a reformulation of Lemma 5.3.1.

($3 \Longrightarrow 4$) Suppose $h: A \xrightarrow{\perp !} A$ defines an endomorphism of the algebra $i: F(A) \to A$. We apply the invariance property of least fixpoints, Lemma 2.1.21, to the diagram (where ϕ now stands for $\phi_{A,A}$)

$$\begin{array}{ccc}
[A \longrightarrow A] & \xrightarrow{H} & [A \longrightarrow A] \\
\phi \downarrow & & \downarrow \phi \\
[A \longrightarrow A] & \xrightarrow{H} & [A \longrightarrow A]
\end{array}$$

where H maps $g \in [A \longrightarrow A]$ to $h \circ g$. This is a strict operation because h is assumed to be strict. The diagram commutes: $H \circ \phi(g) = H(i \circ F(g) \circ i^{-1}) = h \circ i \circ F(g) \circ i^{-1} = i \circ F(h) \circ F(g) \circ i^{-1} = \phi(H(g))$. By Lemma 2.1.21 we have $\mathsf{id}_A = \mathsf{fix}(\phi) = H(\mathsf{fix}(\phi)) = h \circ \mathsf{id}_A = h$.

($4 \Longrightarrow 1$) By the preceding lemma we have homomorphisms between A and $\mathsf{FIX}(F)$. They compose to the least endomorphisms on A, resp. $\mathsf{FIX}(F)$, which we know to be strict. But then they must be equal to the identity as we have just shown for $\mathsf{FIX}(F)$ and assumed for A. ☐

If, in the last third of this proof, we do not assume that id_A is the only strict endomorphism on A, then we still get an embedding-projection pair between $\mathsf{FIX}(F)$ and A. Thus we have:

Theorem 5.3.3. (Minimality, Part 1) *The canonical fixpoint of a locally continuous functor is a sub-domain of every other fixpoint.*

So we have shown that the canonical solution is the *least* fixpoint in a relevant sense. This is clearly a good canonicity result with respect to the first class of examples. For pedagogical reasons we have restricted attention to the covariant case first, but, as we will see in section 5.3.3, this characterization is also true for functors of mixed variance.

5.3.2 Initiality and finality

By a little refinement of the proofs of the preceding subsection we get the desired result that the canonical fixpoint together with fold is an initial F-algebra. One of the adjustments is that we have to pass completely to strict functions, because Lemma 5.3.1

does not guarantee the existence of strict homomorphisms and only of these can we prove unicity.

Theorem 5.3.4. (Initiality) *Let $F\colon \mathbf{D}_{\perp!} \to \mathbf{D}_{\perp!}$ be a locally continuous functor on a category of domains with strict functions. Then $\mathsf{fold}\colon F(D) \to D$ is an initial F-algebra where D is the canonical solution to $X \cong F(X)$.*

Proof. Let $f\colon F(A) \xrightarrow{\perp!} A$ be a strict F-algebra. The homomorphism $h\colon D \to A$ we get from Lemma 5.3.1 is strict as we see by inspecting its definition. That there are no others is shown as in the proof of Theorem 5.3.2, ($3 \Longrightarrow 4$). The relevant diagram for the application of Lemma 2.1.21 is now:

$$\begin{array}{ccc}
[D \longrightarrow D] & \xrightarrow{H} & [D \longrightarrow A] \\
\Big\downarrow{\scriptstyle \phi_{D,D}} & & \Big\downarrow{\scriptstyle \phi_{A,D}} \\
[D \longrightarrow D] & \xrightarrow{H} & [D \longrightarrow A].
\end{array}$$

□

By dualizing Lemma 5.3.1 and the proof of Theorem 5.3.2, ($3 \Longrightarrow 4$), we get the final co-algebra theorem. It is slightly stronger than initiality since it holds for all co-algebras, not only the strict ones.

Theorem 5.3.5. (Finality) *Let $F\colon \mathbf{D} \to \mathbf{D}$ be a locally continuous functor with canonical fixpoint $D = \mathsf{FIX}(F)$. Then $\mathsf{unfold}\colon D \to F(D)$ is a final co-algebra.*

5.3.3 Mixed variance

Let us now tackle the case that we are given an equation in which the variable X occurs both positively and negatively in the body, as in our first example $X \cong [X \longrightarrow X]$. We assume that by separating the negative occurrences from the positive ones, we have a functor in two variables, contravariant in the first and covariant in the second. As the reader will remember, solving such an equation required the somewhat magical passage to adjoints in the first coordinate. We will now see in how far we can extend the results from the previous two subsections to this case. Note that for a mixed variant functor the concept of F-algebra or co-algebra is no longer meaningful, as there are no homomorphisms. The idea is to pass to pairs of mappings. Lemma 5.3.1 is replaced by

Lemma 5.3.6. *Let $F\colon \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{D}$ be a mixed variant, locally continuous functor and let $i\colon F(A,A) \to A$ and $j\colon F(B,B) \to B$ be isomorphisms. Then there exists a least pair of functions $h\colon A \to B$ and $k\colon B \to A$ such that*

$$\begin{array}{ccc}
F(A,A) & \xrightarrow{F(k,h)} & F(B,B) \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\
A & \xrightarrow{\quad h \quad} & B
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
F(B,B) & \xrightarrow{F(h,k)} & F(A,A) \\
\Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle i} \\
B & \xrightarrow{\quad k \quad} & A
\end{array}$$

commute.

The composition of two such least pairs gives another one.

Proof. Define a Scott-continuous function ϕ on $[A \longrightarrow B] \times [B \longrightarrow A]$ by $\phi(f, g) = (j \circ F(g, f) \circ i^{-1}, i \circ F(f, g) \circ j^{-1})$ and let (h, k) be its least fixpoint. Commutativity of the two diagrams is shown as in the proof of Lemma 5.3.1.

Comment: The statement about composition of least pairs of functions is certainly true for constant bottom maps, and this is lifted to the limits by induction over the fixpoint computation.

$\square$

By equating A and B in this lemma, we get a least endofunction h which satisfies $h \circ f = f \circ F(h, h)$. Again, it must be below the identity. Let us call such endofunctions *mixed endomorphisms*.

Theorem 5.3.7. (Invariance, Part 2) *Let* $F\colon \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{D}$ *be a mixed variant and locally continuous functor and let* $i\colon F(A, A) \to A$ *be an isomorphism. Then the following are equivalent:*

1. *A is isomorphic to the canonical fixpoint* $\mathsf{FIX}(F)$*;*
2. id_A *is the least mixed endomorphism of A;*
3. $\mathsf{id}_A = \mathsf{fix}(\phi_{A,A})$ *where* $\phi_{A,A}\colon [A \longrightarrow A] \to [A \longrightarrow A]$ *is defined by* $\phi_{A,A}(g) = i \circ F(g, g) \circ i^{-1}$*;*
4. id_A *is the only strict mixed endomorphism of A.*

Proof. The proof is of course similar to that of Theorem 5.3.2, but let us spell out the parts where mixed variance shows up. Recall from Section 5.2.2 how the expanding sequence defining $D = \mathsf{FIX}(F)$ looks like: $\mathbb{I} \xrightarrow{e} F(\mathbb{I}, \mathbb{I}) \xrightarrow{F(e^*, e)} F(F(\mathbb{I}, \mathbb{I}), F(\mathbb{I}, \mathbb{I})) \longrightarrow \cdots$. If $e_0, e_1, \ldots$ are the colimiting maps into D, then $F(e_0^*, e_0), F(e_1^*, e_1), \ldots$ form the cocone into $F(D, D)$, which, by local continuity, is also colimiting. The equations from Lemma 5.2.2 read: $F(e_n^*, e_n) = \mathsf{unfold} \circ e_{n+1}$ and $F(e_n^*, e_n)^* = F(e_n, e_n^*) = e_{n+1}^* \circ \mathsf{fold}$. We show that the n-th approximation to the least mixed endomorphism equals $e_n \circ e_n^*$. For $n = 0$ we get $\mathsf{c}_\perp = e_0 \circ e_0^*$, and for the induction step:

$$\begin{aligned}
\phi^{n+1}(\mathsf{c}_\perp) &= \phi(\phi^n(\mathsf{c}_\perp)) \\
&= \phi(e_n \circ e_n^*) \\
&= \mathsf{fold} \circ F(e_n \circ e_n^*, e_n \circ e_n^*) \circ \mathsf{unfold} \\
&= \mathsf{fold} \circ F(e_n^*, e_n) \circ F(e_n, e_n^*) \circ \mathsf{unfold} \\
&= e_{n+1} \circ e_{n+1}^*.
\end{aligned}$$

(Note how contravariance in the first argument of F shuffles e_n and e_n^* in just the right way.)

(3 $\Longrightarrow$ 4) The diagram to which Lemma 2.1.21 is applied is as before, but $H\colon [A \longrightarrow A] \to [A \longrightarrow A]$ now maps $g\colon A \to A$ to $h \circ g \circ h$.

The rest can safely be left to the reader. $\square$

Theorem 5.3.8. (Minimality, Part 2) *The canonical fixpoint of a mixed variant and locally continuous functor is a sub-domain of every other fixpoint.*

Now that we have some experience with mixed variance, it is pretty clear how to deal with initiality and finality. The trick is to pass once more to pairs of (strict) functions.

Theorem 5.3.9. (Free mixed variant algebra) *Let* $F\colon {\mathbf{D}_{\perp!}}^{op} \times \mathbf{D}_{\perp!} \to \mathbf{D}_{\perp!}$ *be a mixed variant, locally continuous functor and let* D *be the canonical solution to* $X \cong F(X,X)$*. Then for every pair of strict continuous functions* $f\colon A \xrightarrow{\perp!} F(B,A)$ *and* $g\colon F(A,B) \xrightarrow{\perp!} B$ *there are unique strict functions* $h\colon A \xrightarrow{\perp!} D$ *and* $k\colon D \xrightarrow{\perp!} B$ *such that*

$$\begin{tikzcd} F(B,A) \arrow[r, "F(k,h)"] & F(D,D) \\ A \arrow[u, "f"] \arrow[r, "h"] & D \arrow[u, "\mathsf{unfold}"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} F(D,D) \arrow[r, "F(h,k)"] \arrow[d, "\mathsf{fold}"'] & F(A,B) \arrow[d, "g"] \\ D \arrow[r, "k"] & B \end{tikzcd}$$

commute.

We should mention that the passage from covariant to mixed-variant functors, which we have carried out here concretely, can be done on an abstract, categorical level as was demonstrated by Peter Freyd in [Fre91]. The feature of domain theory which Freyd uses as his sole axiom is the existence and coincidence of initial algebras and final co-algebras for "all" endofunctors ("all" to be interpreted in some suitable enriched sense, in our case as "all locally continuous endofunctors"). Freyd's results are the most striking contribution to date towards Axiomatic Domain Theory, for which see 8.4 below.

5.4 Analysis of solutions

We have worked hard in the last section in order to show that our domain theoretic solutions are canonical in various respects. Besides this being reassuring, the advantage of canonical solutions is that we can establish proof rules for showing properties of them. This is the topic of this section.

5.4.1 Structural induction on terms

This technique is in analogy with universal algebra. While one has no control over arbitrary algebras of a certain signature, we feel quite comfortable with the initial or term algebra. There, every element is described by a term and no identifications are made. The first property carries over to our setting quite easily. For each of the finitary constructions of Section 3.2, we have introduced a notation for the basis elements of the constructed domain, to wit, tuples $\langle d, e\rangle$, variants $(d\colon i)$, one-element constant $\perp \in \mathbb{I}$, and step-functions $(d \searrow e)$. Since our canonical solutions are built as bilimits, starting from $\mathbb{I}$, and since every basis element of a bilimit shows up at a finite iteration already, Theorem 3.3.11, these can be denoted by finite expressions. The proof can then be based on structural induction on the length of these terms.

Unicity, however, is hard to achieve and this is the fault of the function space. One has to define normal forms and prove conversion rules. A treatment along these lines, based on [Abr91b], is given in Chapter 7.3.

5.4.2 Admissible relations

This is a more domain-theoretic formulation of structural induction, based on certain relations. The subject has recently been expanded and re-organized in an elegant way by Andrew Pitts [Pit93b, Pit94]. We follow his treatment closely but do not seek the same generality. We start with admissible relations, which we have met shortly in Chapter 2 already.

Definition 5.4.1. *A relation $R \subseteq D^n$ on a pointed domain D is called* admissible *if it contains the constantly-bottom tuple and if it is closed under suprema of ω-chains. We write $\mathcal{R}^n(D)$ for the set of all admissible n-ary relations on D, ordered by inclusion. Unary relations of this kind are also called* admissible predicates.

This is tailored to applications of the Fixpoint Theorem 2.1.19, whence we preferred the slightly more inclusive concept of ω-chain over directed sets. If we are given a strict continuous function $f\colon D \xrightarrow{\perp!} E$, then we can apply it to relations pointwise in the usual way:

$$f^{rel}(R) = \{\langle f(x_1), \ldots, f(x_n)\rangle \mid \langle x_1, \ldots, x_n\rangle \in R\}.$$

Proposition 5.4.2. *For dcpo's D and E and admissible n-ary relations R on D and S on E the set $\{f \mid f^{rel}(R) \subseteq S\}$ is an admissible predicate on $[D \xrightarrow{\perp!} E]$.*

We also need to say how admissible relations may be transformed by our locally continuous functors. This is a matter of definition because there are several – and equally useful – possibilities.

Definition 5.4.3. *Let $F\colon {\mathbf{D}_{\perp!}}^{op} \times \mathbf{D}_{\perp!} \to \mathbf{D}_{\perp!}$ be a mixed variant and locally continuous functor on a category of domains and strict functions. An* admissible action on (n-ary) relations *for F is given by a function F^{rel} which assigns to each pair $\langle D, E\rangle$ a map $F^{rel}_{\langle D,E\rangle}$ from $\mathcal{R}(D) \times \mathcal{R}(E)$ to $\mathcal{R}(F(D,E))$. These maps have to be compatible with strict morphisms in $\mathbf{D}_{\perp!}$ as follows: If $f\colon D_2 \xrightarrow{\perp!} D_1$ and $g\colon E_1 \xrightarrow{\perp!} E_2$ and if $R_1 \in \mathcal{R}(D_1)$ etc., such that $f^{rel}(R_2) \subseteq R_1$ and $g^{rel}(S_1) \subseteq S_2$, then*

$$F(f,g)^{rel}(F^{rel}_{\langle D_1,E_1\rangle}(R_1,S_1)) \subseteq F^{rel}_{\langle D_2,E_2\rangle}(R_2,S_2).$$

(Admittedly, this is a bit heavy in terms of notation. But in our concrete examples it is simply not the case that the behaviour of $F^{rel}_{\langle D,E\rangle}$ on R and S is the same as – or in a simple way related to – the result of applying the functor to R and S viewed as dcpo's.)

Specializing f and g to identity mappings in this definition, we get:

Proposition 5.4.4. *The maps $F^{rel}_{\langle D,E\rangle}$ are antitone in the first and monotone in the second variable.*

Theorem 5.4.5. *Let $\mathbf{D}_{\perp!}$ be a category of domains and let F be a mixed variant and locally continuous functor from ${\mathbf{D}_{\perp!}}^{op} \times \mathbf{D}_{\perp!}$ to $\mathbf{D}_{\perp!}$ together with an admissible action on relations. Abbreviate $\mathsf{FIX}(F)$ by D. Given two admissible relations $R, S \in \mathcal{R}^n(D)$ such that*

$$\mathsf{unfold}^{rel}(R) \subseteq F^{rel}(S,R) \quad \textit{and} \quad \mathsf{fold}^{rel}(F^{rel}(R,S)) \subseteq S$$

then $R \subseteq S$ holds.

Proof. We know from the invariance theorem that the identity on D is the least fixpoint of ϕ, where $\phi(g) = \mathsf{fold} \circ F(g,g) \circ \mathsf{unfold}$. Let $P = \{f \in [D \xrightarrow{\perp!} D] \mid f^{rel}(R) \subseteq S\}$, which we know is an admissible predicate. We want that the identity on D belongs to P and for this it suffices to show that ϕ maps P into itself. So suppose $g \in P$:

$$\begin{array}{rcll}
\phi(g)^{rel}(R) & = & \mathsf{fold}^{rel} \circ F(g,g)^{rel} \circ \mathsf{unfold}^{rel}(R) & \text{by definition} \\
& \subseteq & \mathsf{fold}^{rel} \circ F(g,g)^{rel}(F^{rel}(S,R)) & \text{by assumption} \\
& \subseteq & \mathsf{fold}^{rel}(F^{rel}(R,S)) & \text{because } g \in P \\
& \subseteq & S & \text{by assumption}
\end{array}$$

Indeed, $\phi(g)$ belongs again to P. □

In order to understand the power of this theorem, we will study two particular actions in the next subsections. They, too, are taken from [Pit93b].

5.4.3 Induction with admissible relations

Definition 5.4.6. *Let F be a mixed variant functor as before. We call an admissible action on (n-ary) relations* logical*, if for all objects D and E and $R \in \mathcal{R}^n(D)$ we have $F^{rel}_{\langle D,E\rangle}(R, E^n) = F(D,E)^n$.*

Specializing R to be the whole D in Theorem 5.4.5 and removing the assumption $\mathsf{unfold}^{rel}(R) \subseteq F^{rel}(S,R)$, which for this choice of R is always satisfied for a logical action, we get:

Theorem 5.4.7. (Induction) *Let $\mathbf{D}_{\perp!}$ be a category of domains and let $F\colon {\mathbf{D}_{\perp!}}^{op} \times \mathbf{D}_{\perp!} \to \mathbf{D}_{\perp!}$ be a mixed variant and locally continuous functor together with a logical action on admissible predicates. Let D be the canonical fixpoint of F. If $S \in \mathcal{R}^1(D)$ is an admissible predicate, for which $x \in F^{rel}(D,S)$ implies $\mathsf{fold}(x) \in S$, then S must be equal to D.*

The reader should take the time to recognize in this the principle of structural induction on term algebras.

We exhibit a particular logical action on admissible predicates for functors which are built from the constructors of Section 3.2. If R, S are admissible predicates on the pointed domains D and E, then we set

$$\begin{array}{rcl} R_\perp & = & \mathsf{up}(R) \cup \{\perp\} \subseteq D_\perp, \\ R \times S & = & \{\langle x, y\rangle \in D \times E \mid x \in R, y \in S\}, \\ [R \longrightarrow S] & = & \{f \in [D \longrightarrow E] \mid f(R) \subseteq S\}, \\ R \oplus S & = & \mathsf{inl}(R) \cup \mathsf{inr}(S) \subseteq D \oplus E, \end{array}$$

and analogously for $\otimes$ and $[\cdot \xrightarrow{\perp!} \cdot]$. (This is not quite in accordance with our notational convention. For example, the correct expression for $[R \longrightarrow S]$ is $[\cdot \longrightarrow \cdot]^{rel}_{\langle D,E\rangle}(R, S)$.) The definition of the action for the function space operator should make it clear why we chose the adjective 'logical' for it.

We get more complicated functors by composing the basic constructors. The actions also compose in a straightforward way: If F, G_1, and G_2 are mixed variant functors on a category of domains then we can define a mixed variant composition $H = F \circ \langle G_1, G_2\rangle$ by setting $H(X, Y) = F(G_1(Y, X), G_2(X, Y))$ for objects and similarly for morphisms. Given admissible actions for each of F, G_1, and G_2, we can define an action for H by setting $H^{rel}(R, S) = F^{rel}(G_1^{rel}(S, R), G_2^{rel}(R, S))$. It is an easy exercise to show that this action is logical if all its constituents are.

5.4.4 Co-induction with admissible relations

In this subsection we work with another canonical relation on domains, namely the order relation. We again require that it is dominant if put in the covariant position.

Definition 5.4.8. *Let F be a mixed variant functor. We call an admissible action on binary relations* extensional*, if for all objects D and E and $R \in \mathcal{R}^n(D)$ we have $F^{rel}_{\langle D,E\rangle}(R, \sqsubseteq_E) = \sqsubseteq_{F(D,E)}$.*

Theorem 5.4.9. (Co-induction) *Let $\mathbf{D}_{\perp!}$ be a category of domains and let $F \colon {\mathbf{D}_{\perp!}}^{op} \times \mathbf{D}_{\perp!} \to \mathbf{D}_{\perp!}$ be a mixed variant and locally continuous functor together with an extensional action on binary relations. Let D be the canonical fixpoint of F. If $R \in \mathcal{R}^2(D)$ is an admissible relation such that for all $\langle x, y\rangle \in R$ we have $\langle \mathsf{unfold}(x), \mathsf{unfold}(y)\rangle \in F^{rel}(\sqsubseteq_D, R)$, then R is contained in $\sqsubseteq_D$.*

If we call an admissible binary relation R on D a *simulation*, if it satisfies the hypothesis of this theorem, then we can formulate quite concisely:

Corollary 5.4.10. *Two elements of the canonical fixpoint of a mixed variant and locally continuous functor are in the order relation if and only if they are related by a simulation.*

We still have to show that extensional actions exist. We proceed as in the last subsection and first give extensional actions for the primitive constructors and then rely on the fact that these compose. So let R, S be admissible binary relations on D, resp. E. We set:

$$\begin{array}{rcl} R_\bot & = & \{\langle x,y\rangle \in D^2 \mid x = \bot \text{ or } \langle x,y\rangle \in R\} \\ R \times S & = & \{\langle\langle x,y\rangle, \langle x',y'\rangle\rangle \in (D\times E)^2 \mid \\ & & \quad \langle x,x'\rangle \in R \text{ and } \langle y,y'\rangle \in S\} \\ [R \longrightarrow S] & = & \{\langle f,g\rangle \in [D \longrightarrow E]^2 \mid \forall x \in D.\ \langle f(x), g(x)\rangle \in S\} \\ R \oplus S & = & \{\langle x,y\rangle \in (D \oplus E)^2 \mid x = \bot \text{ or} \\ & & \quad (x = \mathsf{inl}(x'), y = \mathsf{inl}(y') \text{ and } \langle x',y'\rangle \in R) \text{ or} \\ & & \quad (x = \mathsf{inr}(x'), y = \mathsf{inr}(y') \text{ and } \langle x',y'\rangle \in S)\} \end{array}$$

and similarly for $\otimes$ and $[\cdot \xrightarrow{\bot!} \cdot]$. We call this family of actions 'extensional' because the definition in the case of the function space is the same as for the extensional order on functions.

Exercises 5.4.11. *1. Find recursive domain equations which characterize the three versions of the natural numbers from Figure 2.*

2. *[Ern85] Find an example which demonstrates that the ideal completion functor is not locally continuous. Characterize the solutions to $X \cong \mathsf{Idl}(X, \sqsubseteq)$.*

3. *[DHR71] Prove that only the one-point poset satisfies $P \cong [P \xrightarrow{m} P]$.*

4. *Verify Bekič's rule in the dcpo case. That is, let D, E be pointed dcpo's and let $f\colon D \times E \to D$ and $g\colon D \times E \to E$ be continuous functions. We can solve the equations*

$$x = f(x,y) \qquad y = g(x,y)$$

directly by taking the simultaneous fixpoint $(a,b) = \mathsf{fix}(\langle f,g\rangle)$. Or we can solve for one variable at a time by defining

$$h(y) = \mathsf{fix}(\lambda x.f(x,y)) \qquad k(y) = g(h(y),y)$$

and setting

$$d = \mathsf{fix}(k) \qquad c = h(d)\ .$$

Verify that $(a,b) = (c,d)$ holds by using fixpoint induction.

5. *Find an example which shows that the Initiality Theorem 5.3.4 may fail for non-strict algebras.*

6. *Why does Theorem 5.3.5 hold for arbitrary (non-strict) co-algebras?*

7. *What are initial algebra and final co-algebra for the functor $X \mapsto \mathbb{I} \,\dot\cup\, X$ on the category of sets? Show that they are not isomorphic as algebras.*

8. *(G. Plotkin) Let F be the functor which maps X to $[X \longrightarrow X]_\bot$ and let D be its canonical fixpoint. This gives rise to a model of the (lazy) lambda calculus (see [Bar84, Abr90c, AO93]). Prove that the denotation of the* Y *combinator in this model is the least fixpoint function* fix. *Proceed as follows:*

(a) *Define a multiplication on* D *by* $x \cdot y = \mathsf{unfold}(x)(y)$.

(b) *The interpretation* y_f *of* $\mathsf{Y}f$ *is* $\omega_f \cdot \omega_f$ *where* $\omega_f = \mathsf{fold}(x \mapsto f(x \cdot x))$. *Check that this is a fixpoint of* f. *It follows that* $\mathsf{fix}(f) \sqsubseteq y_f$ *holds.*

(c) *Define a subset* E *of* $[D \longrightarrow D]_\perp$ *by*

$$E = \{e \mid e \sqsubseteq \mathsf{id}_D \textit{ and } e(\omega_f) \cdot \omega_f \sqsubseteq \mathsf{fix}(f)\} .$$

(d) *Use Theorem 5.3.7 to show that* $\mathsf{id}_D \in E$. *Then* $y_f \sqsubseteq \mathsf{fix}(f)$ *is also valid.*

9. *Given an action on relations for a functor in four variables, contravariant in the first two, covariant in the last two, define an action for the functor* $(D, E) \mapsto \mathsf{FIX}(F(D, \cdot, E, \cdot))$. *Prove that the resulting action is logical (extensional) if the original action was logical (extensional).*

6 Equational theories

In the last chapter we saw how we can build initial algebras over domains. It is a natural question to ask whether we can also accommodate equations, i.e. construct free algebras with respect to equational theories. In universal algebra this is done by factoring the initial or term algebra with respect to the congruence generated by the defining equations, and we will see that we can proceed in a similar fashion for domains. Bases will play a prominent role in this approach.

The technique of the previous chapter, namely, to generate the desired algebra in an iterative process, is no longer applicable. A formal proof for this statement may be found in [AT89], Section III.3, but the result is quite intuitive: Recall that an F-algebra $\alpha\colon F(A) \to A$ encodes the algebraic structure on A by giving information about the basic operations on A, where $F(A)$ is the sum of the input domains for each basic operation. Call an equation *flat* if each of the equated terms contains precisely one operation symbol. For example, commutativity of a binary operation is expressed by a flat equation while associativity is not. Flat equations can be incorporated into the concept of F-algebras by including the input, on which the two operations agree, only once in $F(A)$. For non-flat equations such a trick is not available. What we need instead of just the basic operations is a description of all term operations over A. In this case, $F(A)$ will have to be the free algebra over A, the object we wanted to construct!

Thus F-algebras are not the appropriate categorical concept to model equational theories. The correct formalization, rather, is that of monads and Eilenberg-Moore algebras.

We will show the existence of free algebras for dcpo's and continuous domains in the first section of this chapter. For the former, we use the Adjoint Functor Theorem (see [Poi92], for example), for the latter, we construct the basis of the free algebra as a quotient of the term algebra.

Equational theories come up in semantics when non-deterministic languages are studied. They typically contain a commutative, associative, and idempotent binary operation, standing for the union of two possible branches a program may take. The associated algebras are known under the name 'powerdomains' and they have been the subject of detailed studies. We shall present some of their theory in the second section.

6.1 General techniques

6.1.1 Free dcpo-algebras

Let us recall the basic concepts of universal algebra so as to fix the notation for this chapter. A signature $\Sigma = \langle \Omega, \alpha \rangle$ consists of a set Ω of operation symbols and a map $\alpha\colon \Omega \to \mathbb{N}$, assigning to each operation symbol a (finite) arity. A Σ-algebra $\underline{A} = \langle A, I \rangle$ is given by a carrier set A and an interpretation I of the operation symbols, in the sense that for $f \in \Omega$, $I(f)$ is a map from $A^{\alpha(f)}$ to A. We also write f_A or even f for the interpreted operation symbol and speak of the operation f on A. A homomorphism between two Σ-algebras $\underline{A}$ and $\underline{B}$ is a map $h\colon A \to B$ which commutes with the operations:

$$\forall f \in \Omega.\ h(f_A(a_1, \ldots, a_{\alpha(f)})) = f_B(h(a_1), \ldots, h(a_{\alpha(f)}))$$

We denote the term algebra over a set X with respect to a signature Σ by $\underline{T_\Sigma(X)}$. It has the universal property that each map from X to A, where $\underline{A} = \langle A, I \rangle$ is a Σ-algebra, can be extended uniquely to a homomorphism $\bar{h}\colon \underline{T_\Sigma(X)} \to \underline{A}$. Let V be a fixed countable set whose elements we refer to as 'variables'. Pairs of elements of $T_\Sigma(V)$ are used to encode equations. An equation $\tau_1 = \tau_2$ is said to hold in an algebra $\underline{A} = \langle A, I \rangle$ if for each map $h\colon V \to A$ we have $\bar{h}(\tau_1) = \bar{h}(\tau_2)$. The pair $\langle \bar{h}(\tau_1), \bar{h}(\tau_2) \rangle$ is also called an instance of the equation $\tau_1 = \tau_2$. The class of Σ-algebras in which each equation from a set $\mathcal{E} \subseteq T_\Sigma(V) \times T_\Sigma(V)$ holds, is denoted by $\mathbf{Set}(\Sigma, \mathcal{E})$.

Here we are interested in *dcpo-algebras*, characterized by the property that the carrier set is equipped with an order relation such that it becomes a dcpo, and such that each operation is Scott-continuous. Naturally, we also require the homomorphisms to be Scott-continuous. Because of the order we also can incorporate inequalities. So from now on we let a pair $\langle \tau_1, \tau_2 \rangle \in \mathcal{E} \subseteq T_\Sigma(V) \times T_\Sigma(V)$ stand for the inequality $\tau_1 \sqsubseteq \tau_2$. We use the notation $\mathbf{DCPO}(\Sigma, \mathcal{E})$ for the class of all dcpo-algebras over the signature Σ which satisfy the inequalities in $\mathcal{E}$. For these we have:

Proposition 6.1.1. *For every signature Σ and set $\mathcal{E}$ of inequalities, the class $\mathbf{DCPO}(\Sigma, \mathcal{E})$ with Scott-continuous homomorphisms forms a complete category.*

Proof. It is checked without difficulties that $\mathbf{DCPO}(\Sigma, \mathcal{E})$ is closed under products and equalizers, which both are defined as in the ordinary case. $\square$

This proves that we have one ingredient for the Adjoint Functor Theorem, namely, a complete category $\mathbf{DCPO}(\Sigma, \mathcal{E})$ and a (forgetful) functor $U\colon \mathbf{DCPO}(\Sigma, \mathcal{E}) \to \mathbf{DCPO}$ which preserves all limits. The other ingredient is the so-called solution set condition. For this setup it says that each dcpo can generate only set-many non-isomorphic dcpo-algebras. This is indeed the case: Given a dcpo D and a continuous map $i\colon D \to A$, where A is the carrier set of a dcpo-algebra $\underline{A}$, we construct the dcpo-subalgebra of $\underline{A}$ generated by $i(D)$ in two stages. In the first we let S be the (ordinary) subalgebra of $\underline{A}$ which is generated by $i(D)$. Its cardinality is bounded by an expression depending on the cardinality of D and Ω. Then we add to S all suprema of directed subsets until we get a sub-dcpo $\bar{S}$ of the dcpo A. Because we have required the operations on A to be Scott-continuous, $\bar{S}$ remains to be a subalgebra. The crucial step in this argument now is that the cardinality of $\bar{S}$ is bounded by $2^{|S|}$ as we asked you to show in Exercise 2.3.9(34). All in all, given Σ, the cardinality of $\bar{S}$ has a bound depending on $|D|$ and so there is only room for a set of different dcpo-algebras. Thus we have shown:

Theorem 6.1.2. *For every signature Σ and set $\mathcal{E}$ of inequalities, the forgetful functor $U\colon \mathbf{DCPO}(\Sigma, \mathcal{E}) \to \mathbf{DCPO}$ has a left adjoint.*

Equivalently: *For each dcpo D the free dcpo-algebra over D with respect to Σ and $\mathcal{E}$ exists.*

The technique of this subsection is quite robust and has been used in [Nel81] for proving the existence of free algebras under more general notions of convergence than that of directed-completeness. This, however, is not the direction we are interested in, and instead we shall now turn to continuous domains.

6.1.2 Free continuous domain-algebras

None of the categories of approximated dcpo's, or domains, we have met so far is complete. Both infinite products and equalizers may fail to exist. Hence we cannot rely on the Adjoint Functor Theorem. While this will result in a more technical proof, there will also be a clear advantage: we will gain explicit information about the basis of the constructed free algebra, which may help us to find alternative descriptions. In the case of dcpo's, such concrete representations are quite complicated, see [Nel81, ANR82].

We denote the category of dcpo-algebras, whose carriers form a continuous domain, by $\mathbf{CONT}(\Sigma, \mathcal{E})$ and speak of (continuous) *domain-algebras*. Again there is the obvious forgetful functor $U\colon \mathbf{CONT}(\Sigma, \mathcal{E}) \to \mathbf{CONT}$. To keep the notation manageable we shall try to suppress mention of U, in particular, we will write A for $U(\underline{A})$ on objects and make no distinction between h and $U(h)$ on morphisms. Let us write down the condition for adjointness on which we will base our proof:

$$
\begin{tikzcd}
\mathbf{CONT} & D \arrow[r, "\eta"] \arrow[dr, "g"'] & F(D) \arrow[d, "\mathsf{ext}(g)"] & \underline{F(D)} \arrow[d, "\exists!\mathsf{ext}(g)"'] & \mathbf{CONT}(\Sigma, \mathcal{E}) \\
& & A & \underline{A} &
\end{tikzcd}
$$

In words: Suppose a signature Σ and a set $\mathcal{E}$ of inequalities has been fixed. Then given a continuous domain D we must construct a dcpo-algebra $\underline{F(D)}$, whose carrier set $F(D)$ is a continuous domain, and a Scott-continuous function $\eta\colon D \to F(D)$ such that $\underline{F(D)}$ satisfies the inequalities in $\mathcal{E}$ and such that given any such domain-algebra $\underline{A}$ and Scott-continuous map $g\colon D \to A$ there is a unique Scott-continuous homomorphism $\mathsf{ext}(g)\colon \underline{F(D)} \to \underline{A}$ for which $\mathsf{ext}(g) \circ \eta = g$. (It may be instructive to compare this with Definition 3.1.9.)

Comment: In fact, what is shown below is that the free *domain*-algebra is also free for all *dcpo*-algebras, in other words, the adjunction between **CONT** and $\mathbf{CONT}(\Sigma, \mathcal{E})$ is (up to isomorphism) the restriction of the adjunction between **DCPO** and $\mathbf{DCPO}(\Sigma, \mathcal{E})$ established in Theorem 6.1.2.

The idea for solving this problem is to work explicitly with bases (cf. Section 2.2.6). So assume that we have fixed a basis $\langle B, \ll \rangle$ for the continuous domain D. We will construct an abstract basis $\langle FB, \prec \rangle$ for the desired free domain-algebra $\underline{F(D)}$. The underlying set FB is given by the set $T_\Sigma(B)$ of all terms over B. On FB we have two natural order relations. The first, which we denote by $\underset{\sim}{\sqsubset}$, is induced by the defining set $\mathcal{E}$ of inequalities. We can give a precise definition in the form of a deduction scheme:
Axioms:

(A1) $t \underset{\sim}{\sqsubset} t$ for all $t \in FB$.

(A2) $s \underset{\sim}{\sqsubset} t$ if this is an instance of an inequality from $\mathcal{E}$.

Rules:

(R1) If $f \in \Omega$ is an n-ary function symbol and if $s_1 \underset{\sim}{\sqsubset} t_1, \ldots, s_n \underset{\sim}{\sqsubset} t_n$ then $f(s_1, \ldots, s_n) \underset{\sim}{\sqsubset} f(t_1, \ldots, t_n)$.

(R2) If $s \sqsubseteq t$ and $t \sqsubseteq u$ then $s \sqsubseteq u$.

The relation $\sqsubseteq$ is the 'least substitutive preorder' in the terminology of [Sto88]. It is the obvious generalization of the concept of a congruence relation to the preordered case, and indeed, $\langle FB, \sqsubseteq \rangle$ is the free preordered algebra over B. The associated equivalence relation we denote by $\approx$. The factor set $FB/_{\approx}$ is ordered by $\sqsubseteq$ and this is the free ordered algebra over B.

Let us now turn to the second relation on FB, namely, the one which arises from the order of approximation on B. We set $t \prec^s t'$ if t and t' have the same structure and corresponding constants are related by $\ll$. Formally, $\prec^s$ is given through the deduction scheme:
Axioms:

(A) $a \prec^s b$ if $a \ll b$ in B.

Rules:

(R) If $f \in \Omega$ is an n-ary function symbol and if $s_1 \prec^s t_1, \ldots, s_n \prec^s t_n$ then $f(s_1, \ldots, s_n) \prec^s f(t_1, \ldots, t_n)$.

Our first observation is that $\prec^s$ satisfies the interpolation axiom:

Proposition 6.1.3. *$\langle FB, \prec^s \rangle$ is an abstract basis.*

Proof. Since $\prec^s$ relates only terms of the same structure, it is quite obvious that it is a transitive relation. For the interpolation axiom assume that $s \prec^s t$ holds for all elements s of a finite set $M \subseteq FB$. For each occurrence of a constant a in t let M_a be the set of constants which occur in the same location in one of the terms $s \in M$. Since M_a is finite and since $M_a \ll a$ holds by the definition of $\prec^s$, we find interpolating elements a' between M_a and a. Let t' be the term in which all constants are replaced by the corresponding interpolating element. This is a term which interpolates between M and t in the relation $\prec^s$. $\square$

The question now is how to combine $\sqsubseteq$ and $\prec^s$. As a guideline we take Proposition 2.2.2(2). If the inequalities tell us that t_1 should be below s_1 and s_2 should be below t_2 and if s_1 approximates s_2 then it should be the case that t_1 approximates t_2. Hence we define $\prec$, the order of approximation on FB, to be the transitive closure of $\sqsubseteq \circ \prec^s \circ \sqsubseteq$. The following, somewhat technical properties will be instrumental for the free algebra theorem:

Proposition 6.1.4. *1. $\prec^s \circ \sqsubseteq$ is contained in $\prec^s \circ \sqsubseteq \circ \prec^s$.*

2. For every $n \leq m \in \mathbb{N}$ we have $(\sqsubseteq \circ \prec^s \circ \sqsubseteq)^n \subseteq (\sqsubseteq \circ \prec^s \circ \sqsubseteq)^m$.

Proof. (1) Assume $s \prec^s t \sqsubseteq u$. Let $C \subseteq B$ be the set of all constants which appear in the derivation of $t \sqsubseteq u$. For each $c \in C$ let M_c be the set of constants which appear in s at the same place as c appears in t. Of course, c may not occur in t at all; in this case M_c will be empty. If it occurs several times then M_c can contain more than one element. In any case, M_c is finite and $M_c \ll c$ holds. Let c' be an interpolating element between M_c and c. We now replace each constant c in the derivation of $t \sqsubseteq u$ by the corresponding constant c' and we get a valid derivation of a formula $t' \precsim u'$. (The catch is that an instance of an inequality is transformed into an instance of the same inequality.) It is immediate from the construction that $s \prec^s t' \precsim u' \prec^s u$ holds.

(2) Using (1) and the reflexivity of $\precsim$ we get

$$\precsim \circ \prec^s \circ \precsim \;\; \subseteq \;\; \precsim \circ (\prec^s \circ \precsim \circ \prec^s) \;\; \subseteq \;\; \precsim \circ \prec^s \circ \precsim \circ \precsim \circ \prec^s \circ \precsim .$$

The general case follows by induction. ☐

Lemma 6.1.5. *$\langle FB, \prec\rangle$ is an abstract basis.*

Proof. Transitivity has been built in, so it remains to look at the interpolation axiom. Let $M \prec t$ for a finite set M. From the definition of $\prec$ we get for each $s \in M$ a sequence of terms $s \precsim s^1 \prec^s s^2 \precsim \ldots \precsim s^{n(s)-1} \prec^s s^{n(s)} \precsim t$. The last two steps may be replaced by $s^{n(s)-1} \prec^s s' \precsim s'' \prec^s t$ as we have shown in the preceding proposition. The collection of all s'' is finite and we find an interpolating term t' between it and t according to Proposition 6.1.3. Because of the reflexivity of $\precsim$ we have $M \prec t' \prec t$. ☐

So we can take as the carrier set of our free algebra over D the ideal completion of $\langle FB, \prec\rangle$ and from Proposition 2.2.22 we know that this is a continuous domain. The techniques of Section 2.2.6 also help us to fill in the remaining pieces. The operations on $F(D)$ are defined pointwise: If $A_1, \ldots, A_n$ are ideals and if $f \in \Omega$ is an n-ary function symbol then we let $f_{F(D)}(A_1, \ldots, A_n)$ be the ideal which is generated by $\{f(t_1, \ldots, t_n) \mid t_1 \in A_1, \ldots, t_n \in A_n\}$. We need to know that this set is directed. It will follow if the operations on FB are monotone with respect to $\prec$. So assume we are given an operation symbol $f \in \Omega$ and pairs $s_1 \prec t_1, \ldots, s_n \prec t_n$. By definition, each pair translates into a sequence $s_i \precsim s_i^1 \prec^s s_i^2 \precsim \ldots \prec^s s_i^{m(i)} \precsim t_i$. Now we use Proposition 6.1.4(2) to extend all these sequences to the same length m. Then we can apply f step by step, using Rules (R1) and (R) alternately:

$$\begin{array}{rcl} f(s_1, \ldots, s_n) \precsim f(s_1^1, \ldots, s_n^1) & \prec^s & f(s_1^2, \ldots, s_n^2) \precsim \;\ldots \\ \ldots & \prec^s & f(s_1^m, \ldots, s_n^m) \precsim f(t_1, \ldots, t_n). \end{array}$$

Using the remark following Proposition 2.2.24 we infer that the operations $f_{F(D)}$ defined this way are Scott-continuous functions. Thus $\underline{F(D)}$ is a continuous domain-algebra. The generating domain D embeds into $F(D)$ via the extension η of the monotone inclusion of B into FB.

Theorem 6.1.6. *$\underline{F(D)}$ is a continuous domain algebra and is the free continuous dcpo-algebra over D with respect to Σ and $\mathcal{E}$.*

Proof. We already know the first part. For the second we must show that $\underline{F(D)}$ satisfies the inequalities in $\mathcal{E}$ and that it has the universal property with respect to all objects in **DCPO**$(\Sigma, \mathcal{E})$.

For the inequalities let $\langle \tau_1, \tau_2\rangle \in \mathcal{E}$ and let $h\colon V \to F(D)$ be a map. It assigns to each variable an ideal in FB. We must show that $\bar{h}(\tau_1)$ is a subset of $\bar{h}(\tau_2)$. As we have just seen, the ideal $\bar{h}(\tau_1)$ is generated by terms of the form $\bar{k}(\tau_1)$ where k is a map from V to FB, such that for each variable $x \in V$, $k(x) \in h(x)$. So suppose $s \prec \bar{k}(\tau_1)$ for such a k. Then $\bar{k}(\tau_1) \sqsubseteq\!\!\!\!\sim \bar{k}(\tau_2)$ is an instance of the inequality in the term algebra $\underline{FB} = \underline{T_\Sigma(B)}$ and so we know that $s \prec \bar{k}(\tau_2)$ also holds. The term $\bar{k}(\tau_2)$ belongs to $\bar{h}(\tau_2)$, again because the operations on $\underline{F(D)}$ are defined pointwise. So $s \in \bar{h}(\tau_2)$ as desired.

To establish the universal property assume that we are given a continuous map $g\colon D \to A$ for a dcpo-algebra $\underline{A}$ which satisfies the inequalities from $\mathcal{E}$. The restriction of g to the set $B \subseteq D$ has a unique monotone extension $\bar{g}$ to the preordered algebra $\langle FB, \precsim \rangle$. We want to show that $\bar{g}$ also preserves $\prec^s$. For an axiom $a \prec^s b$ this is clear because g is monotone on $\langle B, \ll \rangle$. For the rules (R) we use that $\bar{g}$ is a homomorphism and that the operations on $\underline{A}$ are monotone:

$$\begin{array}{rcl}
\bar{g}(f(s_1, \ldots, s_n)) & = & f_A(\bar{g}(s_1), \ldots, \bar{g}(s_n)) \\
& \sqsubseteq & f_A(\bar{g}(t_1), \ldots, \bar{g}(t_n)) \\
& = & \bar{g}(f(t_1, \ldots, t_n)) \;.
\end{array}$$

Together this says that $\bar{g}$ translates the order of approximation $\prec$ on FB to $\sqsubseteq$ on A, and therefore it can be extended to a homomorphism $\mathsf{ext}(g)$ on the ideal completion $F(D)$. Uniqueness of $\mathsf{ext}(g)$ is obvious. What we have to show is that $\mathsf{ext}(g)$, when restricted to B, equals g, because Proposition 2.2.24 does not give an extension but only a best approximation. We can nevertheless prove it here because g arose as the restriction of a continuous map on D. An element d of D is represented in $F(D)$ as the ideal $\eta(d)$ containing at least all of $B_d = B \cap \mathop{\downarrow\!\!\!\!\downarrow} d$ because of the axioms of our second deductive system. So we have: $\mathsf{ext}(g)(\eta(d)) = \bigsqcup^{\uparrow} \bar{g}(\eta(d)) \sqsupseteq \bigsqcup^{\uparrow} \bar{g}(B_d) = \bigsqcup^{\uparrow} g(B_d) = g(d)$. □

Theorem 6.1.7. *For any signature Σ and set $\mathcal{E}$ of inequalities the forgetful functor $U\colon$ **CONT**$(\Sigma, \mathcal{E}) \to$ **CONT** has a left adjoint F. It is equivalent to the restriction and corestriction of the left adjoint from Theorem 6.1.2 to* **CONT** *and* **CONT**$(\Sigma, \mathcal{E})$*, respectively.*

In other words: *Free continuous domain-algebras exist and they are also free with respect to dcpo-algebras.*

The action of the left adjoint functor on morphisms is obtained by assigning to a continuous function $g\colon D \to E$ the homomorphism which extends $\eta_E \circ g$.

$$\begin{tikzcd}
D \arrow[r, "\eta_D"] \arrow[d, "g"'] & F(D) \arrow[d, "F(g)"] \\
E \arrow[r, "\eta_E"] & F(E)
\end{tikzcd}$$

We want to show that F is locally continuous (Definition 5.2.3). To this end let us first look at the passage from maps to their extension.

Proposition 6.1.8. *The assignment* $g \mapsto \mathsf{ext}(g)$*, as a map from* $[D \longrightarrow A]$ *to* $[F(D) \longrightarrow A]$ *is Scott-continuous.*

Proof. By Proposition 2.2.25 it is sufficient to show this for the restriction of g to the basis B of D. Let G be a directed collection of monotone maps from B to A and let $t \in FB$ be a term in which the constants $a_1, \ldots, a_n \in B$ occur. We calculate:

$$\begin{aligned} \overline{\bigsqcup{}^{\uparrow} G(t)} &= t[\bigsqcup{}^{\uparrow} G(a_1)/a_1, \ldots, \bigsqcup{}^{\uparrow} G(a_n)/a_n] \\ &= \bigsqcup_{g \in G}{}^{\uparrow} t[g(a_1)/a_1, \ldots, g(a_n)/a_n] \\ &= \bigsqcup_{g \in G}{}^{\uparrow} \bar{g}(t), \end{aligned}$$

where we have written $t[b_1/a_1, \ldots, b_n/a_n]$ for the term in which each occurrence of a_i is replaced by b_i. Restriction followed by homomorphic extension followed by extension to the ideal completion gives a sequence of continuous functions $[D \longrightarrow A] \to [B \xrightarrow{m} A] \to [FB \xrightarrow{m} A] \to [F(D) \longrightarrow A]$ which equals ext. ☐

Cartesian closed categories can be viewed as categories in which the Hom-functor can be internalized. The preceding proposition formulates a similar closure property of the free construction: if the free construction can be cut down to a cartesian closed category then there the associated monad and the natural transformations that come with it can be internalized. This concept was introduced by Anders Kock [Koc70, Koc72]. It has recently found much interest under the name 'computational monads' through the work of Eugenio Moggi [Mog91].

Theorem 6.1.9. *For any signature* Σ *and set* $\mathcal{E}$ *of inequalities the composition* $U \circ F$ *is a locally continuous functor on* **CONT**.

Proof. The action of $U \circ F$ on morphisms is the combination of composition with η_E and ext. ☐

If $e \colon D \to E$ is an embedding then we can describe the action of F, respectively $U \circ F$, quite concretely. A basis element of $F(D)$ is the equivalence class of some term s. Its image under $F(e)$ is the equivalence class of the term s', which we get from s by replacing all constants in s by their image under e.

If we start out with an algebraic domain D then we can choose as its basis $\mathsf{K}(D)$, the set of compact elements. The order of approximation on $\mathsf{K}(D)$ is the order relation inherited from D, in particular, it is reflexive. From this it follows that the constructed order of approximation $\prec$ on FB is also reflexive, whence the ideal completion of $\langle FB, \prec\rangle$ is an algebraic domain. This gives us:

Theorem 6.1.10. *For any signature* Σ *and set* $\mathcal{E}$ *of inequalities the forgetful functor from* $\mathbf{ALG}(\Sigma, \mathcal{E})$ *to* **ALG** *has a left adjoint.*

Finally, let us look at η, which maps the generating domain D into the free algebra, and let us study the question of when it is injective. What we can say is that if injectivity fails then it fails completely:

Proposition 6.1.11. *For any in-equational theory the canonical map η from a dcpo D into the free algebra $\underline{F(D)}$ over D is order-reflecting if and only if there exists a dcpo-algebra $\underline{A}$ for this theory for which the carrier dcpo A is non-trivially ordered.*

Proof. Assume that there exists a dcpo-algebra $\underline{A}$ which contains two elements $a \sqsubset b$. Let D be any dcpo and $x \not\sqsubseteq y$ two distinct elements. We can define a continuous map g from D to A, separating x from y by setting

$$g(d) = \begin{cases} a, & \text{if } d \sqsubseteq y; \\ b, & \text{otherwise.} \end{cases}$$

Since g equals $\mathsf{ext}(g) \circ \eta$, where $\mathsf{ext}(g)$ is the unique homomorphism from $\underline{F(D)}$ to $\underline{A}$, it cannot be that $\eta(x) \sqsubseteq \eta(y)$ holds.

The converse is trivial, because η must be monotone. □

6.1.3 Least elements and strict algebras

We have come across strict functions several times already. It therefore seems worthwhile to study the problem of free algebras also in this context. But what should a strict algebra be? There are several possibilities as to what to require of the operations on such an algebra:

1. An operation which is applied to arguments, one of which is bottom, returns bottom.
2. An operation applied to the constantly bottom vector returns bottom.
3. An operation of arity greater than 0 applied to the constantly bottom vector returns bottom.

Luckily, we can leave this open as we shall see shortly. All we need is:

Definition 6.1.12. *A* strict dcpo-algebra *is a dcpo-algebra for which the carrier set contains a least element. A* strict homomorphism *between strict algebras is a Scott-continuous homomorphism which preserves the least element.*

For pointed dcpo's the existence of free strict dcpo-algebras can be established as before through the Adjoint Functor Theorem. For pointed domains the construction of the previous subsection can be adapted by adding a further axiom to the first deduction scheme:

(A3) $\bot \precsim t$ for all $t \in FB$.

Thus we have:

Theorem 6.1.13. *Free strict dcpo- and domain-algebras exist, that is, the forgetful functors*

$$\begin{array}{rcl} \mathbf{DCPO}_{\bot!}(\Sigma, \mathcal{E}) & \longrightarrow & \mathbf{DCPO}_{\bot!}, \\ \mathbf{CONT}_{\bot!}(\Sigma, \mathcal{E}) & \longrightarrow & \mathbf{CONT}_{\bot!}, \\ \textit{and} \quad \mathbf{ALG}_{\bot!}(\Sigma, \mathcal{E}) & \longrightarrow & \mathbf{ALG}_{\bot!} \end{array}$$

have left adjoints.

Let us return to the problem of strict operations. The solution is that we can add a nullary operation 0 to the signature and the inequality $0 \sqsubseteq x$ to $\mathcal{E}$ without changing the free algebras. Because of axiom (A3) we have $\perp \precsim 0$ and because of the new inequality we have $0 \precsim \perp$. Therefore the new operation must be interpreted by the bottom element. The advantage of having bottom explicitly in the signature is that we can now formulate equations about strictness of operations. For example, the first possibility mentioned at the beginning can be enforced by adding to $\mathcal{E}$ the inequality

$$f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{\alpha(f)}) \sqsubseteq 0$$

for all operation symbols f of positive arity and all $1 \leq i \leq \alpha(f)$. The corresponding free algebras then exist by the general theorem.

More problematic is the situation with $\mathbf{DCPO}_\perp$ (respectively $\mathbf{CONT}_\perp$ and $\mathbf{ALG}_\perp$). The existence of a least element in the generating dcpo does not imply the existence of a least element in the free algebra (Exercise 6.2.23(2)). Without it, we cannot make use of local continuity in domain equations. Furthermore, even if the free algebra has a least element, it need not be the case that η is strict (Exercise 6.2.23(3)). The same phenomena appears if we restrict attention to any of the cartesian closed categories exhibited in Chapter 4. The reason is that we require a special structure of the objects of our category but allow morphisms which do not preserve this structure. It is therefore always an interesting fact if the general construction for a particular algebraic theory can be restricted and corestricted to one of these sub-categories. In the case that the general construction does not yield the right objects it may be that a different construction is needed. This has been tried for the Plotkin powerdomain in several attempts by Karel Hrbacek but a satisfactory solution was obtained only at the cost of changing the morphisms between continuous algebras, see [Hrb87, Hrb89, Hrb88].

On a more positive note, we can say:

Proposition 6.1.14. *If the free functor maps finite pointed posets to finite pointed posets then it restricts and corestricts to bifinite domains.*

6.2 Powerdomains

6.2.1 The convex or Plotkin powerdomain

Definition 6.2.1. *The* convex *or* Plotkin powertheory *is defined by a signature with one binary operation $\uplus$ and the equations*

1. $x \uplus y = y \uplus x$ *(Commutativity)*
2. $(x \uplus y) \uplus z = x \uplus (y \uplus z)$ *(Associativity)*
3. $x \uplus x = x$ *(Idempotence)*

The operation $\uplus$ is called formal union.

A dcpo-algebra with respect to this theory is called a dcpo-semilattice. *The free dcpo-semilattice over a dcpo D is called the* Plotkin powerdomain *of D and it is denoted by $\mathsf{P}^P(D)$.*

Every semilattice can be equipped with an order by setting

$$x \le y \text{ if } x \uplus y = y.$$

Formal union then becomes the join in the resulting ordered set. On a dcpo-semilattice this order has little to do with the domain ordering and it is not in the focus of our interest.

The free semilattice over a set X is given by the set of all non-empty finite subsets of X, where formal union is interpreted as actual union of sets. This gives us the first half of an alternative description of the Plotkin powerdomain over a continuous domain D with basis B. Its basis FB, which we constructed as the term algebra over B, is partitioned into equivalence classes by $\approx$, the equivalence relation derived from $\precsim$, that is, from the defining equations. These equivalence classes are in one-to-one correspondence with finite subsets of B. Indeed, given a term from FB, we can re-arrange it because of associativity and commutativity, and because of idempotence we can make sure that each constant occurs just once.

Remember that we have set up the order of approximation $\prec$ on FB as the transitive closure of $\precsim \circ \prec^s \circ \precsim$. This way we have ensured that an ideal in FB contains only full equivalence classes with respect to $\approx$. We may therefore replace FB by $\mathfrak{P}_f(B)$, the set of finite subsets of B, where we associate with a term $t \in FB$ the set $[t]$ of constants appearing in t.

Let us now also transfer the order of approximation to the new basis.

Definition 6.2.2. *Two subsets M and N of a set equipped with a relation* R *are in the* Egli-Milner relation*, written as $M \ \mathsf{R}_{EM} \ N$, if the following two conditions are satisfied:*

$$\forall a \in M \ \ \exists b \in N. \ \ a \ \mathsf{R} \ b$$

$$\forall b \in N \ \ \exists a \in M. \ \ a \ \mathsf{R} \ b.$$

Here we are talking about finite subsets of $\langle B, \ll\rangle$, so we write $\ll_{EM}$ for the Egli-Milner relation between finite subsets of B. Let us establish the connection between $\ll_{EM}$ on $\mathfrak{P}_f(B)$ and $\prec$ on FB. Firstly, if $s \prec^s t$ then by definition each constant in t is matched by a constant in s which approximates it and vice versa. These are just the conditions for $[s] \ll_{EM} [t]$. Since $\ll_{EM}$ is transitive, we find that $s \prec t$ implies $[s] \ll_{EM} [t]$ in general. Conversely, if two finite subsets $M = \{a_1, \ldots, a_m\}$ and $N = \{b_1, \ldots, b_n\}$ of B are related by $\ll_{EM}$ then we can build terms s and t, such that $[s] = M$, $[t] = N$, and $s \prec^s t$ hold. This is done as follows. For each $a_i \in M$ let $b_{j(i)}$ be an element of N such that $a_i \ll b_{j(i)}$ and for each $b_j \in N$ let $a_{i(j)}$ be an element of M such that $a_{i(j)} \ll b_j$. Then we can set

$$\begin{aligned} s = (a_1 \uplus \ldots \uplus a_m) \quad &\uplus \quad (a_{i(1)} \uplus \ldots \uplus a_{i(n)}) \\ \text{and } t = (b_{j(1)} \uplus \ldots \uplus b_{j(m)}) \quad &\uplus \quad (b_1 \uplus \ldots \uplus b_n). \end{aligned}$$

We have proved:

Theorem 6.2.3. *The Plotkin powerdomain of a continuous domain D with basis $\langle B, \ll\rangle$ is given by the ideal completion of $\langle \mathfrak{P}_f(B), \ll_{EM}\rangle$.*

An immediate consequence of this characterization is that the Plotkin powerdomain of a finite pointed poset is again finite and pointed. By Proposition 6.1.14, the Plotkin powerdomain of a bifinite domain is again bifinite. This is almost the best result we can obtain. The Plotkin power construction certainly destroys all properties of being lattice-like, see Exercise 6.2.23(8). It is, on the other hand, not completely haphazard, in the sense that not every finite poset is a sub-domain of a powerdomain of some other poset. This was shown in [Nüß92].

The passage from terms to finite sets has reduced the size of the basis for the powerdomain drastically. Yet, it is still possible to get an even leaner representation. We present this for algebraic domains only. For continuous domains a similar treatment is possible but it is less intuitive. Remember that abstract bases for algebraic domains are preordered sets.

Definition 6.2.4. *For a subset M of a preordered set $\langle B, \sqsubseteq\rangle$ let the* convex hull $\mathsf{Cx}(M)$ *be defined by*

$$\{a \in B \mid \exists m, n \in M.\ m \sqsubseteq a \sqsubseteq n\}.$$

A set which coincides with its convex hull is called convex.

The following properties are easily checked:

Proposition 6.2.5. *Let $\langle B, \sqsubseteq\rangle$ be a preordered set and M, N be subsets of B.*

1. $\mathsf{Cx}(M) = {\uparrow}M \cap {\downarrow}M$.
2. $M \subseteq \mathsf{Cx}(M)$.
3. $\mathsf{Cx}(\mathsf{Cx}(M)) = \mathsf{Cx}(M)$.
4. $M \subseteq N \Longrightarrow \mathsf{Cx}(M) \subseteq \mathsf{Cx}(N)$.
5. $M =_{EM} \mathsf{Cx}(M)$.
6. $M =_{EM} N$ *if and only if* $\mathsf{Cx}(M) = \mathsf{Cx}(N)$.

Comment: In (5) and (6) we have used the notation "$=_{EM}$" as an abbreviation for "$\sqsubseteq_{EM} \cap \sqsupseteq_{EM}$"; it is not the EM-version of equality as defined in 6.2.2 (which would be nothing more than equality on the powerset).

While $\langle \mathfrak{P}_f(\mathsf{K}(D)), \sqsubseteq_{EM}\rangle$ is only a preordered set, parts (5) and (6) of the preceding proposition suggests how to replace it with an ordered set. Writing $\mathfrak{P}_{\mathsf{Cx},f}(\mathsf{K}(D))$ for the set of finitely generated convex subsets of $\mathsf{K}(D)$, we have:

Proposition 6.2.6. *The Plotkin powerdomain of an algebraic domain D is isomorphic to the ideal completion of $\langle \mathfrak{P}_{\mathsf{Cx},f}(\mathsf{K}(D)), \sqsubseteq_{EM}\rangle$.*

This explains the alternative terminology 'convex powerdomain'. We will sharpen this description in 6.2.3 below.

For examples of how the Plotkin powerdomain can be used in semantics, we refer to [HP79, Abr91a].

6.2.2 One-sided powerdomains

Definition 6.2.7. *If the Plotkin powertheory is augmented by the inequality*

$$x \sqsubseteq x \uplus y$$

then we obtain the Hoare *or* lower powertheory. *Algebras for this theory are called* inflationary semilattices. *The free inflationary semilattice over a dcpo D is called the* lower *or* Hoare powerdomain *of D, and it is denoted by $\mathsf{P}^H(D)$.*

Similarly, the terminology concerning the inequality

$$x \sqsupseteq x \uplus y$$

is upper *or* Smyth powerdomain*,* deflationary semilattice*, and $\mathsf{P}^S(D)$.*

It is a consequence of the new inequality that the semilattice ordering and the domain ordering coincide in the case of the Hoare powertheory. For the Smyth powertheory the semilattice ordering is the reverse of the domain ordering. This forces these powerdomains to have additional structure.

Proposition 6.2.8.

1. *The Hoare powerdomain of any dcpo is a lattice which has all non-empty suprema and bounded infima. The sup operation is given by formal union.*
2. *The Smyth powerdomain of any dcpo has binary infima. They are given by formal union.*

Unfortunately, the existence of binary infima does not force a domain into one of the cartesian closed categories of Chapter 4. We take up this question again in the next subsection.

Let us also study the bases of these powerdomains as derived from a given basis $\langle B, \ll \rangle$ of a continuous domain D. The development proceeds along the same lines as for the Plotkin powertheory. The equivalence relation induced by the equations and the new inequality has not changed, so we may again replace FB by the set $\mathfrak{P}_f(B)$ of finite subsets of B. The difference is wholly in the associated preorder on $\mathfrak{P}_f(B)$.

Proposition 6.2.9. *For M and N finite subsets of a basis $\langle B, \ll \rangle$ we have*

$$M \precsim N \text{ if and only if } M \subseteq {\downarrow} N$$

in the case of the Hoare powertheory and

$$M \precsim N \text{ if and only if } N \subseteq {\uparrow} M$$

for the Smyth powertheory.

The restricted order of approximation $\prec^s$ is as before given by the Egli-Milner relation $\ll_{EM}$. As prescribed by the general theory we must combine it with inclusion (for the lower theory) and with reversed inclusion (for the upper theory), respectively. Without difficulties one obtains the following connection

$$s \prec_H t \text{ if and only if } \forall a \in [s]\ \exists b \in [t].\ a \ll b$$

and

$$s \prec_S t \text{ if and only if } \forall b \in [t]\ \exists a \in [s].\ a \ll b.$$

So each of the one-sided theories is characterized by one half of the Egli-Milner ordering. Writing $\ll_H$ and $\ll_S$ for these we can formulate:

Theorem 6.2.10. *Let D be a continuous domain with basis $\langle B, \ll\rangle$.*

1. *The Hoare powerdomain of D is isomorphic to the ideal completion of $\langle \mathfrak{P}_f(B), \ll_H\rangle$.*
2. *The Smyth powerdomain of D is isomorphic to the ideal completion of $\langle \mathfrak{P}_f(B), \ll_S\rangle$.*

For algebraic domains we can replace the preorders on $\mathfrak{P}_f(B)$ by an ordered set in both cases.

Proposition 6.2.11. *For subsets M and N of a preordered set $\langle B, \leq\rangle$ we have*

1. *$M =_H {\downarrow} M$,*
2. *$M \leq_H N$ if and only if ${\downarrow} M \subseteq {\downarrow} N$,*

and

3. *$M =_S {\uparrow} M$,*
4. *$M \leq_S N$ if and only if ${\uparrow} M \supseteq {\uparrow} N$.*

Writing $\mathfrak{P}_{L,f}(B)$ for the set of finitely generated lower subsets of B and $\mathfrak{P}_{U,f}(B)$ for the set of finitely generated upper subsets of B, we have:

Proposition 6.2.12. *Let D be an algebraic domain.*

1. *The Hoare powerdomain $\mathsf{P}^H(D)$ of D is isomorphic to the ideal completion of $\langle \mathfrak{P}_{L,f}(\mathsf{K}(D)), \subseteq\rangle$.*
2. *The Smyth powerdomain $\mathsf{P}^S(D)$ of D is isomorphic to the ideal completion of $\langle \mathfrak{P}_{U,f}(\mathsf{K}(D)), \supseteq\rangle$.*

From this description we can infer through Proposition 6.1.14 that the Smyth powerdomain of a bifinite domain is again bifinite. Since a deflationary semilattice has binary infima anyway, we conclude that the Smyth powerdomain of a bifinite domain is actually a bc-domain. For a more general statement see Corollary 6.2.15.

6.2.3 Topological representation theorems

The objective of this subsection is to describe the powerdomains we have seen so far directly as spaces of certain subsets of the given domain, without recourse to bases and the ideal completion. It will turn out that the characterizations of Proposition 6.2.6 and Proposition 6.2.12 can be extended nicely once we allow ourselves topological methods.

Theorem 6.2.13. *The Hoare powerdomain of a continuous domain D is isomorphic to the lattice of all non-empty Scott-closed subsets of D. Formal union is interpreted by actual union.*

Proof. Let $\langle B, \ll\rangle$ be a basis for D. We establish an isomorphism with the representation of Theorem 6.2.10. Given an ideal I of finite sets in $\mathsf{P}^H(D)$ we map it to $\phi^H(I) = \mathsf{Cl}(\bigcup I)$, the Scott-closure of the union of all these sets. Conversely, for a non-empty Scott-closed set A we let $\psi^H(A) = \mathfrak{P}_f(\mathop{\downarrow\!\!\downarrow} A \cap B)$, the set of finite sets of basis elements approximating some element in A. We first check that $\psi^H(A)$ is indeed an ideal with respect to $\ll_H$. It is surely non-empty as A was assumed to contain elements. Given two finite subsets M and N of $\mathop{\downarrow\!\!\downarrow} A \cap B$ then we can apply the interpolation axiom to get finite subsets M' and N' with $M \ll_{EM} M'$ and $N \ll_{EM} N'$. An upper bound for M and N with respect to $\ll_H$ is then given by $M' \cup N'$. It is also clear that the Scott closure of $\mathop{\downarrow\!\!\downarrow} A \cap B$ gives A back again because every element of D is the directed supremum of basis elements. Hence $\phi^H \circ \psi^H = \mathsf{id}$. Starting out with an ideal I, we must show that we get it back from $\phi^H(I)$. So let $M \in I$. By the roundness of I (see the discussion before Definition 2.2.21) there is another finite set $M' \in I$ with $M \ll_H M'$. So for each $a \in M$ there is $b \in M'$ with $a \ll b$. Since all elements of I are contained in $\phi^H(I)$, we have that a belongs to $\mathop{\downarrow\!\!\downarrow}\phi(I) \cap B$. Conversely, if a is an element of $\mathop{\downarrow\!\!\downarrow}\phi(I) \cap B$ then $\mathop{\uparrow\!\!\uparrow} a \cap \phi(I)$ is not empty and therefore must meet $\bigcup I$ as $D \setminus \mathop{\uparrow\!\!\uparrow} a$ is closed. The set $\{a\}$ is then below some element of I under the $\ll_H$-ordering. Monotonicity of the isomorphisms is trivial and the representation is proved.

Formal union applied to two ideals returns the ideal of unions of the constituting sets. Under the isomorphism this operation is transformed into union of closed subsets. $\square$

This theorem holds not just for continuous domains but also for all dcpo's and even all T_0-spaces. See [Sch93] for this. We can also get the full complete lattice of all closed sets if we add to the Hoare powertheory a nullary operation e and the equations

$$e \uplus x = x \uplus e = x.$$

Alternatively, we can take the strict free algebra with respect to the Hoare powertheory. If the domain has a least element then these adjustments are not necessary, a least element for the Hoare powerdomain is $\{\perp\}$. Homomorphisms, however, will only preserve non-empty suprema.

The characterization of the Smyth powerdomain builds on the material laid out in Section 4.2.3. In particular, recall that a Scott-compact saturated set in a continuous domain has a Scott-open filter of open neighborhoods and that each Scott-open filter in σ_D arises in this way.

Theorem 6.2.14. *The Smyth powerdomain of a continuous domain D is isomorphic to the set $\kappa_D \setminus \{\emptyset\}$ of non-empty Scott-compact saturated subsets ordered by reversed inclusion. Formal union is interpreted as union.*

Proof. Let $\langle B, \ll\rangle$ be a basis for D. We show that $\kappa_D \setminus \{\emptyset\}$ is isomorphic to $\mathsf{P}^S(D) = \mathsf{Idl}(\mathfrak{P}_f(B), \ll_S)$. Given an ideal I we let $\phi^S(I)$ be $\bigcap_{M \in I} \uparrow M$. This constitutes a monotone map from $\mathsf{P}^S(D)$ to $\kappa_D \setminus \{\emptyset\}$ by Proposition 4.2.14. In the other direction, we assign to a compact saturated set A the set $\psi^S(A)$ of all finite sets $M \subseteq B$ such that $A \subseteq \mathop{\Uparrow} M$. Why is this an ideal? For every open neighborhood O of A we find a finite set M of basis elements contained in O such that $A \subseteq \mathop{\Uparrow} M$ because A is compact and $O = \bigcup_{b \in O \cap B} \mathop{\Uparrow} b$ (Proposition 2.3.6). Then given two finite sets M and N in $\psi^S(A)$ an upper bound for them is any such finite set P with $A \subseteq \mathop{\Uparrow} P \subseteq \mathop{\Uparrow} M \cap \mathop{\Uparrow} N$. Clearly, ψ^S is monotone as $\kappa_D \setminus \{\emptyset\}$ is equipped with reversed inclusion.

Let us show that $\psi^S \circ \phi^S$ is the identity on $\mathsf{P}^S(D)$. For $M \in I$ let $M' \in I$ be above M in the $\ll_S$-ordering. Then $\phi^S(I) \subseteq \uparrow M' \subseteq \mathop{\Uparrow} M$ and so M belongs to $\psi^S \circ \phi^S(I)$. Conversely, every neighborhood of $\phi^S(I)$ contains some $\uparrow M$ with $M \in I$ already as we saw in Proposition 4.2.14. So if $\phi^S(I)$ is contained in $\mathop{\Uparrow} N$ for some finite set $N \subseteq B$ then there are M and M' in I with $M \subseteq \mathop{\Uparrow} N$ and $M \ll_S M'$. Hence $N \ll_S M'$ and N belongs to I.

The composition $\phi^S \circ \psi^S$ is clearly the identity as we just saw that every neighborhood of a compact set contains a finitely generated one and as every saturated set is the intersection of its neighborhoods.

The claim about formal union follows because on powersets union and intersection completely distribute: $\phi^S(I \uplus J) = \bigcap_{M \in I, N \in J} \uparrow(M \cup N) = \bigcap_{M \in I, N \in J}(\uparrow M \cup \uparrow N) = \bigcap_{M \in I} \uparrow M \cup \bigcap_{N \in J} \uparrow N = \phi^S(I) \cup \phi^S(J)$. ☐

For this theorem continuity is indispensable. A characterization of the free deflationary semilattice over an arbitrary dcpo is not known. The interested reader may consult [Hec90, Hec93a] and [Sch93] for a discussion of this open problem.

Corollary 6.2.15. *The Smyth powerdomain of a coherent domain with bottom is a bc-domain.*

Proof. That two compact saturated sets A and B are bounded by another one, C, simply means $C \subseteq A \cap B$. In this case $A \cap B$ is not empty. It is compact saturated by the very definition of coherence. ☐

Let us now turn to the Plotkin powerdomain. An ideal I of finite sets ordered by $\ll_{EM}$ will generate ideals with respect to both coarser orders $\ll_H$ and $\ll_S$. We can therefore associate with I a Scott-closed set $\phi^H(I) = \mathsf{Cl}(\bigcup I)$ and a compact saturated set $\phi^S(I) = \bigcap_{M \in I} \uparrow M$. However, not every such pair arises in this way; the Plotkin powerdomain is not simply the product of the two one-sided powerdomains. We will be able to characterize them in two special cases: for countably based domains and for coherent domains. The general situation is quite hopeless, as is illustrated by Exercise 6.2.23(11). In both special cases we do want to show that I is faithfully represented by the intersection $\phi(I) = \phi^H(I) \cap \phi^S(I)$. In the first case we will need the following weakening of the Egli-Milner ordering:

Definition 6.2.16. *For a dcpo D we let* $\mathsf{Lens}(D)$ *be the set of all non-empty subsets of D which arise as the intersection of a Scott-closed and a compact saturated subset. The elements of* $\mathsf{Lens}(D)$ *we call* lenses. *On* $\mathsf{Lens}(D)$ *we define the* topological Egli-Milner ordering, *$\sqsubseteq_{TEM}$, by*

$$K \sqsubseteq_{TEM} L \text{ if } L \subseteq \uparrow K \text{ and } K \subseteq \mathsf{Cl}(L).$$

Proposition 6.2.17. *Let D be a dcpo.*

1. *Every lens is convex and Scott-compact.*
2. *A canonical representation for a lens L is given by $\uparrow L \cap \mathsf{Cl}(L)$.*
3. *The topological Egli-Milner ordering is anti-symmetric on $\mathsf{Lens}(D)$.*

Proof. Convexity is clear as every lens is the intersection of a lower and an upper set. An open covering of a lens $L = C \cap U$, where C is closed and U compact saturated, may be extended to a covering of U by adding the complement of C to the cover. This proves compactness. Since all Scott-open sets are upwards closed, compactness of a set A implies the compactness of $\uparrow A$. Using convexity, we get $L = \uparrow L \cap \downarrow L \subseteq \uparrow L \cap \mathsf{Cl}(L)$ and using boolean algebra we calculate $\uparrow L = \uparrow(C \cap U) \subseteq \uparrow U = U$ and $\mathsf{Cl}(L) = \mathsf{Cl}(C \cap U) \subseteq \mathsf{Cl}(C) = C$, so $\uparrow L \cap \mathsf{Cl}(L) \subseteq U \cap C = L$. Then if $K =_{TEM} L$ we have $\uparrow K = \uparrow L$ and $\mathsf{Cl}(K) = \mathsf{Cl}(L)$. Equality of K and L follows. $\square$

Before we can prove the representation theorem we need yet another description of the lens $\phi(I)$.

Lemma 6.2.18. *Let D be a continuous domain with basis B and let I be an ideal in $\langle \mathfrak{P}_f(B), \ll_{EM} \rangle$. Then $\phi(I) = \{\bigsqcup^{\uparrow} A \mid A \subseteq \bigcup I$ directed and $A \cap M \neq \emptyset$ for all $M \in I\}$.*

Proof. The elements of the set on the right clearly belong to the Scott-closure of $\bigcup I$. They are also contained in $\phi^S(I)$ because $\bigsqcup^{\uparrow} A$ is above some element in $A \cap M$ for each $M \in I$.

Conversely, let $x \in \phi(I)$ and let $a \in A = \mathop{\downarrow\!\!\downarrow} x \cap B$. The set $\mathop{\uparrow\!\!\uparrow} a$ is Scott-open and must therefore meet some $M \in I$. From the roundness of I we get $M' \in I$ with $M \ll_{EM} M'$. The set $M \cup \{a\}$ also approximates M' and so it is contained in I. Hence $a \in \bigcup I$. Furthermore, given any $M \in I$, let again $M' \in I$ be such that $M \ll_{EM} M'$. Then x is above some element of M' as $\phi(I) \subseteq \uparrow M'$ and therefore $m \ll x$ holds for some $m \in M$. $\square$

Theorem 6.2.19. *Let D be an ω-continuous domain. The Plotkin powerdomain $\mathsf{P}^P(D)$ is isomorphic to $\langle \mathsf{Lens}(D), \sqsubseteq_{TEM} \rangle$. Formal union is interpreted as union followed by topological convex closure.*

Proof. Let $\langle B, \ll \rangle$ be a countable basis of D. We have already defined the map $\phi \colon \mathsf{P}^P(D) \to \mathsf{Lens}(D)$. In the other direction we take the function ψ which assigns to a lens K the set $\psi^H(\mathsf{Cl}(K)) \cap \psi^S(\uparrow K)$. Before we can prove that these maps constitute a pair of isomorphisms, we need the following information about reconstructing $\phi^H(I)$ and $\phi^S(I)$ from $\phi(I)$.

1. $\phi^S(I) = \uparrow \phi(I)$: Since $\phi^S(I)$ is an upper set which contains $\phi(I)$, only one inclusion can be in doubt. Let $x \in \phi^S(I)$ and $I' = \{M \cap \mathop{\downarrow\!\!\downarrow} x \mid M \in I\}$. Firstly, each set in I' is non-empty and, secondly, we have $M \cap \mathop{\downarrow\!\!\downarrow} x \ll_S N \cap \mathop{\downarrow\!\!\downarrow} x$ whenever $M \ll_{EM} N$. Calculating $\phi^S(I')$ in the continuous domain $\phi^H(I)$ gives us a non-empty set which is below x and contained in the lens $\phi(I)$.

2. $\phi^H(I) = \mathsf{Cl}(\phi(I))$: Again, only one inclusion needs an argument. We show that every element of $\Downarrow\phi^H(I) \cap B$ belongs to $\downarrow\phi(I)$. Given a basis element a approximating some element of $\phi^H(I)$ then we already know that it belongs to $\bigcup I$. Let $M \in I$ be some set which contains a. Using countability of the basis we may assume that M extends to a cofinal chain in I (Proposition 2.2.13): $M = M_0 \ll_{EM} M_1 \ll_{EM} M_2 \ll_{EM} \ldots$. König's Lemma then tells us that we can find a chain of elements $a = a_0 \ll a_1 \ll a_2 \ll \ldots$ where $a_n \in A_n$. The supremum $x = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} a_n$ belongs to $\phi(I)$ and is above a.

3. ϕ is monotone: Let $I \subseteq I'$ be two ideals in $\langle \mathfrak{P}_f(B), \ll_{EM}\rangle$. The larger ideal results in a bigger lower set $\phi^H(I')$ and a smaller upper set $\phi^S(I')$. Using 1 and 2 we can calculate for the corresponding lenses:

$$\phi(I) \subseteq \phi^H(I) \subseteq \phi^H(I') = \mathsf{Cl}(\phi(I')),$$

$$\phi(I') \subseteq \phi^S(I') \subseteq \phi^S(I) = \uparrow\phi(I).$$

So $\phi(I) \sqsubseteq_{TEM} \phi(I')$ as desired.

4. The monotonicity of ψ follows by construction and one half of the topological Egli-Milner ordering: $K \subseteq \uparrow M$ implies $L \subseteq \uparrow M$ if we assume $K \sqsubseteq_{TEM} L$.

5. $\phi \circ \psi = \mathsf{id}$: Given a lens $L = C \cap U$ we clearly have $\phi^S(\psi(L)) \supseteq L$. Using the continuity of D and the compactness of L we infer that $\phi^S(\psi(L))$ must equal $\uparrow L$. Every basis element approximating some element of L occurs in some set of $\psi(L)$, so $\phi^H(\psi(L)) = \mathsf{Cl}(L)$ is clear. Proposition 6.2.17 above then implies that $\phi \circ \psi(L)$ gives back L.

6. $\psi \circ \phi = \mathsf{id}$: Given an ideal I we know that each $M \in I$ covers the lens $\phi(I)$ in the sense of $\Uparrow M \supseteq \phi(I)$. So M is contained in $\psi^S(\phi(I))$. By (2), we also have that M is contained in $\psi^H(\mathsf{Cl}(\phi(I)))$. Conversely, if $\Uparrow M \supseteq \phi(I)$ for a finite set M of basis elements contained in $\Downarrow\phi(I)$, then for some $N \in I$ we have $\Uparrow M \supseteq N$ by the Hofmann-Mislove Theorem 4.2.14. For this N we have $M \ll_S N$. On the other hand, each element a of M approximates some $x \in \phi(I)$ and hence belongs to some $N_a \in I$. An upper bound for N and all N_a in I, therefore, is above M in $\ll_{EM}$ which shows that M must belong to I.

7. In the representation theorems for the one-sided powerdomains we have shown that formal union translates to actual union. We combine this for the convex setting: $\phi(I \uplus J) = \phi^H(I \uplus J) \cap \phi^S(I \uplus J) = (\phi^H(I) \cup \phi^H(J)) \cap (\phi^S(I) \cup \phi^S(J)) = (\mathsf{Cl}(\phi(I)) \cup \mathsf{Cl}(\phi(J))) \cap (\uparrow\phi(I) \cup \uparrow\phi(J)) = \mathsf{Cl}(\phi(I) \cup \phi(J)) \cap \uparrow(\phi(I) \cup \phi(J))$. $\square$

Note that we used countability of the basis only for showing that $\phi^H(I)$ can be recovered from $\phi(I)$. In general, this is wrong. Exercise 6.2.23(11) discusses an example.

The substitution of topological closure for downward closure was also necessary, as the example in Figure 13 shows. There, the set $A = \uparrow a$ is a lens but its downward closure is not Scott-closed, c is missing. The set $A \cup \{c\}$ is also a lens. It is below A in the topological Egli-Milner order but not in the plain Egli-Milner order. The convex closure of the union of the two lenses $\{\bot\}$ and A is not a lens, c must be added.

A better representation theorem is obtained if we pass to coherent domains (Section 4.2.3). (Note that the example in Figure 13 is not coherent, because the set

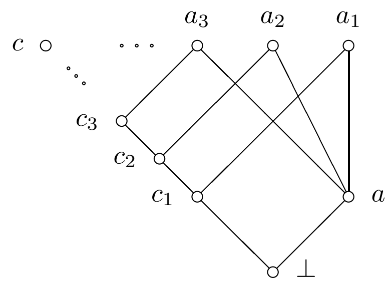


Figure 13: An algebraic domain in which topological Egli-Milner ordering and ordinary Egli-Milner ordering do not coincide.

$\{c_1, a\}$ has infinitely many minimal upper bounds, violating the condition in Proposition 4.2.17.) We first observe that lenses are always Lawson closed sets. If the domain is coherent then this implies that they are also Lawson-compact. Compactness will allow us to use downward closure instead of topological closure.

Lemma 6.2.20. *Let L be a Lawson-compact subset of a continuous domain D. Then $\downarrow L$ is Scott-closed.*

Proof. Let x be an element of D which does not belong to $\downarrow L$. For each $y \in L$ there exists $b_y \ll x$ such that $b_y \not\sqsubseteq y$. The set $D \setminus \uparrow b_y$ is Lawson-open and contains y. By compactness, finitely many such sets cover L. Let b be an upper bound for the associated basis elements approximating x. Then $\twoheaduparrow b$ is an open neighborhood of x which does not intersect L. Hence $\downarrow L$ is closed. $\square$

Corollary 6.2.21. *The lenses of a coherent domain are precisely the convex Lawson-compact subsets. For these, topological Egli-Milner ordering and Egli-Milner ordering coincide.*

Theorem 6.2.22. *Let D be a coherent domain. The Plotkin powerdomain of D is isomorphic to $\langle \mathsf{Lens}(D), \sqsubseteq_{EM} \rangle$. Formal union is interpreted as union followed by convex closure.*

Proof. The differences to the proof of Theorem 6.2.19, which are not taken care of by the preceding corollary, concern part 2. We must show that $\mathsf{Cl}(\phi(I)) = \downarrow\phi(I)$ contains all of $\twoheaddownarrow\phi^H(I) \cap B$. In the presence of coherence this can be done through the Hofmann-Mislove Theorem 4.2.14. The lower set $\phi^H(I)$ is a continuous domain in itself. For an element a of $\twoheaddownarrow\phi^H(I) \cap B$ we look at the filtered collection of upper sets $J = \{\uparrow a \cap \uparrow M \mid M \in I\}$. Each of these is non-empty, because a belongs to some $M \in I$, and compact saturated because of coherence. Hence $\bigcap J$ is non-empty. It is also contained in $\phi(I)$ and above a. $\square$

6.2.4 Hyperspaces and probabilistic powerdomains

In our presentation of powerdomains we have emphasized the feature that they are free algebras with respect to certain (in-)equational theories. From the general existence theorem for such algebras we derived concrete representations as sets of subsets. This is the approach which in the realm of domain theory was suggested first by Matthew Hennessy and Gordon Plotkin in [HP79] but it has a rather long tradition in algebraic semantics (see e.g. [NR85]). However, it is not the only viewpoint one can take. One may also study certain sets of subsets of domains in their own right. In topology, this study of 'hyperspaces', as they are called, is a long-standing tradition, starting with Felix Hausdorff [Hau14] and Leopold Vietoris [Vie21, Vie22]. It is also how the subject started in semantics and, indeed, continues to be developed. A hyperspace can be interesting even if an equational characterization cannot be found or can be found only in restricted settings. Recent examples of this are the set-domains introduced by Peter Buneman [BDW88, Gun92a, Hec90, Puh93, Hec91, Hec93b] in connection with a general theory of relational databases. While these are quite natural from a domain-theoretic point of view, their equational characterizations (which do exist for some of them) are rather bizarre and do not give us much insight. The hyperspace approach is developed in logical form in Section 7.3.

We should also mention the various attempts to define a probabilistic version of the powerdomain construction, see [SD80, Mai85, Gra88, JP89, Jon90]. (As an aside, these cannot be restricted to algebraic domains; the wider concept of continuous domain is forced upon us through the necessary use of the unit interval $[0, 1]$.) They do have an equational description in some sense but this goes beyond the techniques of this chapter.

One can then ask abstractly what constitutes a powerdomain construction and build a theory upon such a definition. This approach was taken in [Hec90, Hec91]. The most notable feature of this work is that under this perspective, too, many of the known powerdomains turn out to be canonical in a precise sense. How this (very natural) formulation of canonicity is connected with concerns in semantics, however, is as yet unclear.

Exercises 6.2.23. *1. For the proof of Theorem 6.1.6 we can equip FB also with the transitive closure of $\prec^s \circ \precsim$. Show:*

- *(a) This relation $\prec'$ satisfies the interpolation axiom.*
- *(b) In general, $\prec'$ is different from $\prec$.*
- *(c) The ideal completions of $\langle FB, \prec\rangle$ and $\langle FB, \prec'\rangle$ are isomorphic. (Use Exercise 2.3.9(27).)*
- *(d) What is the advantage of $\prec$ over $\prec'$?*

2. Describe the free domain algebra for an arbitrary domain D and an arbitrary signature Σ in the case that $\mathcal{E}$ is empty.

3. Set up an algebraic theory such that all its dcpo-algebras have least elements but the embeddings η are not strict.

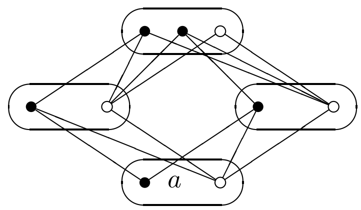


Figure 14: Part of an algebraic domain where Theorem 6.2.19 fails.

4. *Let $\langle\Sigma,\mathcal{E}\rangle$ be the usual equational theory of groups (or boolean algebras). Show that any dcpo-algebra with respect to this theory is trivially ordered. Conclude that the free construction collapses each connected component of the generating dcpo into a single point.*

5. *Given signatures Σ and Σ' and sets of inequalities $\mathcal{E}$ and $\mathcal{E}'$ we call the pair $\langle\Sigma,\mathcal{E}\rangle$ a* reduct *of $\langle\Sigma',\mathcal{E}'\rangle$ if $\Sigma\subseteq\Sigma'$ and $\mathcal{E}\subseteq\mathcal{E}'$. In this case there is an obvious forgetful functor from $\mathbf{C}(\Sigma',\mathcal{E}')$ to $\mathbf{C}(\Sigma,\mathcal{E})$, where $\mathbf{C}$ is any of the categories considered in this chapter. Show that the general techniques of Theorem 6.1.2 and 6.1.7 suffice to prove that this functor has a left adjoint.*

6. *Likewise, show that partial domain algebras can be completed freely.*

7. *Let $\underline{A}$ be a free domain-algebra over an algebraic domain. Is it true that every operation, if applied to compact elements of A, returns a compact element?*

8. *Let $D=\{\bot\sqsubseteq a,b\sqsubseteq\top\}$ be the four-element lattice (Figure 1) and let $E=D\times D$. The sets $\{\langle\bot,a\rangle,\langle\bot,b\rangle\}$ and $\{\langle a,\bot\rangle,\langle b,\bot\rangle\}$ are elements of the Plotkin powerdomain of E. Show that they have two minimal upper bounds. Since $\{\langle\top,\top\rangle\}$ is a top element, $\mathsf{P}^P(E)$ is not an L-domain.*

9. *Is the Plotkin powerdomain closed on* **F-B***, the category whose objects are bilimits of finite (but not necessarily pointed) posets?*

10. *Define a natural isomorphism between $\mathsf{P}^H(D)_\bot\multimap E$ and $[D\longrightarrow E]$ where D is any continuous domain, E is a complete lattice, and $\cdot\multimap\cdot$ stands for the set of functions which preserve all suprema (ordered pointwise).*

11. *We want to construct an algebraic domain D to which Theorem 6.2.19 cannot be extended. The compact elements of D are arranged in finite sets already such that they form a directed collection in the Egli-Milner ordering, generating the ideal I. We take one finite set for each element of $\mathfrak{P}_f(\mathbb{R})$, the finite powerset of the reals (or any other uncountable set), and we will have $M_\alpha\ll_{EM}M_\beta$ if $\alpha\subseteq\beta\subseteq\mathbb{R}$. So we can arrange the M_α in layers according to the cardinality of α. Each M_α contains one 'white' and $|\alpha|!$ many 'black' elements. If $\alpha\subsetneqq\beta$ then the white element of M_α is below every element of M_β. For the order between black elements look at adjacent layers. There are $|\beta|$ many subsets of β*

with cardinality $|\beta| - 1$. The $|\beta|!$ many black elements of M_β we partition into $|\beta|$ many classes of cardinality $(|\beta| - 1)!$. So we can let the black elements of a lower neighbor of M_β be just below the equally many black elements of one of these classes. (The idea being that no two black elements have an upper bound.) Figure 14 shows a tiny fraction of the resulting ordered set $\mathsf{K}(D)$. Establish the following facts about this domain:

(a) *Above a black element there are only black, below a white element there are only white elements.*

(b) *i. An ideal in $\mathsf{K}(D)$ can contain at most one black element from each set.*

  ii. An ideal can contain at most one black element in each layer.

  iii. An ideal can contain at most countably many black elements.

(c) *i. An ideal meeting all sets must contain all white elements.*

  ii. If an ideal contains a black element, then it contains the least black element a.

  iii. If an ideal meeting all sets contains a then it must contain upper bounds for a and the uncountably many white elements of the first layer. These upper bounds must form an uncountable set and consist solely of black elements.

(d) *From the contradiction between b-iii and c-iii conclude that only one ideal in K^D meets all sets, the ideal W of white elements. Therefore, $\phi(I)$ contains precisely one element, say b. Show that $\downarrow b$ equals $W \cup \{b\}$ and that it is Scott-closed. Hence it is far from containing all elements of $\bigcup I = K^D$.*

(e) *Go a step farther and prove that the lenses of D are not even directed-complete by showing that the ideal I we started out with does not have an upper bound.*

12. *(R. Heckmann) Remove idempotence from the Hoare powertheory and study free domain algebras with respect to this theory. These are no longer finite if the generating domain is finite. Show that the free algebra over the four-element lattice (Figure 1) is neither bifinite nor an L-domain.*

7 Domains and logic

There are at least three ways in which the idea of a function can be formalized. The first is via algorithms, which is the Computer Science viewpoint. The second is via value tables or, in more learned words, via graphs. This is the – rather recent – invention of Mathematics. The third, finally, is via propositions: We can either take propositions about the function itself or view a function as something which maps arguments which satisfy ϕ to values which satisfy ψ. The encoding in the latter case is by the set of all such pairs (ϕ, ψ). The beauty of the subject, then, lies in the interplay between these notions.

The passage from algorithms (programs) to the extensional description via graphs is called denotational semantics. It requires sophisticated structures, precisely *domains* in the sense of this text, because of, for example, recursive definitions in programs. The passage from algorithms to propositions about functions is called program logics. If we take the computer scientist's point of view as primary then denotational semantics and program logics are two different ways of describing the behaviour of programs. It is the purpose of this chapter to lay out the connection between these two forms of semantics. As propositions we allow all those formulae whose extensions in the domain under consideration are (compact) Scott-open sets. This choice is well justified because it can be argued that such propositions correspond to properties which can be detected in a finite amount of time [Abr87]. The reader will find lucid explications of this point in [Smy92] and [Vic89].

Mathematically, then, we have to study the relation between domains and their complete lattices of Scott-open sets. Stated for general topological spaces, this is the famous Stone duality. We treat it in Section 7.1. The restriction to domains introduces several extra features which we discuss in a one by one fashion in Section 7.2. The actual domain logic, as a syntactical theory, is laid out in Section 7.3.

The whole open-set lattice, however, is too big to be syntactically represented. We must, on this higher level, once more employ ideas of approximation and bases. There is a wide range of possibilities here, which can be grouped under the heading of *information systems*. We concentrate on one of these, namely, the logic of compact open subsets. This is well motivated by the general framework of Stone duality and also gives the richest logic.

7.1 Stone duality

7.1.1 Approximation and distributivity

We start out with a few observations concerning distributivity. So far, this didn't play a role due to the poor order theoretic properties of domains. Now, in the context of open set lattices, it becomes a central theme, because, as we shall see, it is closely related with the concept of approximation. The earliest account of this connection is probably [Ran53].

A word on notation: We shall try to keep a clear distinction between spaces, which in the end will be our domains, and their open-set lattices. We shall emphasize this by using $\leq$ for the less-than-or-equal-to relation whenever we speak of lattices, even though these do form a special class of domains, too, as you may remember from Section 4.1.

Recall that a lattice L is said to be *distributive* if for all $x, y, z \in L$ the equality

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds. The dual of this axiom is then satisfied as well. For the infinitary version of distributivity, we introduce the following notation for choice functions: If $(A_i)_{i\in I}$ is a family of sets then we write $f\colon I \xrightarrow{\odot} \bigcup A_i$ if $f(i)$ takes its value in A_i for every $i \in I$. *Complete distributivity* can then be expressed by the equation

$$\bigwedge_{i\in I} \bigvee A_i \;=\; \bigvee_{f\colon I \xrightarrow{\odot} \cup A_i} \bigwedge_{i\in I} f(i).$$

It, too, implies its order dual, see Exercise 7.3.19(1). There is a lot of room for variations of this and we shall meet a few of them in this section. Here comes the first:

Theorem 7.1.1. *A complete lattice L is continuous if and only if*

$$\bigwedge_{i\in I} \bigvee{}^{\uparrow} A_i \;=\; \bigvee_{f\colon I \xrightarrow{\odot} \cup A_i}^{\uparrow} \bigwedge_{i\in I} f(i)$$

holds for all families $(A_i)_{i\in I}$ of directed subsets of L.

Proof. The reader should check for himself that the supremum on the right hand side is indeed over a directed set. Let now x be an element approximating the left hand side of the equation. Then for each $i \in I$ we have $x \ll \bigvee^{\uparrow} A_i$ and so there is $a_i \in A_i$ with $x \leq a_i$. Let f be the choice function which selects these a_i. Then $x \leq \bigwedge_{i\in I} f(i)$ and x is below the right hand side as well. Assuming L to be continuous, this proves $\bigwedge_{i\in I} \bigvee^{\uparrow} A_i \leq \bigvee^{\uparrow}_{f\colon I \xrightarrow{\odot} \cup A_i} \bigwedge_{i\in I} f(i)$. The reverse inequality holds in every complete lattice.

For the converse fix an element $x \in L$ and let $(A_i)_{i\in I}$ be the family of all directed sets A for which $x \leq \bigvee^{\uparrow} A$. From the equality, which we now assume to hold, we get that $x = \bigvee^{\uparrow}_{f\colon I \xrightarrow{\odot} \cup A_i} \bigwedge f(i)$. We claim that for each choice function $f\colon I \xrightarrow{\odot} \bigcup A_i$, the corresponding element $y = \bigwedge_{i\in I} f(i)$ is approximating x. Indeed, if A is a directed set with $x \leq \bigvee^{\uparrow} A$ then $A = A_{i_0}$ for some $i \in I$ and so $y \leq f(i_0) \in A$. ☐

Let us now look at completely distributive lattices which, by the preceding theorem, are guaranteed to be continuous. We can go further and express this stronger distributivity by an approximation axiom, too.

Definition 7.1.2. *For a complete lattice L define a relation $\lll$ on L by*

$$x \lll y \text{ if } \forall A \subseteq L.\ (y \leq \bigvee A \Longrightarrow \exists a \in A.\ x \leq a).$$

Call L prime-continuous *if for every $x \in L$, $x = \bigvee \{y \mid y \lll x\}$ holds.*

Note that the relation $\lll$ is defined in just the same way as the order of approximation, except that directed sets are replaced by arbitrary subsets. All our fundamental results about the order of approximation hold, *mutatis mutandis*, for $\lll$ as well. In particular, we shall make use of Proposition 2.2.10 and Lemma 2.2.15. Adapting the previous theorem we get George N. Raney's characterization of complete distributivity [Ran53].

Theorem 7.1.3. *A complete lattice is prime-continuous if and only if it is completely distributive.*

Let us now turn our attention to 'approximation' from above. The right concept for this is:

Definition 7.1.4. *A complete lattice L is said to be* $\wedge$-generated *by a subset A if for every $x \in L$, $x = \bigwedge(\uparrow x \cap A)$ holds. (Dually, we can speak of* $\vee$-generation*.)*

We will study $\wedge$-generation by certain elements only, which we now introduce in somewhat greater generality than actually needed for our purposes.

Definition 7.1.5. *An element x of a lattice L is called* $\wedge$-irreducible *if whenever $x = \bigwedge M$ for a finite set $M \subseteq L$ then it must be the case that $x = m$ for some $m \in M$. We say x is* $\wedge$-prime *if $x \geq \bigwedge M$ implies $x \geq m$ for some $m \in M$, where M is again finite. Stating these conditions for arbitrary $M \subseteq L$ gives rise to the notions of* completely $\wedge$-irreducible *and* completely $\wedge$-prime *element. The dual notions are obtained by exchanging supremum for infimum.*

Note that neither $\wedge$-irreducible nor $\wedge$-prime elements are ever equal to the top element of the lattice, because that is the infimum of the empty set.

Proposition 7.1.6. *A $\wedge$-prime element is also $\wedge$-irreducible. The converse holds if the lattice is distributive.*

Theorem 7.1.7. *A continuous (algebraic) lattice L is $\wedge$-generated by its set of (completely) $\wedge$-irreducible elements.*

Proof. If x and y are elements of L such that x is not below y then there is a Scott-open filter F which contains x but not y, because $\downarrow y$ is closed and the Scott-topology is generated by open filters, Lemma 2.3.8. Employing the Axiom of Choice in the form of Zorn's Lemma, we find a maximal element above y in the inductive set $L \setminus F$. It is clearly $\wedge$-irreducible. In an algebraic lattice we can choose F to be a principal filter generated by a compact element. The maximal elements in the complement are then completely $\wedge$-irreducible. $\square$

Theorem 7.1.8. *If L is a complete lattice which is $\wedge$-generated by $\wedge$-prime elements, then L satisfies the equations*

$$\bigwedge_{m \in M} \bigvee A_m \quad = \quad \bigvee_{f\colon M \xrightarrow{\odot} \cup A_m} \; \bigwedge_{m \in M} f(m)$$

and

$$\bigvee_{i\in I}\bigwedge M_i \;=\; \bigwedge_{f\colon I\xrightarrow{\odot}\cup M_i}\bigvee_{i\in I} f(i)$$

where the sets M and M_i are finite.

A dual statement holds for lattices which are $\vee$-generated by $\vee$-prime elements.

Proof. The right hand side is certainly below the left hand side, so assume that p is a $\wedge$-prime element above $\bigvee_{f\colon M\xrightarrow{\odot}\cup A_m}\bigwedge_{m\in M} f(m)$. Surely, p is above $\bigwedge_{m\in M} f(m)$ for every $f\colon M\xrightarrow{\odot}\cup A_m$ and because it is $\wedge$-prime it is above $f(m_f)$ for some $M_f\in M$. We claim that the set B of all $f(m_f)$ covers at least one A_m. Assume the contrary. Then for each $m\in M$ there exists $a_m\in A_m\setminus B$ and we can define a choice function $f_0\colon m\mapsto a_m$. Then $f_0(m_{f_0})\in B$ contradicts our construction of f_0. So we know that for some $m\in M$ all elements of A_m are below p and hence p is also above $\bigwedge_{m\in M}\bigvee A_m$. The proof for the second equation is similar and simpler. $\square$

Note that the two equations are not derivable from each other because of the side condition on finiteness. The first equation is equivalent to

$$x\wedge\bigvee_{i\in I} y_i \;=\; \bigvee_{i\in I}(x\wedge y_i)$$

which can be stated without choice functions. In this latter form it is known as the *frame distributivity law* and complete lattices, which satisfy it, are called *frames*. The basic operations on a frame are those which appear in this equation, namely, arbitrary join and finite meet.

7.1.2 From spaces to lattices

Given a topology τ on a set X then τ consists of certain subsets of X. We may think of τ as an ordered set where the order relation is set inclusion. This ordered set is a complete lattice because arbitrary joins exist. Let us also look at continuous functions. In connection with open-set lattices it seems right to take the inverse image operation which, for a continuous function, is required to map opens to opens. Set-theoretically, it preserves all unions and intersections of subsets, and hence all joins and finite meets of opens. This motivates the following definition.

Definition 7.1.9. *A* frame-homomorphism *between complete lattices K and L is a map which preserves arbitrary suprema and finite infima.*

We let **CLat** stand for the category of complete lattices and frame-homomorphisms. We want to relate it to **Top**, the category of topological spaces and continuous functions. The first half of this relation is given by the contravariant functor Ω, which assigns to a topological space its lattice of open subsets and to a continuous map the inverse image function.

For an alternative description let $\mathbf{2}$ be the two-element chain $\bot\le\top$ equipped with the Scott-topology. The open sets of a space X are in one-to-one correspondence with continuous functions from X to $\mathbf{2}$, if for each open set $O\subseteq X$ we set χ_O to be the map which assigns $\top$ to an element x if and only if $x\in O$. The action of Ω on morphisms can then be expressed by $\Omega(f)(\chi_O)=\chi_O\circ f$.

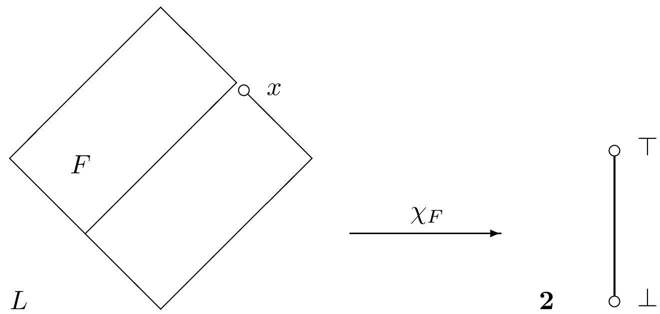


Figure 15: A ‘point’ in a complete lattice.

7.1.3 From lattices to topological spaces

For motivation, let us look at topological spaces first. An element of a topological space X is naturally equipped with the following three pieces of information. We can associate with it its filter $\mathcal{F}_x$ of open neighborhoods, the complement of its closure, or a map from $\mathbf{1}$, the one-element topological space, to X. Taking the filter, for example, we observe that it has the additional property that if a union of open sets belongs to it then so does one of the opens. Also, the closure of a point has the property that it cannot be contained in a union of closed sets without being contained in one of them already. The map $\mathbf{1} \to \mathbf{X}$, which singles out the point, translates to a frame-homomorphism from $\Omega(X)$ to $\Omega(\mathbf{1}) = \mathbf{2}$. Let us fix this new piece of notation:

Definition 7.1.10. *A filter $F \subseteq L$ is called* prime *if $\bigvee M \in F$ implies $F \cap M \neq \emptyset$ for all finite $M \subseteq L$. Allowing M to be an arbitrary subset we arrive at the notion of* completely prime filter. *Dually, we speak of* (completely) prime ideals.

Proposition 7.1.11. *Let L be a complete lattice and let F be a subset of L. The following are equivalent:*

1. *F is a completely prime filter.*
2. *F is a filter and $L \setminus F = {\downarrow}x$ for some $x \in L$.*
3. *$L \setminus F = {\downarrow}x$ for a $\wedge$-prime element $x \in L$.*
4. *χ_F is a frame-homomorphism from L to $\mathbf{2}$.*

This proposition shows that all three ways of characterizing points through opens coincide (see also Figure 15). Each of them has its own virtues and we will take advantage of the coincidence. As our official definition we choose the variant which is closest to our treatment of topological spaces.

Definition 7.1.12. *Let L be a complete lattice. The* points *of L are the completely prime filters of L. The collection $\mathsf{pt}(L)$ of all points is turned into a topological space by requiring all those subsets of $\mathsf{pt}(L)$ to be open which are of the form*

$$\mathcal{O}_x = \{F \in \mathsf{pt}(L) \mid x \in F\}, \quad x \in L\ .$$

Proposition 7.1.13. *The sets $\mathcal{O}_x$, $x \in L$, form a topology on $\mathsf{pt}(L)$.*

Proof. We have $\bigcap_{m\in M} \mathcal{O}_{x_m} = \mathcal{O}_{\bigwedge_{m\in M} x_m}$, M finite, because points are filters and $\bigcup_{i\in I} \mathcal{O}_{x_i} = \mathcal{O}_{\bigvee_{i\in I} x_i}$ because they are completely prime. □

Observe the perfect symmetry of our setup. In a topological space an element x belongs to an open set O if $x \in O$; in a complete lattice a point F belongs to an open set $\mathcal{O}_x$ if $x \in F$.

By assigning to a complete lattice L the topological space of all points, and to a frame-homomorphism $h\colon K \to L$ the map $\mathsf{pt}(h)$ which assigns to a point F the point $h^{-1}(F)$ (which is readily seen to be a completely prime filter), we get a contravariant functor, also denoted by pt, from **CLat** to **Top**.

Again, we give the alternative description based on characteristic functions. The fact is that we can use the same object **2** for this purpose, because it is a complete lattice as well. One speaks of a *schizophrenic object* in such a situation. As we saw in Proposition 7.1.11, a completely prime filter F gives rise to a frame-homomorphism $\chi_F\colon L \to \mathbf{2}$. The action of the functor pt on morphisms can then be expressed, as before, by $\mathsf{pt}(h)(\chi_F) = \chi_F \circ h$.

7.1.4 The basic adjunction

A topological space X can be mapped into the space of points of its open set lattice, simply map $x \in X$ to the completely prime filter $\mathcal{F}_x$ of its open neighborhoods. This assignment, which we denote by $\eta_X\colon X \to \mathsf{pt}(\Omega(X))$, is continuous and open onto its image: Let U be an open set in X. Then we get by simply unwinding the definitions: $\mathcal{F}_x \in \mathcal{O}_U \iff U \in \mathcal{F}_x \iff x \in U$. It also commutes with continuous functions $f\colon X \to Y$: $\mathsf{pt}(\Omega(f))(\eta_X(x)) = \Omega(f)^{-1}(\mathcal{F}_x) = \mathcal{F}_{f(x)} = \eta_Y \circ f(x)$. So the family of all η_X constitutes a natural transformation from the identity functor to $\mathsf{pt} \circ \Omega$.

The same holds for complete lattices. We let $\varepsilon_L\colon L \to \Omega(\mathsf{pt}(L))$ be the map which assigns $\mathcal{O}_x$ to $x \in L$. It is a frame-homomorphism as we have seen in the proof of Proposition 7.1.13. To see that this, too, is a natural transformation, we check that it commutes with frame-homomorphisms $h\colon K \to L$: $\Omega(\mathsf{pt}(h))(\varepsilon_K(x)) = \mathsf{pt}(h)^{-1}(\mathcal{O}_x) = \mathcal{O}_{h(x)} = \varepsilon_L \circ h(x)$, which is essentially the same calculation as for η. We have all the ingredients to formulate the Stone Duality Theorem:

Theorem 7.1.14. *The functors $\Omega\colon$ **Top** $\to$ **CLat** and $\mathsf{pt}\colon$ **CLat** $\to$ **Top** are dual adjoints of each other. The units are η and ε.*

Proof. It remains to check the triangle equalities

$$\begin{tikzcd}
\Omega(X) \arrow[r, "\varepsilon_{\Omega(X)}"] \arrow[dr, "\mathsf{id}"'] & \Omega(\mathsf{pt}(\Omega(X))) \arrow[d, "\Omega(\eta_X)"] \\
& \Omega(X)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathsf{pt}(L) \arrow[r, "\eta_{\mathsf{pt}(L)}"] \arrow[dr, "\mathsf{id}"'] & \mathsf{pt}(\Omega(\mathsf{pt}(L))) \arrow[d, "\mathsf{pt}(\varepsilon_L)"] \\
& \mathsf{pt}(L)
\end{tikzcd}$$

For the left diagram let O be an open set in X.

$$\begin{array}{rcl}
\Omega(\eta_X)(\varepsilon_{\Omega(X)}(O)) = \eta_X^{-1}(\mathcal{O}_O) & = & \{x \in X \mid \eta_X(x) \in \mathcal{O}_O\} \\
& = & \{x \in X \mid \mathcal{F}_x \in \mathcal{O}_O\} \\
& = & \{x \in X \mid O \in \mathcal{F}_x\} \\
& = & \{x \in X \mid x \in O\} = O.
\end{array}$$

The calculation for the right diagram is verbatim the same if we exchange η and ε, Ω and pt, X and L, and $\mathcal{O}$ and $\mathcal{F}$. $\square$

While our concrete representation through open sets and completely prime filters, respectively, allowed us a very concise proof of this theorem, it is nevertheless instructive to see how the units behave in terms of characteristic functions. Their type is from X to $(X \to \mathbf{2}) \to \mathbf{2}$ and from L to $(L \to \mathbf{2}) \to \mathbf{2}$, whereby the right hand sides are revealed to be second duals. The canonical mapping into a second dual is, of course, point evaluation: $x \mapsto \mathsf{ev}_x$, where $\mathsf{ev}_x(\chi) = \chi(x)$. This is indeed what both η and ε do.

7.2 Some equivalences

7.2.1 Sober spaces and spatial lattices

In this subsection we look more closely at the units η and ϵ. We will need the following concept:

Definition 7.2.1. *A closed subset of a topological space is called* irreducible *if it is non-empty and cannot be written as the union of two closed proper subsets.*

Clearly, an irreducible closed set corresponds via complementation to a $\wedge$-irreducible (and hence $\wedge$-prime) element in the lattice of all open sets.

Proposition 7.2.2. *Let X be a topological space. Then $\eta_X\colon X \to \mathsf{pt}(\Omega(X))$ is injective if and only if X satisfies the T_0-separation axiom. It is surjective if and only if every irreducible closed set is the closure of an element of X.*

Proof. The first half is just one of the various equivalent definitions of T_0-separation: different elements have different sets of open neighborhoods.

For the second statement observe that the $\wedge$-prime elements of $\Omega(X)$ are in one-to-one correspondence with completely prime filters of open sets. The condition then simply says that every such filter arises as the neighborhood filter of an element of X. $\square$

Definition 7.2.3. *A topological space X is called* sober *if η_X is bijective.*

Note that if η_X is bijective then it must be a homeomorphism because we know from Section 7.1.4 that it is always continuous and open onto the image. By the preceding proposition, a space is sober if and only if it is T_0 and every irreducible closed set is the closure of a point. The intuitive meaning is, of course, that a space is sober if it can be recovered from its lattice of open sets.

Proposition 7.2.4. *For any complete lattice L the unit $\varepsilon_L\colon L \to \Omega(\mathsf{pt}(L))$ is surjective and monotone. Furthermore, the following are equivalent:*

1. *ε_L is injective.*
2. *The elements of L are separated by completely prime filters.*
3. *L is $\wedge$-generated by $\wedge$-prime elements.*
4. *If $x \not\leq y$ then there exists a completely prime filter F such that $x \in F$ and $y \notin F$.*
5. *ε_L is order-reflecting.*

Proof. We have seen in Proposition 7.1.13 that all open sets on $\mathsf{pt}(L)$ are of the form $\mathcal{O}_x$ for some $x \in L$. This proves surjectivity. Monotonicity is clear because filters are upper sets.

Turning to the equivalent conditions for injectivity, we note that $\mathcal{O}_x = \mathcal{O}_y$ is equivalent to $x \in F \iff y \in F$ for all completely prime filters F. In other words, ε_L is injective if and only if the elements of L are separated by completely prime filters. Given $x \in L$ let x' be the infimum of all $\wedge$-primes above x. We want to show that $x = x'$. If x' is strictly above x then there exists a completely prime filter containing x' but not x. Using the equivalence of Proposition 7.1.11, we see that this is the same as the existence of a $\wedge$-prime element in $\uparrow x \setminus \uparrow x'$, a contradiction. From (3) the last two statements follow easily. They, in turn, imply injectivity (which, in a general order-theoretic setting, is strictly weaker than order-reflection). $\square$

Definition 7.2.5. *A complete lattice L is called* spatial *if ε_L is bijective.*

The intuitive meaning in this case is that a spatial lattice can be thought of as a lattice of open sets for some topological space. A direct consequence of Theorem 7.1.8 is the following:

Theorem 7.2.6. *A spatial lattice is a frame. In particular, it is distributive.*

Theorem 7.2.7. *For any complete lattice L the topological space $\mathsf{pt}(L)$ is sober. For any topological space X the lattice $\Omega(X)$ is spatial.*

Proof. The space of points of a lattice L is certainly T_0, because if we are given different completely prime filters then there is $x \in L$ which belongs to one of them but not the other. Hence, $\mathcal{O}_x$ contains one but not the other. For surjectivity of $\eta_{\mathsf{pt}(L)}$ let $\mathcal{A}$ be an irreducible closed set of filters. First of all, the union A of all filters in $\mathcal{A}$ is a non-empty upper set in L which is unreachable by joins. Hence the complement of A

is a principal ideal $\downarrow x$. Also, the complement of $\mathcal{A}$ in $\mathsf{pt}(L)$ certainly contains $\mathcal{O}_x$. We claim that x must be $\wedge$-prime. Indeed, if $y \wedge z \leq x$ then $\mathcal{A}$ is covered by the complements of $\mathcal{O}_y$ and $\mathcal{O}_z$, whence it is covered by one of them, say the complement of $\mathcal{O}_y$, which means nothing else than $y \leq x$. It follows that $\mathcal{A}$ is contained in the closure of the point $L \setminus \downarrow x$. On the other hand, $L \setminus \downarrow x$ belongs to the closed set $\mathcal{A}$ as each of its open neighborhoods contains an element of $\mathcal{A}$.

The second statement is rather easier to argue for. If O and O' are different open sets then there is an element x of X contained in one but not the other. Hence the neighborhood filter of x, which is always completely prime, separates O and O'. □

Corollary 7.2.8. *The functors* Ω *and* pt *form a dual equivalence between the category of sober spaces and the category of spatial lattices.*

This result may suggest that a reasonable universe of topological spaces ought to consist of sober spaces, or, if one prefers the lattice-theoretic side, of spatial lattices. This is indeed true as far as spaces are concerned. For the lattice side, however, it has been argued forcefully that the right choice is the larger category of *frames* (which are defined to be those complete lattices which satisfy the frame distributivity law, Section 7.1.1). The basis of these arguments is the fact that free frames exist, see [Joh82], Theorem II.1.2, a property which holds neither for complete lattices nor for spatial lattices. (More information on this is in [Isb72, Joh82, Joh83].) The choice of using frames for doing topology has more recently found support from theoretical computer science, because it is precisely the frame distributivity law which can be expected to hold for observable properties of processes. Even though this connection is to a large extent the *raison d'être* for this chapter, we must refer to [Abr87, Abr91b, Vic89, Smy92] for an in-depth discussion.

7.2.2 Properties of sober spaces

Because application of $\mathsf{pt} \circ \Omega$ to a space X is an essentially idempotent operation, it is best to think of $\mathsf{pt}(\Omega(X))$ as a completion of X. It is commonly called the *soberification* of X. Completeness of this particular kind is also at the heart of the Hofmann-Mislove Theorem, which we have met in Section 4.2.3 already and which we are now able to state in its full generality.

Theorem 7.2.9. *Let X be a sober space. The sets of open neighborhoods of compact saturated sets are precisely the Scott-open filters in $\Omega(X)$.*

Proof. It is pretty obvious that the neighborhoods of compact subsets are Scott-open filters in $\Omega(X)$. We are interested in the other direction. Given a Scott-open filter $\mathcal{F} \subseteq \Omega(X)$ then the candidate for the corresponding compact set is $K = \bigcap \mathcal{F}$. We must show that each open neighborhood of K belongs to $\mathcal{F}$ already. For the sake of contradiction assume that there exists an open neighborhood $O \notin \mathcal{F}$. By Zorn's Lemma we may further assume that O is maximal with this property. Because $\mathcal{F}$ is a filter, O is $\wedge$-prime as an element of $\Omega(X)$ and this is tantamount to saying that its complement A is irreducible as a closed set. By sobriety it must be the closure of a single point $x \in X$. The open sets which do not contain x are precisely those which are contained in O. Hence every open set from the filter $\mathcal{F}$ contains x and so x belongs to K. This, finally, contradicts our assumption that O is a neighborhood of K. $\square$

This appeared first in [HM81]. Our proof is taken from [KP94]. Note that it relies, like almost everything else in this chapter, on the Axiom of Choice.

Saturated sets are uniquely determined by their open neighborhoods, so we can reformulate the preceding theorem as follows:

Corollary 7.2.10. *Let X be a sober space. The poset of compact saturated sets ordered by inclusion is dually isomorphic to the poset of Scott-open filters in $\Omega(X)$ (also ordered by inclusion).*

Corollary 7.2.11. *Let X be a sober space. The filtered intersection of a family of (non-empty) compact saturated subsets is compact (and non-empty). If such a filtered intersection is contained in an open set O then some element of the family belongs to O already.*

Proof. By the Hofmann-Mislove Theorem we can switch freely between compact saturated sets and open filters in $\Omega(X)$. Clearly, the directed union of open filters is another such. This proves the first statement. For the intersection of a filtered family to be contained in O means that O belongs to the directed union of the corresponding filters. Then O must be contained in one of these already. The claim about the intersection of non-empty sets follows from this directly because we can take $O = \emptyset$. $\square$

Every T_0-space can be equipped with an order relation, called the *specialization order*, by setting $x \sqsubseteq y$ if for all open sets O, $x \in O$ implies $y \in O$. We may then compare the given topology with topologies defined on ordered sets. One of these which plays a role in this context, is the *weak upper topology*. It is defined as the coarsest topology for which all sets of the form $\downarrow x$ are closed.

Proposition 7.2.12. *For an T_0-space X the topology on X is finer than the weak upper topology derived from the specialization order.*

Proposition 7.2.13. *A sober space is a dcpo in its specialization order and its topology is coarser than the Scott-topology derived from this order.*

Proof. By the equivalence between sober spaces and spatial lattices we may think of X as the points of a complete lattice L. It is seen without difficulties that the specialization order on X then translates to the inclusion order of completely prime filters. That a directed union of completely prime filters is again a completely prime filter is immediate.

Let $\bigcup^{\uparrow}_{i\in I} F_i$ be such a directed union. It belongs to an open set $\mathcal{O}_x$ if and only if $x \in F_i$ for some $i \in I$. This shows that each $\mathcal{O}_x$ is Scott-open. $\square$

A dcpo equipped with the Scott-topology, on the other hand, is not necessarily sober, see Exercise 7.3.19(7). We also record the following fact although we shall not make use of it.

Theorem 7.2.14. *The category of sober spaces is complete and cocomplete. It is also closed under retracts formed in the ambient category* **Top**.

For the reader's convenience we sum up our considerations in a table comparing concepts in topological spaces to concepts in $\mathsf{pt}(L)$ for L a complete lattice.

space	$\mathsf{pt}(L)$
point	completely prime filter (c. p. filter)
specialization order	inclusion order
open set	c. p. filters containing some $x \in L$
saturated set	c. p. filters containing some upper set
compact saturated set	c. p. filters containing a Scott-open filter

7.2.3 Locally compact spaces and continuous lattices

We already know that sober spaces may be seen as dcpo's with an order-consistent topology. We move on to more special kinds of spaces with the aim to characterize our various kinds of domains through their open-set lattices. Our first step in this direction is to introduce local compactness. We have:

Lemma 7.2.15. *Distributive continuous lattices are spatial.*

Proof. We have shown in Theorem 7.1.7 that continuous lattices are $\wedge$-generated by $\wedge$-irreducible elements. In a distributive lattice these are also $\wedge$-prime. □

Now recall that a topological space is called locally compact if every element has a fundamental system of compact neighborhoods. This alone does not imply sobriety, as the ascending chain of natural numbers, equipped with the weak upper topology, shows. But in combination with sobriety we get the following beautiful result:

Theorem 7.2.16. *The functors Ω and pt restrict to a dual equivalence between the category of sober locally compact spaces and the category of distributive continuous lattices.*

Proof. We have seen in Section 4.2.3 already that $O \ll O'$ holds in $\Omega(X)$ if there is a compact set between O and O'. This proves that the open-set lattice of a locally compact space is continuous.

For the converse, let F be a point in an open set $\mathcal{O}_x$, that is, $x \in F$. A completely prime filter is Scott-open, therefore there is a further element $y \in F$ with $y \ll x$. Lemma 2.3.8 tells us that there is a Scott-open filter G contained in $\mathord{\uparrow\!\!\!\uparrow} y$ which contains x. We know by the previous lemma that a distributive continuous lattice can be thought of as the open-set lattice of its space of points, which, furthermore, is guaranteed to be sober. So we can apply the Hofmann-Mislove Theorem 7.2.9 and get that the set $\mathcal{A}$ of points of L, which are supersets of G, is compact saturated. In summary, F is contained in $\mathcal{O}_y$ which is a subset of $\mathcal{A}$ and this is a subset of $\mathcal{O}_x$. □

From now on, all our spaces are locally compact and sober. The three properties introduced in the next three subsections, however, are independent of each other.

7.2.4 Coherence

We have introduced coherence in Section 4.2.3 for the special case of continuous domains. The general definition reads as follows:

Definition 7.2.17. *A topological space is called* coherent*, if it is sober, locally compact, and the intersection of two compact saturated subsets is compact.*

Definition 7.2.18. *The order of approximation on a complete lattice is called* multiplicative *if $x \ll y$ and $x \ll z$ imply $x \ll y \wedge z$. A distributive continuous lattice for which the order of approximation is multiplicative is called* arithmetic.

As a generalization of Proposition 4.2.16 we have:

Theorem 7.2.19. *The functors Ω and pt restrict to a dual equivalence between the category of coherent spaces and the category of arithmetic lattices.*

Proof. The same arguments as in Proposition 4.2.15 apply, so it is clear that the open-set lattice of a coherent space is arithmetic. For the converse we may, just as in the proof of Theorem 7.2.16, invoke the Hofmann-Mislove Theorem. It tells us that compact saturated sets of $\mathsf{pt}(L)$ are in one-to-one correspondence with Scott-open filters. Multiplicativity of the order of approximation is just what we need to prove that the pointwise infimum of two Scott-open filters is again Scott-open. $\square$

7.2.5 Compact-open sets and spectral spaces

By passing from continuous lattices to algebraic ones we get:

Theorem 7.2.20. *The functors Ω and pt restrict to a dual equivalence between the category of sober spaces, in which every element has a fundamental system of compact-open neighborhoods, and the category of distributive algebraic lattices.*

The proof is the same as for distributive continuous lattices, Theorem 7.2.16. We now combine this with coherence.

Definition 7.2.21. *A topological space, which is coherent and in which every element has a fundamental system of compact-open neighborhoods, is called a* spectral space.

Theorem 7.2.22. *The functors Ω and pt restrict to a dual equivalence between the category of spectral spaces and the category of algebraic arithmetic lattices.*

Having arrived at this level, we can replace the open-set lattice with the sublattice of compact-open subsets. Our next task then is to reformulate Stone-duality with bases of open-set lattices. For objects we have:

Proposition 7.2.23. *Let L be an algebraic arithmetic lattice. The completely prime filters of L are in one-to-one correspondence with the prime filters of $\mathsf{K}(L)$. The topology on $\mathsf{pt}(L)$ is generated by the set of all $\mathcal{O}_x$, where x is compact in L.*

Proof. Given a completely prime filter F in L, we let $F \cap \mathsf{K}(L)$ be the set of compact elements contained in it. This is clearly an upwards closed set in $\mathsf{K}(L)$. It is a filter, because L is arithmetic. Primeness, finally, follows from the fact that F is Scott-open and hence equal to $\uparrow(F \cap \mathsf{K}(L))$. Conversely, a filter G in $\mathsf{K}(L)$ generates a filter $\uparrow G$ in L. For complete primeness let A be a subset of L with join in $\uparrow G$. L is algebraic. So we may replace A by $B = \downarrow A \cap \mathsf{K}(L)$ and $\bigvee B \in \uparrow G$ will still hold. Because $\uparrow G$ is Scott-open, there is a finite subset M of B with $\bigvee M \in \uparrow G$. Some element of G must be below $\bigvee M$ and primeness then gives us that some element of M belongs to G already.

The statement about the topology on $\mathsf{pt}(L)$ follows from the fact that every element of L is a join of compact elements. □

A frame-homomorphism between algebraic arithmetic lattices need not preserve compact elements, so in order to represent it through bases we need to resort to relations, as in Section 2.2.6, Definition 2.2.27. Two additional axioms are needed, however, because frame-homomorphisms are more special than Scott-continuous functions.

Definition 7.2.24. *A relation R between lattices V and W is called* join-approximable *if the following conditions are satisfied:*

1. $\forall x, x' \in V\ \forall y, y' \in W.\ (x' \geq x\ R\ y \geq y' \Longrightarrow x'\ R\ y')$*;*
2. $\forall x \in V\ \forall N \subseteq_{\mathit{fin}} W.\ (\forall y \in N.\ x\ R\ y \Longrightarrow x\ R\ (\bigvee N))$*;*
3. $\forall M \subseteq_{\mathit{fin}} V\ \forall y \in W.\ (\forall x \in M.\ x\ R\ y \Longrightarrow (\bigwedge M)\ R\ y)$*;*
4. $\forall M \subseteq_{\mathit{fin}} V\ \forall x \in W.\ ((\bigvee M)\ R\ x \Longrightarrow \exists N \subseteq_{\mathit{fin}} W.$
   $(x = \bigvee N \wedge \forall n \in N \exists m \in M.\ m\ R\ n))$.

The following is then easily established:

Proposition 7.2.25. *The category of algebraic arithmetic lattices and frame-homomorphisms is equivalent to the category of distributive lattices and join-approximable relations.*

By Proposition 7.2.23 we can replace the compound functor $\mathsf{pt} \circ \mathsf{Idl}$ by a direct construction of a topological space out of a distributive lattice. We denote this functor by spec, standing for the *spectrum* of a distributive lattice. We also contract $\mathsf{K} \circ \Omega$ to $\mathsf{K}\Omega$. Then we can say:

Theorem 7.2.26. *The category of spectral spaces and continuous functions is dually equivalent to the category of distributive lattices and join-approximable relations via the contravariant functors* $\mathsf{K}\Omega$ *and* spec.

We supplement the table in Section 7.2.2 with the following comparison of concepts in a topological space and concepts in the spectrum of a distributive lattice.

space	$\mathsf{spec}(L)$
point	prime filter
specialization order	inclusion order
compact-open set	prime filters containing some $x \in L$
open set	union of compact open sets
saturated set	prime filters containing some upper set
compact saturated set	prime filters containing a filter

It has been argued that the category of spectral spaces is the right setting for denotational semantics, precisely because these have a finitary 'logical' description through their distributive lattices of compact-open subsets, see [Smy92], for example. However, this category is neither cartesian closed, nor does it have fixpoints for endofunctions, and hence does not provide an adequate universe for the semantics of computation. An intriguing question arises, of how the kinds of spaces traditionally studied in topology and analysis can best be reconciled with the computational intuitions reflected in the very different kinds of spaces which arise in Domain Theory. An interesting recent development is Abbas Edalat's use of Domain Theory as the basis for a novel approach to the theory of integration [Eda93a].

7.2.6 Domains

Let us now see how continuous domains come into the picture. First we note that sobriety no longer needs to be assumed:

Proposition 7.2.27. *Continuous domains eqipped with the Scott-topology are sober spaces.*

Proof. Let A be an irreducible closed set in a continuous domain D and let $B = \mathord{\downarrow\!\!\downarrow} A$. We show that B is directed. Indeed, given x and y in B, then neither $D \setminus \mathord{\uparrow\!\!\uparrow} x$ nor $D \setminus \mathord{\uparrow\!\!\uparrow} y$ contain all of A. By irreducibility, then, they can't cover A. Hence there is $a \in A \cap \mathord{\uparrow\!\!\uparrow} x \cap \mathord{\uparrow\!\!\uparrow} y$. But since $\mathord{\uparrow\!\!\uparrow} x \cap \mathord{\uparrow\!\!\uparrow} y$ is Scott-open, there is also some $b \ll a$ in this set. This gives us the desired upper bound for x and y. It is plain from Proposition 2.2.10 that A is the closure of $\bigsqcup^{\uparrow} B$. $\square$

The following result of Jimmie Lawson and Rudolf-Eberhard Hoffmann, [Law79, Hof81], demonstrates once again the central role played by continuous domains.

Theorem 7.2.28. *The functors Ω and pt restrict to a dual equivalence between* **CONT** *and the category of completely distributive lattices.*

Proof. A Scott-open set O in a continuous domain D is a union of sets of the form $\mathord{\uparrow\!\!\uparrow} x$ where $x \in O$. For each of these we have $\mathord{\uparrow\!\!\uparrow} x \lll O$ in σ_D. This proves complete distributivity, as we have seen in Theorem 7.1.3.

For the converse, let L be completely distributive. We already know that the points of L form a dcpo (where the order is given by inclusion of filters) and that the topology on $\mathsf{pt}(L)$ is contained in the Scott-topology of this dcpo. Now we show that every completely prime filter F has enough approximants. Observe that $F' \ll F$ certainly holds in all those cases where $\bigwedge F'$ is an element of F as directed suprema of points are unions of filters. Now given $x \in F$ we get from prime-continuity that $x = \bigvee\{y \mid y \lll x\}$ and so there must be some $y \in F$ with $y \lll x$. Successively interpolating between y and x gives us a sequence of elements such that $y \lll \ldots \lll y_n \lll \ldots \lll y_1 \lll x$, just as in the proof of Lemma 2.3.8. The set $\bigcup_{n\in\mathbb{N}} \uparrow y_n$ then is a completely prime filter containing x with infimum in F. The directedness of these approximants is clear because F is filtered. As a consequence, we have that $F' \ll F$ holds if and only if $\bigwedge F'$ belongs to F.

We are not quite finished, though, because we also need to show that we get the Scott-topology back. To this end let $\mathcal{O}$ be a Scott-open set of points, that is, $F \supseteq F' \in \mathcal{O}$ implies $F \in \mathcal{O}$ and $\bigcup^{\uparrow}_{i\in I} F_i \in \mathcal{O}$ implies $F_i \in \mathcal{O}$ for some $i \in I$. Let x be the supremum of all elements of the form $\bigwedge F$, $F \in \mathcal{O}$. We claim that $\mathcal{O} = \mathcal{O}_x$. First of all, for each $F \in \mathcal{O}$ there is $F' \in \mathcal{O}$ with $F' \ll F$, which, as we have just seen, is tantamount to $\bigwedge F' \in F$, hence x belongs to all F and $\mathcal{O} \subseteq \mathcal{O}_x$ is proved.

Conversely, if a point G contains x then it must contain some $\bigwedge F$, $F \in \mathcal{O}$, because it is completely prime. Hence G belongs to $\mathcal{O}$, too, and we have shown $\mathcal{O}_x \subseteq \mathcal{O}$. □

To this we can add coherence and we get a dual equivalence between coherent domains and completely distributive arithmetic lattices. Or we can add algebraicity and get a dual equivalence between algebraic domains and algebraic completely distributive lattices. Adding both properties characterizes what can be called 2/3-bifinite domains in the light of Proposition 4.2.17. We prefer to speak of coherent algebraic domains. As these are spectral spaces, we may also ask how they can be characterized through the lattice of compact open subsets. The answer is rather simple: A compact open set in an algebraic domain D is a finite union of sets of the form $\uparrow c$ for $c \in \mathsf{K}(D)$. These, in turn, are characterized by being $\vee$-irreducible and also $\vee$-prime.

Theorem 7.2.29. *The dual equivalence of Theorem 7.2.26 cuts down to a dual equivalence of coherent algebraic domains and lattices in which every element is the join of finitely many $\vee$-primes.*

Proof. We only need to show that if a lattice satisfies the condition stated in the theorem, then its ideal completion is completely distributive. But this is trivial because a principal ideal generated by a $\vee$-prime is completely $\vee$-prime in the ideal completion and so the result follows from Theorem 7.1.3. □

All the combined strength of complete distributivity, algebraicity and multiplicativity of the order of approximation, however, does still not restrict the corresponding spaces far enough so as to bring us into one of our cartesian closed categories of domains. Let us therefore see what we have to add in order to characterize bifinite domains. The only solution in this setting appears to be a translation of mub-closures into the lattice of compact-open subsets, that is to say, the subset of $\vee$-primes has the upside-down finite mub property (Definition 4.2.1). Let us sum up these considerations in a theorem:

Theorem 7.2.30. *A lattice V is isomorphic to the lattice of compact-open subsets of an F-B-domain* (Definition 4.3.7) *if and only if, firstly, V has a least element, secondly,* *each element of V is the supremum of finitely many $\vee$-primes and, thirdly, for every finite set M of $\vee$-primes there is a finite superset N of $\vee$-primes such that*

$$\forall A \subseteq M\ \exists B \subseteq N.\ \bigwedge A = \bigvee B.$$

The additional requirement that there be a largest element which is also $\vee$-prime, characterizes the lattices of compact-open subsets of bifinite domains.

The extra condition about finite mub-closures is not a first-order axiom and cannot be replaced by one as was shown by Carl Gunter in [Gun86]. The smaller class of algebraic bc-domains has a rather nicer description:

Theorem 7.2.31. *A lattice V is isomorphic to the lattice of compact-open subsets of an algebraic bc-domain if and only if it has a least element, each element of V is the supremum of finitely many $\vee$-primes and the set of $\vee$-primes plus least element is closed under finite infima.*

7.2.7 Summary

We have summarized the results of this section in Figure 16 and Table 1. As labels we have invented a few mnemonic names for categories. We won't use them outside this subsection. The filled dots correspond to categories for which there is also a characterization in terms of compact-open subsets (spectral spaces). A similar diagram appears in [GHK$^+$80] but there not everything, which appears to be an intersection of categories, really is one.

7.3 The logical viewpoint

This material is based on [Abr91b].

7.3.1 Working with lattices of compact-open subsets

Having established the duality between algebraic domains and their lattices of compact-open subsets we can now ask to what extent we can do domain theory through these lattices. We have already indicated that such an approach offers many new insights but for the moment our motivation could simply be that working with lattices is a lot easier than working with dcpo's. 'Doing domain theory' refers to performing the domain constructions of Sections 3.2, 3.3, 5 and 6, at least in a first approximation.

Let us try this out. Suppose you know $\mathsf{K}\Omega(D)$ for some bifinite domain D, how do you construct $\mathsf{K}\Omega(D_\perp)$, the lattice of compact-open subsets of the lifted domain? The answer is simple, just add a new top element: $\mathsf{K}\Omega(D_\perp) = \mathsf{K}\Omega(D)^\top$. Coalesced sum also works fine:

$$\mathsf{K}\Omega(D \oplus E) = (\mathsf{K}\Omega(D) \setminus \{D\}) \times (\mathsf{K}\Omega(E) \setminus \{E\}) \cup \{D \oplus E\}.$$

We encounter the first problems when we look at the cartesian product. While it is clear that every compact-open subset of $D \times E$ is a finite union of products of compact-open

TOP	Topological spaces. No Stone-dual.
SOB	Sober spaces vs. spatial lattices.
L-C	Locally-compact sober spaces vs. continuous distributive lattices.
COH	Coherent spaces (= locally compact, sober, and intersection of compact saturated is compact) vs. arithmetic lattices (= distributive, continuous, and order of approximation is multiplicative).
C-O	Sober spaces with a base of compact-open sets vs. distributive algebraic lattices.
CONT	Continuous domains with Scott-topology vs. completely distributive lattices.
SPEC	Spectral spaces vs. algebraic arithmetic lattices vs. distributive lattices.
C-CONT	Coherent domains vs. arithmetic completely distributive lattices.
ALG	Algebraic domains vs. algebraic completely distributive lattices.
C-ALG	Coherent algebraic domains vs. algebraic arithmetic completely distributive lattices vs. distributive lattices in which every element is the finite join of $\vee$-primes.
F-B	F-B-domains (Definition 4.3.7) (= bilimits of finite posets). Stone-dual only described through the basis (or base) of compact-open subsets, which is a distributive lattice with extra properties as stated in Theorem 7.2.30.
B	Bifinite domains. Stone-dual only described through the basis of compact-open subsets, which is a distributive lattice with extra properties as stated in Theorem 7.2.30.
aBC	Algebraic bounded-complete domains. Stone-dual only described through the basis of compact-open subsets, which is a distributive lattice with extra properties as stated in Theorem 7.2.31.

Table 1: The categories and their Stone-duals.

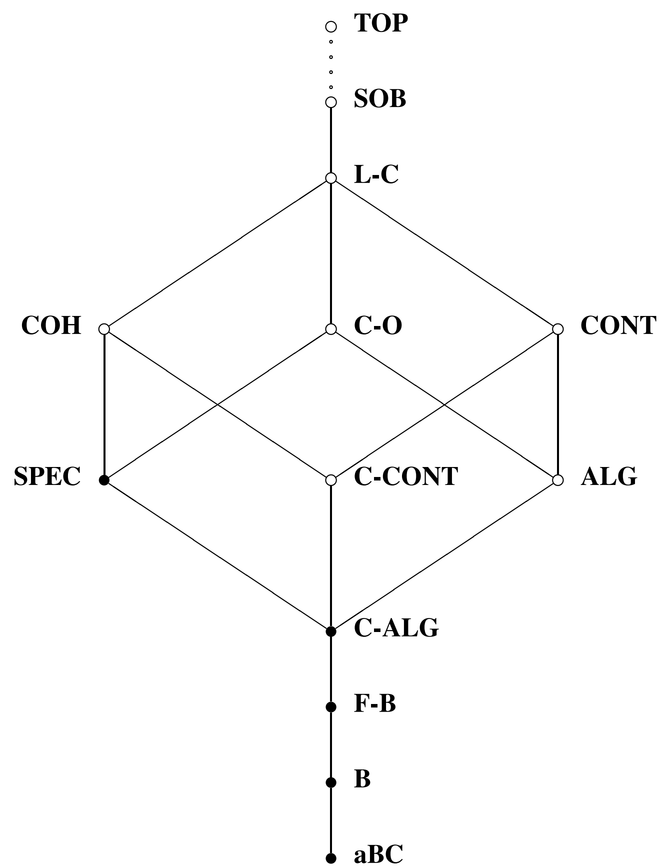


Figure 16: An overview of Stone-dualities in domain theory.

subsets in the factors, there seems to be no simple criterion on such unions which would guarantee unique representation.

The moral then is that we must allow for multiple representations of compact-open subsets. Instead of lattices we shall study certain preordered structures. At first glance this may seem as an unwanted complication but we will soon see that it really makes the whole programme work much more smoothly.

Lattices are determined by either their order structure or their algebraic structure but this equivalence no longer holds in the preordered case. Instead we must mention both preorder and lattice operations. We also make $\vee$-primeness explicit in our axiomatization. The reason for this is that we want to keep all our definitions inductive. This point will become clearer when we discuss the function space construction below.

Definition 7.3.1. *A* coherent algebraic prelocale *A is a preordered algebra with two* *binary operations $\vee$ and $\wedge$, two nullary operations 0 and 1, and a unary predicate C on A, such that $a \vee b$ is a supremum for $\{a, b\}$, $a \wedge b$ is an infimum for $\{a, b\}$, 0 is a least, and 1 is a largest element. The preorder on A is denoted by $\lesssim$, the corresponding equivalence relation by $\approx$. The predicate $\mathsf{C}(a)$ is required to hold if and only if a is $\vee$-prime. Finally, every element of A must be equivalent to a finite join of $\vee$-primes.*

We will not distinguish between a prelocale and its underlying set. The set $\{a \in A \mid \mathsf{C}(a)\}$ is abbreviated as $\mathsf{C}(A)$.

This is essentially the definition which appears in [Abr91b]. There another predicate is included. We can omit this because we will not look at the coalesced sum construction. The expressions 'a supremum', 'an infimum', etc., may seem contradictory but they are exactly appropriate in the preordered universe. It is seen without difficulties that every coherent algebraic prelocale A gives rise to a lattice $A/_{\approx}$ which is $\vee$-generated by $\vee$-primes and hence distributive.

A *domain prelocale* is gotten by incorporating the two extra conditions from Theorem 7.2.30:

- $\forall u \subseteq_{\text{fin}} \mathsf{C}(A)\ \exists v \subseteq_{\text{fin}} \mathsf{C}(A).\ u \subseteq v$ and $(\forall w \subseteq u\ \exists z \subseteq v.\ \bigwedge w = \bigvee z)$;
- $\mathsf{C}(1)$.

Definition 7.3.2. *Let A and B be domain prelocales. A function $\phi\colon A \to B$ is called a* pre-isomorphism *if it is surjective, order-preserving and order-reflecting. If A is a domain prelocale and D is a bifinite domain and if further there is a pre-isomorphism $\llbracket\cdot\rrbracket\colon A \to \mathsf{K}\Omega(D)$ then we say that A is a* localic description *of D via $\llbracket\cdot\rrbracket$.*

A pre-isomorphism $\phi\colon A \to B$ must preserve suprema, infima, and least and largest element (up to equivalence). Furthermore, it restricts and corestricts to a surjective map $\phi^0\colon \mathsf{C}(A) \to \mathsf{C}(B)$. Let us look more closely at the case of a pre-isomorphism $\llbracket\cdot\rrbracket\colon A \to \mathsf{K}\Omega(D)$. A diagram may be quite helpful:

$$\begin{tikzcd}
 & \mathsf{C}(A) \arrow[r, hook] \arrow[d, "\llbracket\cdot\rrbracket^0"'] & A \arrow[d, "\llbracket\cdot\rrbracket"] \\
\mathsf{K}(D) \arrow[r, phantom, "\cong_{\text{dual}}"] & \mathsf{C}(\mathsf{K}\Omega(D)) \arrow[r, hook] & \mathsf{K}\Omega(D)
\end{tikzcd}$$

Remember that $\mathsf{C}(\mathsf{K}\Omega(D))$ are just those compact-open subsets which are of the form $\uparrow c$ for $c \in \mathsf{K}(D)$. The inclusion order between such principal filters is dual to the usual order on $\mathsf{K}(D)$.

Let us now lift the pre-isomorphism to the domain level. In the previous chapters, the natural approach would have been to apply the ideal completion functor to the pre-isomorphism between $\mathsf{C}(A)^{op}$ and $\mathsf{K}(D)$. Here we use Stone-duality and apply spec to $\llbracket\cdot\rrbracket$. This yields an isomorphism between $\mathsf{spec}(A)$ and $\mathsf{spec}(\mathsf{K}\Omega(D))$. Composed with the inverse of the unit η it gives us the isomorphism $\tau\colon \mathsf{spec}(A) \to D$.

$$\begin{tikzcd}
\mathsf{spec}(A) \arrow[d, "\mathsf{spec}(\llbracket\cdot\rrbracket)^{-1}"'] \arrow[dr, "\tau"] & \\
\mathsf{spec}(\mathsf{K}\Omega(D)) \arrow[r, "\eta^{-1}"] & D
\end{tikzcd}$$

It will be good to have a concrete idea of the behaviour of τ, at least for compact elements of $\mathsf{spec}(A)$. These are filters in A which are generated by $\vee$-prime elements. So let $F = \uparrow a$ with $a \in \mathsf{C}(A)$. It is easily checked that $\tau(F)$ equals that compact element c of D which is least in the compact-open subset $\llbracket a\rrbracket^0$.

Proposition 7.3.3. *There exists a map $\llbracket\cdot\rrbracket\colon A \to \mathsf{K}\Omega(D)$ such that the domain prelocale A is a localic description of the bifinite domain D if and only if $\mathsf{spec}(A)$ and D are isomorphic.*

Proof. We have just described how to derive an isomorphism from a pre-isomorphism. For the converse observe that the unit $\varepsilon\colon A \to \mathsf{K}\Omega(\mathsf{spec}(A))$ is surjective, order-preserving and order-reflecting (Proposition 7.2.4). $\square$

For more general functions between domains, we can translate join-approximable relations into the language of domain prelocales. The following is then just a slight extension of Theorem 7.2.30.

Theorem 7.3.4. *The category of domain prelocales and join-approximable relations is dually equivalent to the category of bifinite domains and Scott-continuous functions.*

Our attempt to mimic the cartesian product construction forced us to pass to preordered structures but once we have accepted this we can go one step farther and make the prelocales syntactic objects in which no identifications are made at all. More precisely, it is no loss of generality to assume that the underlying algebra is a term algebra with respect to the operations $\vee, \wedge, 0$, and 1. As an example, let us describe the one-point domain $\mathbb{I}$ in this fashion. We take the term algebra on no generators, that is, every term is a combination of 0's and 1's. The preorder is the smallest relation compatible with the requirements in Definition 7.3.1. The effect of this is that there are exactly two equivalence classes with respect to $\approx$, the terms equivalent to 1 and the terms equivalent to 0. The former are precisely the $\vee$-prime terms. We denote the resulting domain prelocale by **1**.

The syntactic approach also suggests that we look at the following relation between domain prelocales:

Definition 7.3.5. *Let A and B be domain prelocales. We say that A is a* sub-prelocale *of B if the following conditions are satisfied:*

1. *A is a subalgebra of B with respect to $\vee, \wedge, 0$ and 1.*
2. *The preorder on A is the restriction of the preorder on B to A.*

3. $\mathsf{C}(A)$ *equals* $A \cap \mathsf{C}(B)$.

We write $A \trianglelefteq B$ *if* A *is a sub-prelocale of* B.

Proposition 7.3.6. *If* A *is a sub-prelocale of* B *then the following defines an embedding projection pair between* $\mathsf{spec}(A)$ *and* $\mathsf{spec}(B)$*:*

$$
\begin{aligned}
e\colon \mathsf{spec}(A) &\to \mathsf{spec}(B), \qquad & e(F) &= \uparrow_B(F);\\
p\colon \mathsf{spec}(B) &\to \mathsf{spec}(A), \qquad & p(F) &= F \cap A.
\end{aligned}
$$

Proof. It is clear that both e and p are continuous because directed joins of elements in $\mathsf{spec}(A)$, resp. $\mathsf{spec}(B)$, are just directed unions of prime filters. We have $p \circ e = \mathsf{id}$ because the preorder on A is the restriction of that on B. For $e \circ p \sqsubseteq \mathsf{id}$ we don't need any special assumptions.

The crucial point is that the two functions are well-defined in the sense that they indeed produce prime filters. The filter part follows again from the fact that both operations and preorder on A are the restrictions of those on B. For primeness assume that $\bigvee M \in \uparrow_B(F)$ for some finite $M \subseteq B$. This means $x \lesssim \bigvee M$ for some $x \in F$. This element itself is a supremum of $\vee$-primes of A and because F is a prime filter in A we have some $\vee$-prime element x' below $\bigvee M$ in F. But we have also required that the $\vee$-prime elements of A are precisely those $\vee$-prime elements of B which lie in A and therefore some $m \in M$ must be above x'.

Primeness of $F \cap A$, on the other hand, follows easily because suprema in A are also suprema in B. $\square$

Corollary 7.3.7. *Assume that* A *is a localic description of* D *via* $[\![\cdot]\!]_A$*, that* B *describes* E *via* $[\![\cdot]\!]_B$*, and that* $A \trianglelefteq B$*. Then the following defines an embedding* e *of* D *into* E*:*

If $c \in \mathsf{K}(D)$*,* $a \in \mathsf{C}(A)$*,* $[\![a]\!]^0_A = \uparrow c$*,* $[\![a]\!]^0_B = \uparrow d$*, then* $e(c) = d$*.*

Proof. If we denote by e' the embedding from $\mathsf{spec}(A)$ into $\mathsf{spec}(B)$ as defined in the preceding proposition, then the embedding $e\colon D \to E$ is nothing else but $\tau_B \circ e' \circ \tau_A^{-1}$. $\square$

Of course, it happens more often that $\mathsf{spec}(A)$ is a sub-domain of $\mathsf{spec}(B)$ than that A is a sub-prelocale of B but the fact is that it will be fully sufficient and even advantageous to work with the stronger relation when it comes to solving recursive domain equations.

7.3.2 Constructions: The general technique

Before we demonstrate how function space and Plotkin powerdomain can be constructed through prelocales, let us outline the general technique. The overall picture is in the following diagram. We explain how to get its ingredients step by step below.

$$
\begin{tikzcd}
 & \mathsf{C}(T(A,A')) \arrow[r, hook] \arrow[d, "{[\![\cdot]\!]^0}"'] & T(A,A') \arrow[d, "{[\![\cdot]\!]}"] \\
\mathsf{K}(F_T(D,D')) \arrow[r, phantom, "\cong_{\text{dual}}"] & \mathsf{C}(\mathsf{K}\Omega(F_T(D,D'))) \arrow[r, hook] & \mathsf{K}\Omega(F_T(D,D'))
\end{tikzcd}
$$

1. The set-up. We want to study a construction T on (bifinite) domains. This could be any one from the table in Section 3.2.6 or a bilimit or one of the powerdomain constructions from Section 6.2. The diagram illustrates a binary construction. We can assume that we understand the action of the associated functor F_T on bifinite domains. In particular, we know what the compact elements of $F_T(D, D')$ are, how they compare and how F_T acts on embeddings (Proposition 5.2.6). Thus we should have a clear understanding of the bottom row of the diagram, in detail:

- $F_T(D, D')$ is the effect of the functor F_T on objects D and D'.
- $\mathsf{K}(F_T(D, D'))$ are the compact elements of $F_T(D, D')$.
- $\mathsf{K}\Omega(F_T(D, D'))$ are the compact-open subsets of $F_T(D, D')$ and these are precisely those upper sets which are of the form $\uparrow u$ for a finite set u of compact elements.
- $\mathsf{C}(\mathsf{K}\Omega(F_T(D, D')))$ are the $\vee$-prime elements of $\mathsf{K}\Omega(F_T(D, D'))$ and these are precisely those subsets of $F_T(D, D')$ which are of the form $\uparrow c$ for c a compact element. The order is inclusion which is dual to the usual order on compact elements.

Furthermore, we assume that we are given domain prelocales A and A' which describe the bifinite domains D and D', respectively. These descriptions are encoded in pre-isomorphisms $[\![\cdot]\!]_A \colon A \to \mathsf{K}\Omega(D)$ and $[\![\cdot]\!]_{A'} \colon A' \to \mathsf{K}\Omega(D')$.

2.The goal. We want to define a domain prelocale $T(A, A')$ which is a localic description of $F_T(D, D')$. This is achieved in the following series of steps.

3. Definition of $T(A, A')$. This is the creative part of the enterprise. We search for a description of compact-open subsets of $F_T(D, D')$ based on our knowledge of the compact-open subsets of D and D'. The point is to do this directly, *not* via the compact elements of D, D', and $F_T(D, D')$. There will be an immediate payoff, as we will gain an understanding of the construction in terms of properties rather than points. Our treatment of the Plotkin powerdomain below illustrates this most convincingly.

The definition of $T(A, A')$ will proceed uniformly in all concrete instances. First a set G_T of generators is defined and then $T(A, A')$ is taken to be the term algebra over G_T with respect to $\vee, \wedge, 0$, and 1. An interpretation function $[\![\cdot]\!] \colon G_T \to \mathsf{K}\Omega(F_T(D, D'))$ is defined based on the interpretations $[\![\cdot]\!]_A$ and $[\![\cdot]\!]_{A'}$. It is extended to all of $T(A, A')$ as a lattice homomorphism: $[\![a \vee b]\!] = [\![a]\!] \cup [\![b]\!]$, etc. Finally, axioms and rules are given which govern the preorder and $\vee$-primeness predicate.

Next we have to check that our definitions work. This task is also broken into a series of steps as follows.

4. Soundness. We check that axioms and rules translate via $[\![\cdot]\!]$ into valid statements about compact-open subsets of $F_T(D, D')$. This is usually quite easy. From soundness we infer that $[\![\cdot]\!]$ is monotone and can be restricted and corestricted to a map $[\![\cdot]\!]^0 \colon \mathsf{C}(T(A, A')) \to \mathsf{C}(\mathsf{K}\Omega(F_T(D, D')))$.

5. Prime generation. Using the axioms and rules, we prove that every element of $T(A, A')$ can be transformed (effectively) into an equivalent term which is a finite supremum of expressions which are asserted to be $\vee$-prime. This is the crucial step and usually contains the main technical work. It allows us to prove the remaining properties of $\llbracket\cdot\rrbracket$ through $\llbracket\cdot\rrbracket^0$ and for the latter we can use our knowledge of the basis of $F_T(D, D')$.

6. Completeness for $\vee$-primes. We show that $\llbracket\cdot\rrbracket^0$ is order reflecting.

7. Definability for $\vee$-primes. We show that $\llbracket\cdot\rrbracket^0$ is surjective.

At this point we can fill in the remaining pieces without reference to the concrete construction under consideration.

8. Completeness. The interpretation function $\llbracket\cdot\rrbracket$ itself is order-reflecting.

Proof. Let $a, b \in T(A, A')$ be such that $\llbracket a\rrbracket \subseteq \llbracket b\rrbracket$. By 5 we can replace these expressions by formal joins of $\vee$-primes: $a \approx a_1 \vee \ldots \vee a_n$ and $b \approx b_1 \vee \ldots \vee b_m$. Soundness ensures that the value under the interpretation function remains unchanged and that each $\llbracket a_i\rrbracket$ (resp. $\llbracket b_j\rrbracket$) is of the form $\uparrow c_i$ (resp. $\uparrow d_j$) for c_i, d_j compact elements in $F_T(D, D')$. The inclusion order on $\mathsf{K}\Omega(F_T(D, D'))$ translates into the formula $\forall i\ \exists j.\ \uparrow c_i \subseteq \uparrow d_j$ which by the completeness for $\vee$-primes can be pulled back into $T(A, A')$: $\forall i\ \exists j.\ a_i \lesssim b_j$. In every preordered lattice it must follow that $a \lesssim b$ holds. $\square$

9. Definability. The surjectivity of $\llbracket\cdot\rrbracket$ is an easy consequence of the surjectivity of $\llbracket\cdot\rrbracket^0$ because we know that compact-open subsets in an algebraic domain are finite unions of compactly generated principal filters.

10. Well-definedness. Of course, $\mathsf{K}\Omega(F_T(D, D'))$ is a domain prelocale and we have just shown that preorder and primeness predicate on $T(A, A')$ are preserved and reflected by $\llbracket\cdot\rrbracket$. This constitutes a semantic proof that $T(A, A')$ satisfies the two extra conditions for domain prelocales. In other words, T is a well-defined operation on domain prelocales.

11. Stone-duality. At this point we have shown that $\llbracket\cdot\rrbracket$ is a pre-isomorphism. As in the previous subsection we lift it to an isomorphism τ between $\mathsf{spec}(T(A, A'))$ and $F_T(D, D')$ via Stone duality:

$$
\begin{tikzcd}
\mathsf{spec}(T(A,A')) \arrow[d, "\mathsf{spec}(\llbracket\cdot\rrbracket)^{-1}"'] \arrow[drr, "\tau"] & & \\
\mathsf{spec}(\mathsf{K}\Omega(F_T(D,D'))) \arrow[rr, "\eta^{-1}"] & & F_T(D,D')
\end{tikzcd}
$$

So much for the correspondence on the object level. We also want to see how the construction T harmonizes with the sub-prelocale relation, one the one hand, and the isomorphism τ, on the other hand. Thus we assume that we are given two more prelocales, B and B', which are localic descriptions of bifinite domains E and E', such that $A \trianglelefteq B$ and $A' \trianglelefteq B'$ hold. In Corollary 7.3.7 we have seen how to define from this embeddings $e\colon D \to E$ and $e'\colon D' \to E'$. In Proposition 5.2.6 we have shown how the functors associated with different constructions act on embeddings, hence we may unambiguously write $F_T(e, e')$ for the result of this action, which is an embedding from $F_T(D, D')$ to $F_T(E, E')$. Embeddings preserve compact elements so $F_T(e, e')$ restricts and corestricts to a monotone function $F_T(e, e')^0\colon \mathsf{K}(F_T(D, D')) \to \mathsf{K}(F_T(E, E'))$. Now for both $T(A, A')$ and $T(B, B')$

we have a diagram such as depicted at the beginning of this subsection. We connect the lower left corners of these by $F_T(e,e')^0$. This gives rise also to a map i from $\mathsf{C}(\mathsf{K}\Omega(F_T(D,D')))$ to $\mathsf{C}(\mathsf{K}\Omega(F_T(E,E')))$. Our way of defining $T(A,A')$ will be such that it is immediate that $\mathsf{C}(T(A,A'))$ is a subset of $\mathsf{C}(T(B,B'))$ and hence there is an inclusion map connecting the upper left corners. Our next technical step then is the following.

12. Naturality. We show that the diagram

$$\begin{tikzcd}
\mathsf{C}(T(A,A')) \arrow[r, hook] \arrow[d, "\llbracket\cdot\rrbracket^0_{T(A,A')}"] & \mathsf{C}(T(B,B')) \arrow[d, "\llbracket\cdot\rrbracket^0_{T(B,B')}"] \\
\mathsf{C}(\mathsf{K}\Omega(F_T(D,D'))) \arrow[r, "i"] & \mathsf{C}(\mathsf{K}\Omega(F_T(E,E')))
\end{tikzcd}$$

commutes. On the element level this reads: If $a \in \mathsf{C}(T(A,A'))$ and $\llbracket a\rrbracket^0_{T(A,A')} = {\uparrow}c$ and $\llbracket a\rrbracket^0_{T(B,B')} = {\uparrow}d$ then $F_T(e,e')^0(c) = d$. Now we can again get the remaining missing information in a general manner.

13. Monotonicity. We show that $T(A,A') \trianglelefteqslant T(B,B')$. From the form of our construction it will be clear that $T(A,A')$ is a subset of $T(B,B')$ and the axioms and rules will be such that whatever can be derived in $T(A,A')$ can also be derived in $T(B,B')$. We must show that in the larger prelocale nothing extra can be proved for elements of $T(A,A')$. The argument is a semantic one.

Proof. Let $a,a' \in \mathsf{C}(T(A,A'))$ such that $a \lesssim a'$ holds in $T(B,B')$. Let $\llbracket a\rrbracket^0_{T(A,A')} = {\uparrow}c$, $\llbracket a\rrbracket^0_{T(B,B')} = {\uparrow}d$ and similarly for a'. Correctness says that ${\uparrow}d \subseteq {\uparrow}d'$ and hence $d \sqsupseteq d'$. By naturality we have $F_T(e,e')^0(c) = d \sqsupseteq d' = F_T(e,e')^0(c')$. Embeddings are order reflecting so $c \sqsupseteq c'$ follows. Completeness then allows us to conclude that $a \lesssim a'$ holds in $T(A,A')$ as well.

In the same way it is seen that the predicate C on $T(A,A')$ is the restriction of that on $T(B,B')$. □

14. Least prelocale. It follows from the correctness of the construction that $\mathbf{1} \trianglelefteqslant T(A,A')$ holds.

15. Naturality of τ. Having established the relation $T(A,A') \trianglelefteqslant T(B,B')$ we can look at the embedding $I\colon \mathsf{spec}(T(A,A')) \to \mathsf{spec}(T(B,B'))$ which we defined in Proposition 7.3.6. We claim that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{spec}(T(A,A')) \arrow[r, "I"] \arrow[d, "\tau_A"] & \mathsf{spec}(T(B,B')) \arrow[d, "\tau_B"] \\
F_T(D,D') \arrow[r, "F_T(e,e')"] & F_T(E,E')
\end{tikzcd}$$

In other words, $F_T(e,e')$ equals the embedding which can be derived from $T(A,A') \trianglelefteqslant T(B,B')$ in the general manner of Corollary 7.3.7.

Proof. This is a diagram of bifinite domains and Scott-continuous functions. It therefore suffices to check commutativity for compact elements. A compact element in $\mathsf{spec}(T(A, A'))$ is a filter F generated by a term $a \in \mathsf{C}(T(A, A'))$. Its image under τ_A is the compact element c which generates the compact-open subset $[\![a]\!]^0_{T(A,A')}$. The filter $I(F)$ is generated by the same term a. Applying τ_B to it gives us a compact element d which is least in $[\![a]\!]^0_{T(A,A')}$. Step 12 ensures that $F_T(e, e')$ maps c to d. $\square$

7.3.3 The function space construction

We start out with two preparatory lemmas. The following notation will be helpful. We write $(A \Rightarrow B)$ for the set of functions which map all of A into B.

Lemma 7.3.8. *The Scott-topology on the function space $[D \longrightarrow D']$ for bifinite domains D and D' equals the compact-open topology.*

Proof. Let $A \subseteq D$ be compact and $O \subseteq D'$ be open and let $F \subseteq [D \longrightarrow D']$ be a directed set of continuous functions for which $\bigsqcup^{\uparrow} F$ maps A into O. For every $x \in A$ we have $(\bigsqcup^{\uparrow} F)(x) \in O$ and because O is open, there is $f_x \in F$ with $f_x(x) \in O$. The collection of open sets of the form $f_x^{-1}(O)$, $x \in A$, covers A. By compactness, this is true for finitely many $f_x^{-1}(O)$ already. If we let f be an upper bound in F for these f_x, then $A \subseteq f^{-1}(O)$ holds which is equivalent to $f(A) \subseteq O$. Hence $(A \Rightarrow O)$ is a Scott-open set in $[D \longrightarrow D']$.

If, on the other hand, f belongs to a Scott-open open set $O \subseteq [D \longrightarrow D']$ then this is true also for some approximation $g'_m \circ f \circ g_n$ with g_n an idempotent deflation on D, g'_m an idempotent deflation on D'. For each element x in the image of g_n we have the set $(\uparrow x \Rightarrow (\Uparrow g'_m \circ f \circ g_n(x)))$. The intersection of all these belongs to the compact-open topology, contains f, and is contained in O. $\square$

Lemma 7.3.9. *Let D and D' be bifinite and let $A \subseteq D$ and $A' \subseteq D'$ be compact-open. Then $(A \Rightarrow A')$ is compact-open in $[D \longrightarrow D']$.*

Proof. We know that $(A \Rightarrow A')$ defines an open set by the previous lemma. From bifiniteness we get idempotent deflations g_n on D and g'_m on D' such that $A = \uparrow g_n(A)$ and $A' = \uparrow g'_m(A')$. It follows that $(A \Rightarrow A') = \uparrow G_{nm}(A \Rightarrow A')$ for the idempotent deflation G_{nm} on $[D \longrightarrow D']$ which maps f to $g'_m \circ f \circ g_n$. $\square$

Now let A and A' be domain prelocales describing bifinite domains D and D', as outlined in the general scheme in the previous subsection. The two lemmas justify the following choice of generators and interpretation function for our localic function space construction:

$$\begin{array}{rcl} G_{\rightarrow} & = & \{(a \rightarrow a') \mid a \in A, a' \in A'\}; \\ [\![(a \rightarrow a')]\!] & = & ([\![a]\!]_A \Rightarrow [\![a']\!]_{A'}) \end{array}$$

Note that the elements $(a \rightarrow a')$ are just syntactic expressions. Here are axioms and rules for the preorder and C-predicate.

Axioms.

$(\to-\wedge)$	$(a \to \bigwedge_{i\in I} a_i') \approx \bigwedge_{i\in I}(a \to a_i')$.
$(\to-\vee-l)$	$(\bigvee_{i\in I} a_i \to a') \approx \bigwedge_{i\in I}(a_i \to a')$.
(dist)	$a \wedge (b \vee c) \approx (a \wedge b) \vee (a \wedge c)$.

Rules.

$(\to-\vee-r)$	If $\mathsf{C}(a)$ then $(a \to \bigvee_{i\in I} a_i') \approx \bigvee_{i\in I}(a \to a_i')$.
$(\to-\lesssim)$	If $b \lesssim a$ and $a' \lesssim b'$ then $(a \to a') \lesssim (b \to b')$.
$(\to-\mathsf{C})$	If $\forall i \in I.\ (\mathsf{C}(a_i)$ and $\mathsf{C}(a_i'))$ and if $\forall K \subseteq I\ \ \exists L \subseteq I.$ $(\bigwedge_{k\in K} a_k \approx \bigvee_{l\in L} a_l$ and $(\forall k \in K, l \in L.\ a_k' \lesssim a_l'))$ then $\mathsf{C}(\bigwedge_{i\in I}(a_i \to a_i'))$.

A few comments about these formulae are in place. First a convention: we assume that all index sets are finite, so that the expressions $\bigwedge_{i\in I} a_i$, etc., do indeed belong to the term algebra over $G_\to$. Observe the use of the C-predicate in the rule $(\to-\vee-r)$. Without it, it would be very difficult to express this property. Also note that we enforce distributivity. This will be a prerequisite to prove prime generation below.

It is clear that the rules are sound for the given interpretation, in particular, $(\to-\mathsf{C})$ is the exact mirror image of our definition of joinable families of step functions, Definition 4.2.2. Let us therefore immediately turn to the crucial step 5. We cannot use Lemma 7.3.9 directly because we have not encoded the idempotent deflations. We must find the minimal elements of a compact-open subset explicitly. We illustrate the general technique in an example.

Suppose $[\![a]\!]_A$ is of the form $\uparrow c \cup \uparrow d$ and $[\![a']\!]_{A'}$ is of the form $\uparrow c' \cup \uparrow d'$. We get a minimal element of $((\uparrow c \cup \uparrow d) \Rightarrow (\uparrow c' \cup \uparrow d'))$ by choosing a value $f(c)$ and a value $f(d)$ from $\{c', d'\}$. Then we must look at the intersection $\uparrow c \cap \uparrow d$ which again is of the form $\uparrow e_1 \cup \ldots \cup \uparrow e_n$ by coherence. For each e_i we must choose a value from $\mathsf{mub}\{f(c), f(d)\} = \{e_1', \ldots, e_m'\}$. And so on. Bifiniteness of the argument domain ensures that this process stops after a finite number of iterations and that the result is a joinable family of pairs $\langle x, f(x)\rangle$. Coherence of the result domain guarantees that all in all only finitely many choices are possible. (Note that it can happen that a set of minimal upper bounds in the image domain is empty. In this case we have just been unlucky with our choices. If $[\![a']\!]_{A'}$ is not empty then some minimal function exists.)

We can mimic this procedure in the prelocale as follows. For simplicity and to make the analogy apparent, we let c, d stand for terms such that $\mathsf{C}(c), \mathsf{C}(d)$ and $a \approx c \vee d$. Similarly for a'. We get:

$$\begin{array}{lll}
 & (a \to a') \ \approx & \\
\approx & ((c \vee d) \to (c' \vee d')) & (\to-\lesssim) \\
\approx & (c \to (c' \vee d')) \wedge (d \to (c' \vee d')) & (\to-\vee-l) \\
\approx & ((c \to c') \vee (c \to d')) \wedge ((d \to c') \vee (d \to d')) & (\to-\vee-r) \\
\approx & ((c \to c') \wedge (d \to d')) \vee \ldots \text{(3 more terms)} & \text{(dist)}
\end{array}$$

We follow up only the first of these four terms. The trick is to smuggle in the $\vee$-prime terms $e_1, \dots, e_n$ whose join equals $c \wedge d$.

$$\begin{array}{ll} & (c \to c') \wedge (d \to d') \approx \\ \approx & ((c \vee e_1 \vee \dots \vee e_n) \to c') \wedge ((d \vee e_1 \vee \dots \vee e_n) \to d') \quad (\to - \lesssim) \\ \approx & (c \to c') \wedge (d \to d') \wedge ((e_1 \vee \dots \vee e_n) \to (c' \wedge d')) \qquad (\to - \vee - l) \\ \approx & (c \to c') \wedge (d \to d') \wedge ((e_1 \vee \dots \vee e_n) \to (e'_1 \vee \dots \vee e'_m)) \end{array}$$

and now induction may do its job. Eventually we will have transformed $(a \to a')$ into a disjunction of joinable families. For these, $\vee$-primeness may be inferred through rule $(\to - \mathsf{C})$. Note that distributivity allows us to replace every term by an equivalent term of the form $\bigvee(\bigwedge(a_i \to a'_i))$ and for each term of the form $\bigwedge(a_i \to a'_i)$ the transformation works as illustrated.

Next we show completeness for $\vee$-primes. So assume a and b are terms for which the C-predicate holds and for which $[\![a]\!] \subseteq [\![b]\!]$. It must be the case that a and b are equivalent to joinable families $\bigwedge_{i \in I}(a_i \to a'_i)$ and $\bigwedge_{j \in J}(b_j \to b'_j)$ as there is no other way of deriving $\vee$-primeness in $[A \longrightarrow A']$. The order relation between joinable families has been characterized in Lemma 4.2.3. Here it says: $\forall i \in I\ \exists j \in J.\ ([\![b_j]\!] \subseteq [\![a_i]\!]$ and $[\![a'_i]\!] \subseteq [\![b'_j]\!])$. Since we assume completeness for the constituting prelocales A and A', we may infer $\forall i \in I\ \exists j \in J.\ (b_j \lesssim a_i$ and $a'_i \lesssim b'_j)$. The relation $a \lesssim b$ is now easily derived from $(\to - \lesssim)$.

Definability for $\vee$-primes is immediate because we know that all compact functions arise from joinable families (Lemma 4.2.3 and Proposition 4.2.4).

Properties 8 through 11 follow for all constructions uniformly. We are left with proving Naturality, Property 12. To this end, let us first see how the embedding $[e \longrightarrow e']$ transforms a step function $(a \searrow a')$. We have: $[e \longrightarrow e']((a \searrow a')) = (a \searrow e'(a')) \circ e^*$ and $(a \searrow e'(a')) \circ e^*(x) = e'(a') \iff a \sqsubseteq e^*(x) \iff e(a) \sqsubseteq x$. We get the step function $(e(a) \searrow e'(a'))$.

Now let $a \approx \bigwedge_{i \in I}(a_i \to a'_i)$ be an element of $[A \longrightarrow A']$ for which $\mathsf{C}(a)$ holds. The interpretation $[\![a]\!]^0_{[A \longrightarrow A']}$ of a is the upper set generated by the joinable family of step functions $(c_i \searrow c'_i)$, where $[\![a_i]\!]^0_A = \uparrow c_i$ and $[\![a'_i]\!]^0_{A'} = \uparrow c'_i$ for all $i \in I$. Applying the embedding $[e \longrightarrow e']$ to these gives us the step functions $(e(c_i) \searrow e'(c'_i))$ as we have just seen. By Corollary 7.3.7 we can rewrite these as $(d_i \searrow d'_i)$, where $[\![a_i]\!]^0_B = \uparrow d_i$ and $[\![a'_i]\!]^0_{B'} = \uparrow d'_i$. The supremum of the joinable family $((d_i \searrow d'_i))_{i \in I}$ is least in $[\![a]\!]^0_{[B \longrightarrow B']}$. This was to be proved.

Taking D to be $\mathsf{spec}(A)$ and E to be $\mathsf{spec}(B)$ we can express the faithfulness of our localic construction quite concisely as follows:

Theorem 7.3.10. *Let A and B be domain prelocales. Then*

$$[\mathsf{spec}(A) \longrightarrow \mathsf{spec}(B)] \cong \mathsf{spec}([A \longrightarrow B])$$

and this isomorphism is natural with respect to the sub-prelocale relation.

7.3.4 The Plotkin powerlocale

Next we want to describe the lattice of compact-open subsets of the Plotkin powerdomain of a bifinite domain D. By Theorem 6.2.22 we know that $\mathsf{P}^P(D)$ is concretely represented as the set of lenses in D, ordered by the Egli-Milner ordering (Definition 6.2.2). The compact elements in $\mathsf{P}^P(D)$ are those lenses which are convex closures of finite non-empty subsets of $\mathsf{K}(D)$ (Proposition 6.2.6). Idempotent deflations d on D can be lifted to $\mathsf{P}^P(D)$ because P^P is a functor. They map a lens L to the convex closure of $d(L)$.

The compact-open subsets of $\mathsf{P}^P(D)$, however, are not so readily described. The problem is that one half of the Egli-Milner ordering refers to closed lower sets rather than upper sets. We do not follow this up as there is no logical pathway from the order theory to the axiomatization we are aiming for. It is much more efficient to either consult the mathematical literature on hyperspaces (see [Vie21, Vie22, Smy83b]) or to remind ourselves that powerdomains were introduced to model non-deterministic behaviour. If we think of the compact-open subsets in D as observations that can be made about outcomes of a computation, then it is pretty clear that there are two ways of using these to make statements about non-deterministic programs: It could be the case that all runs of the program satisfy the property or it could be that at least one run satisfies it. Let us check the mathematics:

Lemma 7.3.11. *If D is a bifinite domain and O is compact-open in D, then the following are compact-open subsets in $\mathsf{P}^P(D)$:*

$$\begin{array}{rcl} \mathsf{A}(O) & = & \{L \in \mathsf{Lens}(D) \mid L \subseteq O\}, \\ \mathsf{E}(O) & = & \{L \in \mathsf{Lens}(D) \mid L \cap O \neq \emptyset\}, \end{array}$$

Furthermore, if we let O range over all compact-open subsets in D then the collection of all $\mathsf{A}(O)$ and $\mathsf{E}(O)$ forms a base for the Scott-topology on $\mathsf{P}^P(D)$.

Proof. Let O be compact-open. Then O is the upper set of finitely many compact elements and we find an idempotent deflation d such that $O = {\uparrow} d(O)$. It is clear that for $\hat{d} = \mathsf{P}^P(d)$ we have both $\mathsf{A}(O) = {\uparrow}\hat{d}(\mathsf{A}(O))$ and $\mathsf{E}(O) = {\uparrow}\hat{d}(\mathsf{E}(O))$. Hence these sets are compact-open, too.

Let K be a compact lens, that is, of the form $\mathsf{Cx}(u)$ for $u \subseteq_{\text{fin}} \mathsf{K}(D)$. The upper set of K in $\mathsf{P}^P(D)$ can be written as $\mathsf{A}({\uparrow}u) \cap \bigcap_{c \in u} \mathsf{E}({\uparrow}c)$. ☐

The following definition then comes as no surprise:

Definition 7.3.12. *Let A be a domain prelocale which is a localic description of the bifinite domain D. We define the* Plotkin powerlocale $\mathsf{P}^P(A)$ *over A as the term algebra over the generators*

$$G_P = \{\Box a \mid a \in A\} \cup \{\Diamond a \mid a \in A\}$$

with the interpretation function $[\![\cdot]\!] \colon \mathsf{P}^P(A) \to \mathsf{K}\Omega(\mathsf{P}^P(D))$ defined by

$$[\![\Box a]\!] = \mathsf{A}([\![a]\!]), \quad [\![\Diamond a]\!] = \mathsf{E}([\![a]\!])$$

on the generators and extended to $\mathsf{P}^P(A)$ as a lattice homomorphism.

Preorder and C-predicate are defined as follows

Axioms.

$(\Box - \wedge)$	$\Box(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} \Box a_i$,
$(\Box - 0)$	$\Box 0 = 0$,
$(\Diamond - \vee)$	$\Diamond(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \Diamond a_i$,
$(\Diamond - 1)$	$\Diamond 1 = 1$,
$(\Box - \vee)$	$\Box(a \vee b) \lesssim \Box a \vee \Diamond b$,
$(\Diamond - \wedge)$	$\Box a \wedge \Diamond b \lesssim \Diamond(a \wedge b)$,
(dist)	$a \wedge (b \vee c) \approx (a \wedge b) \vee (a \wedge c)$.

Rules.

$(P - \lesssim)$	*If* $a \lesssim b$ *then* $\Box a \lesssim \Box b$ *and* $\Diamond a \lesssim \Diamond b$,
$(P - \mathsf{C})$	*If* $\mathsf{C}(a_i)$ *holds for all* $i \in I$ *and* I *is non-empty, then* $\mathsf{C}(\Box(\bigvee_{i\in I} a_i) \wedge \bigwedge_{i\in I} \Diamond a_i)$.

Note that we again require distributivity explicitly. The derivation scheme is almost minimal (in combination with the rest, $(\Box - 0)$ and $(\Diamond - 1)$ are equivalent). The following derived axioms are more useful than $(\Box - \vee)$ and $(\Diamond - \wedge)$:

(D1) $\Box(a \vee b) \approx \Box a \vee (\Box(a \vee b) \wedge \Diamond b)$,
(D2) $\Box a \wedge \Diamond b \approx \Box a \wedge \Diamond(a \wedge b)$.

We leave it to the interested reader to check soundness and pass straight on to the central Step 5, which is generation by $\vee$-prime elements.

Proof. Given an expression in $\mathsf{P}^P(A)$ we first transform it into a disjunction of conjunctions by using the distributivity axiom. Thus it suffices to represent a term of the form

$$\bigwedge_{i\in I} \Box a_i \wedge \bigwedge_{j\in J} \Diamond b_j$$

as a disjunction of $\vee$-primes. But we can simplify further. Using $(\Box - \wedge)$ we can pack all $\Box$-generators into a single term $\Box a$ and by (D2) we can assume that for each $j \in J$ we have $b_j \lesssim a$. We represent each b_j as a disjunction of $\vee$-primes of A and applying $(\Diamond - \vee)$ and distributivity again we arrive at a disjunction of terms of the form

$$\Box a \wedge \bigwedge_{j=1}^{m} \Diamond d_j$$

where each $d_j \in \mathsf{C}(A)$. Now we write a as a disjunction of $\vee$-primes c_i. Since each d_j is below a, it doesn't hurt to add these, too. We get:

$$\Box(c_1 \vee \ldots \vee c_n \vee d_1 \vee \ldots \vee d_m) \wedge \bigwedge_{j=1}^{m} \Diamond d_j.$$

As yet we can not apply the $\vee$-primeness rule $(P - \mathsf{C})$ because the two sets $\{c_1, \ldots, c_n, d_1, \ldots, d_m\}$ and $\{d_1, \ldots, d_m\}$ may fail to coincide. Looking at the semantics for a moment, we see that in the compact-open subset thus described the minimal lenses are (the convex closures of) the least elements from each $[\![d_j]\!]^0_A$ plus some of the generators of the $[\![c_i]\!]^0_A$. We therefore take our term further apart so as to have a $\vee$-prime expression for each subset of $\{c_1, \ldots, c_n\}$. For this we use (D1). One application (plus some distributivity) yields

$$\big(\Box(c_2 \vee \ldots \vee c_n \vee d_1 \vee \ldots \vee d_m) \wedge \bigwedge_{j=1}^{m} \Diamond d_j\big) \quad \vee$$

$$\big(\Box(c_1 \vee \ldots \vee c_n \vee d_1 \vee \ldots \vee d_m) \wedge \Diamond c_1 \wedge \bigwedge_{j=1}^{m} \Diamond d_j\big)$$

and the picture becomes obvious. $\square$

Next we check that $[\![\cdot]\!]^0$ is order-reflecting.

Proof. Assume $[\![\Box(\bigvee_{i\in I} a_i) \wedge \bigwedge_{i\in I} \Diamond a_i]\!]^0 \subseteq [\![\Box(\bigvee_{i\in I} b_j) \wedge \bigwedge_{j\in J} \Diamond b_j]\!]^0$ and let c_i and d_j be the least compact elements in $[\![a_i]\!]^0_A$, respectively $[\![b_j]\!]^0_A$. Then we have $\{d_j \mid j \in J\} \sqsubseteq_{EM} \{c_i \mid i \in I\}$, that is,

$$\begin{array}{ll} \forall i \in I\ \exists j \in J. & \uparrow c_i \subseteq \uparrow d_j, \\ \forall j \in J\ \exists i \in I. & \uparrow c_i \subseteq \uparrow d_j. \end{array}$$

Since we assume that $[\![\cdot]\!]^0_A$ is order-reflecting, we get from the first equation $\bigvee_{i\in I} a_i \lesssim \bigvee_{j\in J} b_j$ and from the second $\bigwedge_{i\in I} \Diamond a_i \lesssim \bigwedge_{j\in J} \Diamond b_j$. $\square$

The definability for $\vee$-primes was shown in Lemma 7.3.11 already. Hence we are left with checking Naturality, which is Step 12.

Proof. Let $t = \Box(\bigvee_{i\in I} a_i) \wedge \bigwedge_{i\in I} \Diamond a_i$ be a $\vee$-prime element in $\mathsf{P}^P(A)$ and let A be a sub-prelocale of B. Let e be the associated embedding from D to E. The least element in $[\![t]\!]^0_{\mathsf{P}^P(A)}$ is the convex closure of the set of minimal elements c_i in $[\![a_i]\!]^0_A$. Applying $\mathsf{P}^P(e)$ to it gives the convex closure of $\{e(c_i) \mid i \in I\}$, as we have argued in the remark following Theorem 6.1.9. Corollary 7.3.7 tells us that this is the least element in $[\![t]\!]^0_{\mathsf{P}^P(B)}$. $\square$

As in the case of the function space construction we summarize:

Theorem 7.3.13. *Let A be a domain prelocale. Then*

$$\mathsf{P}^P(\mathsf{spec}(A)) \cong \mathsf{spec}(\mathsf{P}^P(A))$$

and this isomorphism is natural with respect to the sub-prelocale relation.

The prelocales for Hoare and Smyth powerdomain are much easier to describe. All we have to do is to elide all generators and rules which refer to $\Box$, respectively $\Diamond$.

7.3.5 Recursive domain equations

In this subsection we will treat bilimits in the same fashion as we have studied finitary constructions. We assume that we are given domain prelocales $A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \ldots$ such that each A_n describes some bifinite domain D_n. Corollary 7.3.7 states how the sub-prelocale relation between A_n and A_m, for $n \leq m$, translates into an embedding $e_{mn} \colon D_n \to D_m$. It is seen easily that $\langle (D_n)_{n \in \mathbb{N}}, (e_{mn})_{n \leq m} \rangle$ is an expanding system, that is, for $n \leq m \leq k$, $e_{kn} = e_{km} \circ e_{mn}$ holds. We claim that the directed union $A = \bigcup_{n \in \mathbb{N}} A_n$ is a domain prelocale which describes $D = \mathsf{bilim} D_n$. The first claim is fairly obvious as all requirements about prelocales refer to finitely many elements only and hence a property of A can be inferred from its validity in some A_n. For the second claim we need to specify the interpretation function. To this end let l_m be the embedding of D_m into the bilimit (as defined in Theorem 3.3.7). Then we can set $[\![a]\!] = l_m([\![a]\!]_{A_m})$ where $m \in \mathbb{N}$ is such that a is contained in A_m. The exact choice of m does not matter; if $m \leq k$ then by Corollary 7.3.7 we have: $[\![a]\!]_{A_k} = e_{km}([\![a]\!]_{A_m})$ and applying l_k to this yields $l_k([\![a]\!]_{A_k}) = l_k \circ e_{km}([\![a]\!]_{A_m}) = l_m([\![a]\!]_{A_m})$. The interpretation function is well-defined because embeddings preserve the order of approximation (Proposition 3.1.14), hence compact elements and compact-open subsets are also preserved.

In order to see that $[\![\cdot]\!]$ is a pre-isomorphism we proceed as before, checking Steps 4, 5, 6, 7, and 12. It is, actually, rather simple. Soundness holds because the l_m are monotone and map compact elements to compact elements. Prime generation holds because it holds in each A_m. Since the l_m are also order-reflecting we get completeness from the completeness of the $[\![\cdot]\!]_{A_m}$. Definability follows from Theorem 3.3.11; the only compact elements in D are the images (under l_n) of compact elements in the approximating D_n. If we are given a second sequence $B_0 \trianglelefteq B_1 \trianglelefteq B_2 \trianglelefteq \ldots$ of prelocales (describing $E_0, E_1, \ldots$) such that for each $n \in \mathbb{N}$ we have $A_n \trianglelefteq B_n$ then it is clear that $A \trianglelefteq B = \bigcup_{n \in \mathbb{N}} B_n$ holds, too. For Naturality (Step 12) we must relate this to the embedding e from D to $E = \mathsf{bilim} E_n$. The exact form of the latter can be extracted from Theorem 3.3.7: $e = \bigsqcup_{n \in \mathbb{N}} k_n \circ e_n \circ l_n^*$, where k_n is the embedding of E_n into E and $e_n \colon D_n \to E_n$ is the embedding derived from $A_n \trianglelefteq B_n$. Now let a be $\vee$-prime in A. We have

$$\begin{aligned} e([\![a]\!]^0_A) &= (\bigsqcup_{n \in \mathbb{N}} k_n \circ e_n \circ l_n^*)(l_m([\![a]\!]^0_{A_m})) \\ &= \bigsqcup_{n \geq m} k_n \circ e_n([\![a]\!]^0_{A_m}) \\ &= \bigsqcup_{n \geq m} k_n([\![a]\!]^0_{B_m}) \\ &= [\![a]\!]^0_B \ , \end{aligned}$$

and our proof is complete.

Theorem 7.3.14. *If $A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \ldots$ is a chain of domain prelocales, then*

$$\mathsf{spec}(\bigcup_{n \in \mathbb{N}} A_n) \cong \mathsf{bilim}(\mathsf{spec}(A_n))_{n \in \mathbb{N}} \ .$$

Observe how simple the limit operation for prelocales is if compared with a bilimit. This comes to full flower if we look at recursive domain equations. If T is a construction built from those which can be treated localically (we have seen function space, Plotkin powerdomain, and bilimit, but all the others from Section 3.2 can also be included) then we can find the initial fixpoint of the functor F_T on the localic side by simply taking the union of $\mathbf{1} \trianglelefteq T(\mathbf{1}) \trianglelefteq T(T(\mathbf{1})) \trianglelefteq \ldots$. Why does this work and why does the result describe the canonical fixpoint of F_T? First of all, we have $\mathbf{1} \trianglelefteq T(\mathbf{1})$ by Step 14. Successively applying T to this relation gives us $T^n(\mathbf{1}) \trianglelefteq T^{n+1}(\mathbf{1})$ by Monotonicity (Step 13). Hence we do have a chain $\mathbf{1} \trianglelefteq T(\mathbf{1}) \trianglelefteq T(T(\mathbf{1})) \trianglelefteq \ldots$ as stated and we can form its union A. It obviously is a fixpoint of the construction T and therefore the domain D described by it is a fixpoint of the functor F_T. But notice that we have $T(A) = A$ rather than merely $T(A) \cong A$. This is not so surprising as it may seem at first sight. Domain prelocales are only representations of domains and what we are exploiting here is the simple idea that we can let A represent both D and $F_T(D)$ via two *different* interpretation functions. Let us now address the question about canonicity. It suffices to check that the embedding corresponding to $T(\mathbf{1}) \trianglelefteq T^2(\mathbf{1})$ is equal to $F_T(e)$ where $e\colon \mathbb{I} \to F_T(\mathbb{I})$ corresponds to $\mathbf{1} \trianglelefteq T(\mathbf{1})$. This is precisely the naturality of τ which we listed as Step 15. It follows that the bilimit is the same as the one constructed in Chapter 5.

7.3.6 Languages for types, properties, and points

We define a formal language of *type* expressions by the following grammar:

$$\sigma \quad ::= \quad \mathbf{1} \mid X \mid (\sigma{\to}\sigma) \mid (\sigma{\times}\sigma) \mid (\sigma{\oplus}\sigma) \mid (\sigma)_\perp \mid \mathsf{P}^P(\sigma) \mid \mathsf{rec}X.\sigma$$

where X ranges over a set TV of type variables. More constructions can be added to this list, of course, such as strict function space, smash product, Hoare powerdomain, and Smyth powerdomain. On the other hand, we do not include expressions for basic types, such as integers and booleans, as these can be encoded in our language by simple formulae.

We have seen two ways to interpret type expressions. The first interpretation takes values directly in $\mathbf{B}$, the category of bifinite domains, and is based on the constructions in Sections 3.2, 3.3, 5.1, and 6.2. Since a type expression may contain free variables, the interpretation can be defined only relative to an *environment* $\rho_D\colon TV \to \mathbf{B}$, which assigns to each type variable a bifinite domain. The semantic clauses corresponding to the individual rules of the grammar are as follows:

$$\begin{array}{rcl}
\mathfrak{I}_D(\mathbf{1};\rho_D) & = & \mathbb{I}; \\
\mathfrak{I}_D(X;\rho_D) & = & \rho_D(X); \\
\mathfrak{I}_D((\sigma\to\tau);\rho_D) & = & [\mathfrak{I}_D(\sigma;\rho_D) \longrightarrow \mathfrak{I}_D(\tau;\rho_D)]; \\
\text{etc.} & & \\
\mathfrak{I}_D(\mathsf{rec}X.\sigma;\rho_D) & = & \mathsf{FIX}(F_T), \\
 & & \text{where } F_T(D) = \mathfrak{I}_D(\sigma;\rho_D[X \mapsto D]).
\end{array}$$

The expression $\rho_D[X \mapsto D]$ denotes the environment which maps X to D and coincides with ρ_D at all other variables .

Our work in the preceding subsections suggests that we can also interpret type expressions in the category **DomPreloc** of domain prelocales. Call the corresponding mappings $\mathfrak{I}_L$ and ρ_L. The semantic clauses for this localic interpretation are:

$$\begin{array}{rcl} \mathfrak{I}_L(\mathbf{1};\rho_L) & = & \mathbf{1}; \\ \mathfrak{I}_L(X;\rho_L) & = & \rho_L(X); \\ \mathfrak{I}_L((\sigma\to\tau);\rho_L) & = & [\mathfrak{I}_L(\sigma;\rho_L)\to\mathfrak{I}_L(\tau;\rho_L)]; \\ \text{etc.} & & \\ \mathfrak{I}_L(\mathsf{rec}X.\sigma;\rho_L) & = & \bigcup T^n(\mathbf{1}), \\ & & \text{where } T(A)=\mathfrak{I}_L(\sigma;\rho_L[X\mapsto A]). \end{array}$$

The preceding subsections were meant to convince the reader of the following:

Theorem 7.3.15. *If ρ_L and ρ_D are environments such that for each $X \in TV$ the domain prelocale $\rho_L(X)$ is a localic description of $\rho_D(X)$, then for every type expression σ it holds that $\mathfrak{I}_L(\sigma;\rho_L)$ is a localic description of $\mathfrak{I}_D(\sigma;\rho_D)$. As a formula:*

$$\mathsf{spec}(\mathfrak{I}_L(\sigma;\rho_L)) \cong \mathfrak{I}_D(\sigma;\rho_D)\ .$$

The next step is to define for each type expression σ a formal language $\mathfrak{L}(\sigma)$ of (computational or observational) *properties*. This is done through the following inductive definition:

$$\begin{array}{rcl} & \Longrightarrow & \mathsf{true},\mathsf{false}\in\mathfrak{L}(\sigma); \\ \phi,\psi\in\mathfrak{L}(\sigma) & \Longrightarrow & \phi\wedge\psi,\phi\vee\psi\in\mathfrak{L}(\sigma); \\ \phi\in\mathfrak{L}(\sigma),\psi\in\mathfrak{L}(\tau) & \Longrightarrow & (\phi{\to}\psi)\in\mathfrak{L}(\sigma{\to}\tau), \\ \phi\in\mathfrak{L}(\sigma),\psi\in\mathfrak{L}(\tau) & \Longrightarrow & (\phi{\times}\psi)\in\mathfrak{L}(\sigma{\times}\tau); \\ \phi\in\mathfrak{L}(\sigma) & \Longrightarrow & (\phi{\oplus}\mathsf{false})\in\mathfrak{L}(\sigma{\oplus}\tau); \\ \psi\in\mathfrak{L}(\tau) & \Longrightarrow & (\mathsf{false}{\oplus}\psi)\in\mathfrak{L}(\sigma{\oplus}\tau); \\ \phi\in\mathfrak{L}(\sigma) & \Longrightarrow & (\phi)_\perp\in\mathfrak{L}((\sigma)_\perp); \\ \phi\in\mathfrak{L}(\sigma) & \Longrightarrow & \Box\phi,\Diamond\phi\in\mathfrak{L}(\mathsf{P}^P(\sigma)); \\ \phi\in\mathfrak{L}(\sigma[\mathsf{rec}X.\sigma/X]) & \Longrightarrow & \phi\in\mathfrak{L}(\sigma). \end{array}$$

Here we have used the expression $\sigma[\tau/X]$ to denote the substitution of τ for X in σ. The usual *caveat* about capture of free variables applies but let us not dwell on this. The rules exhibited above will generate for each σ the carrier set of a (syntactical) domain prelocale in the style of the previous subsections. Note that we don't need special properties for a recursively defined type as these are just the properties of the approximating domains bundled together (Theorem 7.3.14).

On each $\mathfrak{L}(\sigma)$ we define a preorder $\lesssim$ and predicates C and T (the latter is needed for the coalesced sum construction) through yet another inductive definition. For example, the following axioms and rules enforce that each $\mathfrak{L}(\sigma)$ is a preordered distributive lattice.

$$
\begin{array}{rcl}
 & \Longrightarrow & \phi \lesssim \phi; \\
\phi \lesssim \psi, \psi \lesssim \chi & \Longrightarrow & \phi \lesssim \chi; \\
 & \Longrightarrow & \phi \lesssim \mathsf{true}; \\
\phi \lesssim \psi_1, \phi \lesssim \psi_2 & \Longrightarrow & \phi \lesssim \psi_1 \wedge \psi_2; \\
 & \Longrightarrow & \phi \wedge \psi \lesssim \phi; \\
 & \Longrightarrow & \phi \wedge \psi \lesssim \psi; \\
 & \Longrightarrow & \mathsf{false} \lesssim \phi; \\
\phi_1 \lesssim \psi, \phi_2 \lesssim \psi & \Longrightarrow & \phi_1 \vee \phi_2 \lesssim \psi; \\
 & \Longrightarrow & \phi \lesssim \phi \vee \psi; \\
 & \Longrightarrow & \psi \lesssim \phi \vee \psi; \\
 & \Longrightarrow & \phi \wedge (\psi \vee \chi) \lesssim (\phi \wedge \psi) \vee (\phi \wedge \chi);
\end{array}
$$

We have seen some type specific axioms and rules in the definition of the function space prelocale and the Plotkin powerlocale. For the full list we refer to [Abr91b], p. 49ff. If σ is a closed type expression then the domain prelocale $\mathfrak{L}(\sigma)$ describes the intended bifinite domain:

Theorem 7.3.16. *If σ is a closed type expression then*

$$\mathsf{spec}(\mathfrak{L}(\sigma)) \cong \mathfrak{I}_D(\sigma) \ .$$

(Note that this is a special case of Theorem 7.3.15.)

The whole scheme for deriving $\lesssim$, C, and T is designed carefully so as to have finite positive information in the premise of each rule only. Hence the whole system can be seen as a monotone inductive definition (in the technical sense of e.g. [Acz77]). Furthermore, we have already established close connections between the syntactical rules and properties of the described domains. This is the basis of the following result.

Theorem 7.3.17. *The language of properties is decidable.*

Proof. The statement is trivial for the domain prelocale $\mathbf{1}$ because only combinations of true and false occur in $\mathfrak{L}(\mathbf{1})$. For composite types we rely on the general development in Section 7.3.2, which at least for three concrete instances we have verified in Sections 7.3.3–5. First of all, every expression in $\mathfrak{L}(\sigma)$ can be effectively transformed into a finite disjunction of $\vee$-primes (i.e. expressions satisfying the C-predicate); this is Step 5, 'prime generation'. Soundness and completeness ensure that the expressions satisfying the C-predicate are precisely the $\vee$-primes in the preordered lattice $\mathfrak{L}(\sigma)$. Hence we can decide the preorder between arbitrary expressions if we can decide the preorder between $\vee$-primes. For the latter we note that our constructions accomplish more than we have stated so far. All $\vee$-primes, which are produced by the transformation algorithms, are of the explicit form occuring in the rules for deriving the C-predicate; rather than merely expressions which happen to be equivalent to $\vee$-primes. The preorder between these explicit $\vee$-primes is (for each construction) easily characterized through the semantic interpretation function $[\![\cdot]\!]^0$. The task of establishing the preorder between these primes is then reduced to establishing some formula defined by structural induction on the type σ. Since every expression in $\mathfrak{L}(\sigma)$ is derived from true and false in finitely many steps, we will eventually have reduced our task to checking the preorder between certain expressions in $\mathfrak{L}(\mathbf{1})$. $\square$

Finally, we introduce a formal language to speak about points of domains. So far, we have done this in a rather roundabout way, trusting in the reader's experience with sets and functions. Doing it formally will allow us to establish a precise relationship between (expressions for) points and (expressions for) properties.

We assume that for each (closed) type expression σ we have a denumerable set $V(\sigma) = \{x^\sigma, y^\sigma, z^\sigma, \ldots\}$ of typed variables. The terms are defined as follows (where $M : \sigma$ stands for 'M is a term of type σ'):

$$
\begin{array}{rcl}
 & \Longrightarrow & *_\sigma : \sigma; \\
 & \Longrightarrow & x^\sigma : \sigma; \\
M : \tau & \Longrightarrow & \lambda x^\sigma.M : (\sigma{\to}\tau); \\
M : (\sigma{\to}\tau), N : \sigma & \Longrightarrow & (MN) : \tau; \\
M : \sigma, N : \tau & \Longrightarrow & \langle M, N\rangle : (\sigma{\times}\tau); \\
M : (\sigma{\times}\tau), N : \nu & \Longrightarrow & \mathsf{let}\ M\ \mathsf{be}\ \langle x^\sigma, y^\tau\rangle.N : \nu; \\
M : \sigma & \Longrightarrow & \mathsf{inl}(M) : (\sigma{\oplus}\tau)\ \text{and}\ \mathsf{inr}(M) : (\tau{\oplus}\sigma); \\
M : (\sigma{\oplus}\tau), N_1 : \nu, N_2 : \nu & \Longrightarrow & \mathsf{cases}\ M\ \mathsf{of}\ \mathsf{inl}(x^\sigma).N_1\ \mathsf{else}\ \mathsf{inr}(y^\tau).N_2 : \nu; \\
M : \sigma & \Longrightarrow & \mathsf{up}(M) : (\sigma)_\bot; \\
M : (\sigma)_\bot, N : \tau & \Longrightarrow & \mathsf{lift}\ M\ \mathsf{to}\ \mathsf{up}(x^\sigma).N : \tau; \\
M : \sigma & \Longrightarrow & \{\!|M|\!\} : \mathsf{P}^P(\sigma); \\
M : \mathsf{P}^P(\sigma), N : \mathsf{P}^P(\tau) & \Longrightarrow & \mathsf{over}\ M\ \mathsf{extend}\ \{\!|x^\sigma|\!\}.N : \mathsf{P}^P(\tau); \\
M : \mathsf{P}^P(\sigma), N : \mathsf{P}^P(\sigma) & \Longrightarrow & M \uplus N : \mathsf{P}^P(\sigma); \\
M : \mathsf{P}^P(\sigma), N : \mathsf{P}^P(\tau) & \Longrightarrow & M \otimes N : \mathsf{P}^P(\sigma \times \tau); \\
M : \sigma[\mathsf{rec}X.\sigma/X] & \Longrightarrow & \mathsf{fold}(M) : \mathsf{rec}X.\sigma; \\
M : \mathsf{rec}X.\sigma & \Longrightarrow & \mathsf{unfold}(M) : \sigma[\mathsf{rec}X.\sigma/X]; \\
M : \sigma & \Longrightarrow & \mu x^\sigma.M : \sigma.
\end{array}
$$

In the same fashion as for type expressions we have two alternatives for interpreting a term M of type σ. We can either give a direct denotational semantics in the bifinite domain $\mathfrak{I}_D(\sigma)$ or we can specify a prime filter in the corresponding domain prelocale $\mathfrak{L}(\sigma)$. The denotational semantics suffers from the fact that in order to single out a particular element in a domain we use a mathematical language which looks embarrassingly similar to the formal language we intend to interpret. Some of the semantic clauses to follow will therefore appear to be circular.

Again we need environments to deal with free variables. They are maps $\rho\colon \bigcup_\sigma V(\sigma) \to \dot{\bigcup}_\sigma \mathfrak{I}_D(\sigma)$ which we assume to respect the typing. In the following clauses we will also suppress the type information.

$$
\begin{array}{rcl}
[\![*_\sigma]\!]\rho & = & \bot,\ \text{the least element in } \mathfrak{I}_D(\sigma);\\
[\![x]\!]\rho & = & \rho(x);\\
[\![\lambda x.M]\!]\rho & = & (d \mapsto [\![M]\!]\rho[x{\mapsto}d]);\\
[\![(MN)]\!]\rho & = & [\![M]\!]\rho([\![N]\!]\rho);\\
[\![\langle M, N\rangle]\!]\rho & = & \langle [\![M]\!]\rho, [\![N]\!]\rho\rangle;\\
[\![\mathsf{let}\ M\ \mathsf{be}\ \langle x, y\rangle.N]\!]\rho & = & [\![N]\!]\rho[x{\mapsto}d, y{\mapsto}e],\\
\text{where}\quad d & = & \pi_1([\![M]\!]\rho),\\
e & = & \pi_2([\![M]\!]\rho);\\
[\![\mathsf{inl}(M)]\!]\rho & = & \mathsf{inl}([\![M]\!]\rho);\\
[\![\mathsf{inr}(M)]\!]\rho & = & \mathsf{inr}([\![M]\!]\rho);\\
[\![\mathsf{cases}\ M\ \mathsf{of}\ \mathsf{inl}(x).N_1\ \mathsf{else}\ \mathsf{inr}(y).N_2]\!]\rho & = & \begin{cases} [\![N_1]\!]\rho[x{\mapsto}d], & [\![M]\!]\rho = (d\colon 1);\\ [\![N_2]\!]\rho[y{\mapsto}e], & [\![M]\!]\rho = (e\colon 2);\\ \bot, & [\![M]\!]\rho = \bot; \end{cases}\\
[\![\mathsf{up}(M)]\!]\rho & = & \mathsf{up}([\![M]\!]\rho);\\
[\![\mathsf{lift}\ M\ \mathsf{to}\ \mathsf{up}(x^\sigma).N]\!]\rho & = & \begin{cases} [\![N]\!]\rho[x{\mapsto}d], & [\![M]\!]\rho = \mathsf{up}(d);\\ \bot, & [\![M]\!]\rho = \bot; \end{cases}\\
[\![\{\!|M|\!\}]\!]\rho & = & \{[\![M]\!]\rho\};\\
[\![\mathsf{over}\ M\ \mathsf{extend}\ \{\!|x^\sigma|\!\}.N]\!]\rho & = & \uparrow X \cap \mathsf{Cl}(X),\\
\text{where } X & = & \bigcup\{[\![N]\!]\rho[x{\mapsto}d] \mid d \in [\![M]\!]\rho\};\\
[\![M \uplus N]\!]\rho & = & [\![M]\!]\rho \uplus [\![N]\!]\rho;\\
[\![M \otimes N]\!]\rho & = & \{\langle d, e\rangle \mid d \in [\![M]\!]\rho, e \in [\![N]\!]\rho\};\\
[\![\mathsf{fold}(M)]\!]\rho & = & \mathsf{fold}([\![M]\!]\rho);\\
[\![\mathsf{unfold}(M)]\!]\rho & = & \mathsf{unfold}([\![M]\!]\rho);\\
[\![\mu x.M]\!]\rho & = & \mathsf{fix}(f),\\
\text{where } f(d) & = & [\![M]\!]\rho[x{\mapsto}d].
\end{array}
$$

Now let us give the localic, or, as we are now justified in saying, *logical* interpretation. We use a sequent calculus style of presenting this *domain logic*. The problem of free variables is dealt with this time by including a finite list Γ of assumptions on variables. We write them in the form $x{\mapsto}\phi$ and assume that Γ contains at most one of these for each variable x. A sequent then takes the form $\Gamma \vdash M : \phi$ and should be read as 'M satisfies ϕ under the assumptions in Γ'.

$$
\begin{array}{rcl}
\{\Gamma \vdash M : \phi_i\}_{i \in I} & \Longrightarrow & \Gamma \vdash M : \bigwedge_{i \in I} \phi; \\
\phi' \lesssim \phi, \psi \lesssim \psi', & & \\
(\Gamma, x \mapsto \phi \vdash M : \psi) & \Longrightarrow & \Gamma, x \mapsto \phi' \vdash M : \psi'; \\
\{\Gamma, x \mapsto \phi_i \vdash M : \psi\}_{i \in I} & \Longrightarrow & \Gamma, x \mapsto \bigvee_{i \in I} \phi_i \vdash M : \psi; \\
\Gamma \vdash M : \psi & \Longrightarrow & \Gamma, x \mapsto \phi \vdash M : \psi; \\
 & \Longrightarrow & x \mapsto \phi \vdash x : \phi; \\
\Gamma, x \mapsto \phi \vdash M : \psi & \Longrightarrow & \Gamma \vdash \lambda x.M : (\phi \rightarrow \psi); \\
\Gamma \vdash M : (\phi \rightarrow \psi); \Gamma \vdash N : \phi & \Longrightarrow & \Gamma \vdash (MN) : \psi; \\
\Gamma \vdash M : \phi; \Gamma \vdash N : \psi & \Longrightarrow & \Gamma \vdash \langle M, N \rangle : (\phi \times \psi); \\
\Gamma \vdash M : (\phi \times \psi), & & \\
\Gamma, x \mapsto \phi, y \mapsto \psi \vdash N : \chi & \Longrightarrow & \Gamma \vdash \textsf{let } M \textsf{ be } \langle x, y \rangle.N : \chi; \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \textsf{inl}(M) : (\phi \oplus \textsf{false}); \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \textsf{inr}(M) : (\textsf{false} \oplus \phi); \\
\Gamma \vdash M : (\phi \oplus \textsf{false}), \textsf{T}(\phi), & & \\
\Gamma, x \mapsto \phi \vdash N_1 : \psi & \Longrightarrow & \Gamma \vdash \textsf{cases } M \textsf{ of } \textsf{inl}(x).N_1 \\
 & & \qquad \textsf{else } \textsf{inr}(y).N_2 : \psi; \\
\Gamma \vdash M : (\textsf{false} \oplus \phi), \textsf{T}(\phi), & & \\
\Gamma, y \mapsto \phi \vdash N_2 : \psi & \Longrightarrow & \Gamma \vdash \textsf{cases } M \textsf{ of } \textsf{inl}(x).N_1 \\
 & & \qquad \textsf{else } \textsf{inr}(y).N_2 : \psi; \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \textsf{up}(M) : (\phi)_\perp; \\
\Gamma \vdash M : (\phi)_\perp; \Gamma, x \mapsto \phi \vdash N : \psi & \Longrightarrow & \Gamma \vdash \textsf{lift } M \textsf{ to } \textsf{up}(x^\sigma).N : \psi; \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \{\!|M|\!\} : \Box\phi; \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \{\!|M|\!\} : \Diamond\phi; \\
\Gamma \vdash M : \Box\phi; \Gamma, x \mapsto \phi \vdash N : \Box\psi & \Longrightarrow & \Gamma \vdash \textsf{over } M \textsf{ extend } \{\!|x^\sigma|\!\}.N : \Box\psi; \\
\Gamma \vdash M : \Diamond\phi; \Gamma, x \mapsto \phi \vdash N : \Diamond\psi & \Longrightarrow & \Gamma \vdash \textsf{over } M \textsf{ extend } \{\!|x^\sigma|\!\}.N : \Diamond\psi; \\
\Gamma \vdash M : \Box\phi; \Gamma \vdash N : \Box\phi & \Longrightarrow & \Gamma \vdash M \uplus N : \Box\phi; \\
\Gamma \vdash M : \Diamond\phi & \Longrightarrow & \Gamma \vdash M \uplus N : \Diamond\phi; \\
\Gamma \vdash N : \Diamond\phi & \Longrightarrow & \Gamma \vdash M \uplus N : \Diamond\phi; \\
\Gamma \vdash M : \Diamond\phi; \Gamma \vdash N : \Diamond\psi & \Longrightarrow & \Gamma \vdash M \otimes N : \Diamond(\phi \times \psi); \\
\Gamma \vdash M : \Box\phi; \Gamma \vdash N : \Box\psi & \Longrightarrow & \Gamma \vdash M \otimes N : \Box(\phi \times \psi); \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \textsf{fold}(M) : \phi; \\
\Gamma \vdash M : \phi & \Longrightarrow & \Gamma \vdash \textsf{unfold}(M) : \phi; \\
\Gamma \vdash \mu x.M : \phi; \Gamma, x \mapsto \phi \vdash M : \psi & \Longrightarrow & \Gamma \vdash \mu x.M : \psi.
\end{array}
$$

A few comments may help in reading these clauses. The first two rules guarantee that the set of properties which can be deduced for a term M forms a filter in the domain prelocale. The third rule expresses the fact that every particular x will satisfy properties from a prime filter. In particular, it entails that $\Gamma, x{\mapsto}\mathsf{false} \vdash M : \phi$ is always true. The fourth rule (which is the last of the structural rules) is ordinary weakening. We need it to get started in a derivation. In the two rules for the cases-construct the predicate T shows up. Instead of $\mathsf{T}(\phi)$ we could have written $\phi \not\approx \mathsf{false}$ but as we said before, we want to keep the whole logic positive, that is to say, we want to use inductive definitions only. The two rules for fold and unfold may seem a bit boring, but it is precisely at this point where we take advantage of the fact that in the world of domain prelocales we solve domain equation up to equality. The last rule, finally, has to be applied finitely many times, starting from $\Gamma \vdash \mu x.M : \mathsf{true}$, in order to yield something interesting. Here we may note with regret that our whole system is based on the logic of observable properties. A standard proof principle such as fixpoint induction for admissible predicates, Lemma 2.1.20, does not fit into the framework. On the other hand, it is hopefully apparent how canonical the whole approach is. For applications, see [Abr90c, Abr91a, Bou91, Hen93, Ong93, Jen91, Jen92].

Let us now compare denotational and logical semantics. We need to say how environments ρ and assumptions Γ fit together. First of all, we assume that ρ maps each variable x^σ into $\mathsf{spec}(\mathfrak{L}(\sigma))$. Secondly, we want that $\rho(x)$ belongs to the compact-open subset described by the corresponding entry in Γ. But since environments are functions defined on the whole set of variables while assumptions are finite lists, the following definition is a bit delicate. We write $\rho \vDash \Gamma$ if for all entries $x{\mapsto}\phi$ in Γ we have $\rho(x) \in [\![\phi]\!]$. Using this convention, we can formulate validity for assertions about terms:

$$\Gamma \vDash M : \phi \text{ if and only if } \forall \rho.(\rho \vDash \Gamma \Longrightarrow [\![M]\!]\rho \in [\![\phi]\!]) \ .$$

The final tie-up between the two interpretations of type expressions and terms then is the following:

Theorem 7.3.18. *The domain logic is sound and complete. As a formula:*

$$\forall M, \Gamma, \phi. \quad \Gamma \vdash M : \phi \quad \textit{if and only if} \quad \Gamma \vDash M : \phi \ .$$

Exercises 7.3.19. 1. *Prove that a completely distributive lattice also satisfies the dual distributivity axiom:* $\bigvee_{i\in I} \bigwedge A_i = \bigwedge_{f\colon I \xrightarrow{\odot} \cup A_i} \bigvee_{i \in I} f(i)$.

2. *[Ran60] Prove that a complete lattice L is completely distributive if and only if the following holds for all $x \in L$:*

$$x = \bigvee_{a \not\geq x} \ \bigwedge_{b \not\leq a} b \ .$$

(Hint: Use Theorem 7.1.3.)

3. *Show that a topological space is sober if and only if every irreducible closed set is the closure of a unique point.*

4. *Find a complete lattice L for which* $\mathsf{pt}(L)$ *is empty.*

5. *Show that every Hausdorff space is sober. Find a T_1-space which is not sober. The converse, a sober space, which is not T_1, ought to be easy to find.*

6. *Find a dcpo which is not sober in the Scott-topology. (Reference: [Joh81]. For an example which is a complete lattice, see [Isb82]. There is no known example which is a distributive lattice.)*

7. *Describe the topological space $\mathsf{pt}(L)$ in terms of $\wedge$-prime elements of the complete lattice L.*

8. *Let D be a continuous domain. Identify D with the set of $\wedge$-prime elements in $\Omega(D)$. Prove that the Lawson-topology on D is the restriction of the Lawson-topology on $\Omega(D)$ to D.*

9. *Suppose $f\colon V \to W$ is a lattice homomorphism. Show that R defined by xRy if $y \leq f(x)$ is a join-approximable relation. Characterize the continuous functions between spectral spaces which arise from these particular join-approximable relations.*

10. *Extend Lemma 7.3.8 to other classes of domains.*

11. *Try to give a localic description of the coalesced sum construction.*

8 Further directions

Our coverage of Domain Theory is by no means comprehensive. Twenty-five years after its inception, the field remains extremely active and vital. We shall try in this Section to give a map of the parts of the subject we have not covered.

8.1 Further topics in "Classical Domain Theory"

We mention four topics which the reader is likely to encounter elsewhere in the literature.

8.1.1 Effectively given domains

As we mentioned in the Introduction, domain-theoretic continuity provides a qualitative substitute for explicit computability considerations. In order to evaluate this claim rigorously, one should give an effective version of Domain Theory, and check that the key constructions on domains such as product, function space, least fixpoints, and solutions of recursive domain equations, all "lift" to this effective setting. For this purpose, the use of abstract bases becomes quite crucial; we say (simplifying a little for this thumbnail sketch) that an ω-continuous domain is *effectively given* if it has an abstract basis $(B, \prec)$ which is numbered as $B = \{b_n\}_{n\in\omega}$ in such a way that $\prec$ is recursive in the indices. Similarly, a continuous function $f\colon D \to E$ between effectively given domains is effective if the corresponding approximable mapping is recursively enumerable. We refer to [Smy77, Kan79, WD80] and the chapter on Effective Structures in this Handbook for developments of effective domain theory on these lines.

There have also been some more sophisticated approaches which aim at making effectivity "intrinsic" by working inside a constructive universe for set theory based on recursive realizability [McC84, Ros86, Pho91]. We shall return to this idea in subsection 8.5.

8.1.2 Universal Domains

Let $\mathbf{C}$ be a cartesian closed category of domains, and U a domain in $\mathbf{C}$. We say that U is *universal* for $\mathbf{C}$ if, for every D in $\mathbf{C}$, there is an embedding $e\colon D \to U$. Thus universality means that we can, in effect, replace the category $\mathbf{C}$ by the single domain U. More precisely, we can regard the domain D as represented by the idempotent $e_D = e \circ p$, where p is the projection corresponding to e. Since $e_D\colon U \to U$, and $[U \longrightarrow U]$ is again in $\mathbf{C}$ and hence embeddable in U, we can ultimately identify D with an *element* $u_D \in U$, which we can think of as a "code" for D. Moreover, constructions such as product and function space induce continuous functions

$$\mathsf{fun}, \mathsf{prod} : U^2 \longrightarrow U$$

which act on these codes, so that e.g.

$$\mathsf{fun}(u_D, u_E) = u_{[D \longrightarrow E]}\ .$$

In this way, the whole functorial level of Domain Theory which we developed as a basis for the solution of recursive domain equations in Section 5 can be eliminated, and we can solve domain equations *up to equality on the codes* by finding fixpoints of continuous functions over U.

This approach was introduced by Scott in [Sco76], and followed in the first textbook on denotational semantics [Sto77]. However, it must be said that, as regards applications, universal domains have almost fallen into disuse. The main reason is probably that the coding involved in the transition from D to u_D is confusing and unappealing; while more attractive ways of simplifying the treatment of domain equations, based on information systems, have been found (see 8.1.4). However, there have been two recent developments of interest. Firstly, a general approach to the construction of universal domains, using tools from Model Theory, has been developed by Gunter and Jung and Droste and Göbel, and used to construct universal domains for many categories, and to prove their non-existence in some cases [GJ88, DG90, DG91, DG93].

Secondly, there is one application where universal domains do play an important rôle: to provide models for type theories with a type of all types. Again, the original idea goes back to [Sco76]. We say that a univeral domain U admits a universal type if the subdomain V of all u_D for D in **C** is itself a domain in **C**—and hence admits a representation $u_V \in U$. We can think of u_V as a code for the type of all types. In [Sco76], Scott studied the powerset $\mathfrak{P}(\omega)$ as a univeral domain for two categories: the category of ω-continuous lattices (for which domains are taken to be represented by idempotents on $\mathfrak{P}(\omega)$), and the category of ω-algebraic lattices (for which domains are represented by closures). Ershov [Ers75] and Hosono and Sato [HS77] independently proved that $\mathfrak{P}(\omega)$ does not admit a universe for the former category; Hancock and Martin-Löf proved that it does for the latter (reported in [Sco76]). For recent examples of the use of universal domains to model a type of all types see [Tay87, Coq89, Ber91].

8.1.3 Domain-theoretic semantics of polymorphism

We have seen the use of continuity in Domain Theory to circumvent cardinality problems in finding solutions to domain equations such as

$$D \cong [D \longrightarrow D] .$$

A much more recent development makes equally impressive use of continuity to give a finitary semantics for impredicative polymorphism, as in the second-order $lambda$-calculus (Girard's "System F") [Gir86, CGW87, Coq89]. This semantics makes essential use of the functorial aspects of Domain Theory. There have also been semantics for implicit polymorphism based on ideals [MPS86] and partial equivalence relations [AP90] over domains. We refer to the chapter in this volume of the Handbook on Semantics of Types for comprehensive coverage and references.

8.1.4 Information Systems

Scott introduced information systems for bounded-complete ω-algebraic dcpo's ("Scott domains") in [Sco82]. The idea is, roughly, to represent a category of domains by a category of abstract bases and approximable mappings as in Theorems 2.2.28 and

2.2.29. One can then define constructions on domains in terms of the bases, as in Propositions 3.2.4 and 4.2.4. This gives a natural setting for effective domain theory as in 8.1.1 above. Moreover, bilimits are given by unions of information systems, and domain equations solved up to equality, much as in 7.3.5. More generally, information systems correspond to presenting just the coprime elements from the domain prelocales of 7.3. Information system representations of various categories of domains can be found in [Win88, Zha91, Cur93]. A general theory of information systems applicable to a wide class of topological and metric structures can be found in [ES93].

8.2 Stability and Sequentiality

Recall the ϵ-δ style definition of continuity given in Proposition 2.2.11: given $e \in C_{f(x)}$ it provides $d \in B_x$ with $f(d) \sqsubseteq e$. However, there is no *canonical* choice of d from e. In an order-theoretic setting, it is natural to ask for there to be a *least* such d. This leads to the idea of the *modulus of stability*: $M(f, x, e)$, where $f(x) \sqsupseteq e$, is the least such d, if it exists. We say that a continuous function is *stable* if the modulus always exists, and define the *stable ordering* on such functions by

$$f \sqsubseteq_s g \iff f \sqsubseteq g \;\wedge\; \forall x, e.\ e \in C_{f(x)}.\ M(f, x, e) = M(g, x, e).$$

We can think of the modulus as specifying the minimum information actually required of a given input x in order that the function f yields a given information y on the output; the stable ordering refines the usual pointwise order by taking this intensional information into account.

It turns out that these definitions are equivalent to elegant algebraic notions in the setting of the lattice-like domains introduced (for completely different purposes!) in Section 4.1. Let D, E be domains in $\mathbf{L}$. Then a continuous function $f\colon D \to E$ is stable iff it preserves bounded non-empty infima (which always exist in $\mathbf{L}$; cf. Proposition 4.1.2), and $f \sqsubseteq_s g$ iff for all $x \sqsubseteq y$, $f(x) = f(y) \sqcap g(x)$. This is the first step in an extensive development of "Stable Domain Theory" in which stable functions under the stable ordering take the place which continuous functions play in standard Domain Theory. Stable Domain theory was introduced by Berry [Ber78, Ber79]. Some more recent references are [Gir86, CGW87, Tay90, Ehr93].

Berry's motivation in introducing stable functions was actually to try to capture the notion of sequentially computable function at higher types. For the theory of sequential functions on concrete domains, we refer to [KP93, Cur93].

8.3 Reformulations of Domain Theory

At various points in our development of Domain Theory (see e.g. Section 3.2), we have referred to the need to switch between different versions $\mathbf{C}$, $\mathbf{C}_\perp$, $\mathbf{C}_{\perp!}$ of some category of domains, depending on whether bottom elements are required, and if so whether functions are required to preserve them. In some sense $\mathbf{C}$ and $\mathbf{C}_{\perp!}$ are the mathematically natural categories, since what the morphisms must preserve matches the structure that the objects are required to have; while $\mathbf{C}_\perp$ is the preferred category for semantics, since endomorphisms $f\colon D \to D$ need not have fixpoints at all in $\mathbf{C}$, while least fixpoints in $\mathbf{C}_{\perp!}$ are necessarily trivial.

All this suggests that something is lacking from the mathematical framework in order to get a really satisfactory tie-up with the applications. We shall describe a number of attempts to make good this deficiency. While no definitive solution has yet emerged, these proposals have contributed important insights to Domain Theory and its applications.

8.3.1 Predomains and partial functions

The first proposal is due to Gordon Plotkin [Plo85]. The idea is to use the *objects* of $\mathbf{C}$ ("predomains", i.e. domains without any requirement of bottom elements), but to change the notion of morphism to *partial continuous function*: where we say that a partial function $f : D \rightharpoonup E$ is continuous if its domain of definition is a Scott-open subset of D, and its restriction to this subset is a (total) continuous function. The resulting category is denoted by $\mathbf{C}_\partial$. This switch to partial continuous functions carries with it a change in the type structure we can expect to have in our categories of domains: they should be *partial* cartesian closed categories, as defined e.g. in [RR88, Ros86].

One advantage of this approach is that it brings the usage of Domain Theory closer to that of recursion theory. For example, the hierarchy of (strict) partial continuous functionals over the natural numbers will be given by

$$\mathbb{N}, [\mathbb{N} \rightharpoonup \mathbb{N}], [[\mathbb{N} \rightharpoonup \mathbb{N}] \rightharpoonup \mathbb{N}], \ldots$$

rather than

$$\mathbb{N}_\perp, [\mathbb{N}_\perp \xrightarrow{\perp!} \mathbb{N}_\perp], [[\mathbb{N}_\perp \xrightarrow{\perp!} \mathbb{N}_\perp] \xrightarrow{\perp!} \mathbb{N}_\perp], \ldots.$$

This avoidance of bottom elements also leads to a simpler presentation of product and sum types. For example, there is just one notion of sum, the disjoint union $D \mathbin{\dot\cup} E$, which is indeed the coproduct in $\mathbf{C}_\partial$.

An important point is that there is a good correspondence between the operational behaviour of functions with a call-by-value parameter-passing mechanism and the partial function type $[\_ \rightharpoonup \_]$. For example, there is a good fit between $[\_ \rightharpoonup \_]$ and the function type constructor in Standard ML [MT91, MTH90].

To balance these advantages, we have the complication of dealing with partially defined expressions and partial cartesian closure; and also a less straightforward treatment of fixpoints. It is not the case that an arbitrary partial continuous function $f : D \rightharpoonup D$ has a well-defined least fixpoint. However, if D itself is a partial function type, e.g. $D = [E \rightharpoonup E]$, then f does have a well-defined least fixpoint. This is in accord with computational intuition for call-by-value programming languages, but not so pleasant mathematically.

As a final remark, note that in fact $\mathbf{C}_\partial$ is equivalent to $\mathbf{C}_{\perp!}$! Thus, in a sense, this approach brings nothing new. However, there is a distinct conceptual difference, and also $\mathbf{C}_\partial$ is more amenable to constructive proof and categorical axiomatization [Ros86].

8.3.2 Computational Monads

Computational monads have been proposed by Eugenio Moggi as a general structuring mechanism for denotational semantics [Mog91]. A computational monad on a cartesian category **C** is a monad (T, η, μ) together with a "tensorial strength", i.e. a natural transformation

$$t_{A,B} : A \times TB \rightarrow T(A \times B)$$

satisfying some equational axioms. The import of the strength is that the monad can be internalised along the lines mentioned after Proposition 6.1.8. Now let **C** be a category of (pre)domains and total continuous functions. Moggi's proposal is to make a distinction between *values* and (denotations of) *computations*. An element of A is a value, an element of TA is a computation. A (call-by-value) procedure will denote a morphism $A \rightarrow TB$ which accepts an input value of type A and produces a computation over B. Composition of such morphisms is by Kleisli extension: if $f\colon A \rightarrow TB, g\colon B \rightarrow TC$, then composition is defined by

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC,$$

with identities given by the unit $\eta_A : A \rightarrow TA$.

In particular, partiality can be captured in this way using the *lifting* monad, for which see 3.2.5. Note that this particular example is really just another way of presenting the category $\mathbf{C}_\partial$ of the previous subsection; there is a natural isomorphism

$$[D \longrightarrow E_\perp] \cong [D \rightharpoonup E] \ .$$

The value of the monadic approach lies in its generality and in the type distinction it introduces between values and computations. To illustrate the first point, note that the various powerdomain constructions presented in Section 7.2 all have a natural structure as strong monads, with the monad unit and multiplication given by suitable versions of the singleton and big union operations. For the second point, we refer to the elegant axiomatization of general recursion in terms of fixpoint objects given by Crole and Pitts [CP92], which makes strong use of the monadic approach. This work really belongs to Axiomatic Domain Theory, to which we will return in subsection 4 below.

8.3.3 Linear Types

Another proposal by Gordon Plotkin is to use Linear Types (in the sense of Linear Logic [Gir87]) as a metalanguage for Domain Theory [Plo93]. This is based on the following observation. Consider a category $\mathbf{C}_{\perp !}$ of domains with bottom elements and strict continuous functions. This category has products and coproducts, given by cartesian products and coalesced sums. It also has a monoidal closed structure given by smash product and strict function space, as mentioned in 3.2.4. Now lifting, which is a monad on **C** by virtue of the adjunction mentioned in 3.2.5, is dually a *comonad* on $\mathbf{C}_{\perp !}$; and the co-Kelisli category for this comonad is $\mathbf{C}_\perp$.

Indeed, Linear Logic has broader connections with Domain Theory. A key idea of Linear Logic is the linear decomposition of the function space:

$$[A \longrightarrow B] \cong [!A \multimap B] \ .$$

One of the cardinal principles of Domain Theory, as we have seen, is to look for cartesian closed categories of domains as convenient universes for the semantics of computation. Linear Logic leads us to look for linear decompositions of these cartesian closed structures. For example, the cartesian closed category of complete lattices and continuous maps has a linear decomposition via the category of complete lattices and sup-lattice homomorphisms—i.e. maps preserving *all* joins, with $!L = \mathsf{P}^H(L)$, the Hoare powerdomain of L. There are many other examples [Hoo92, Ehr93, Hut94].

8.4 Axiomatic Domain Theory

We began our account of Domain Theory with requirements to interpret certain forms of recursive definitions, and to abstract some key structural features of computable partial functions. We then introduced some quite specific structures for convergence and approximation. The elaboration of the resulting theory showed that these structures do indeed *work*; they meet the requirements with which we began. The question remains whether another class of structures might have served as well or better. To address this question, we should try to axiomatize the key features of a category of domains which make it suitable to serve as a universe for the semantics of computation. Such an exercise may be expected to yield the following benefits:

- By making it clearer what the essential structure is, it should lead to an improved meta-language and logic, a refinement of Scott's Logic of Computable Functions [Sco93].
- Having a clear axiomatization might lead to the discovery of different models, which might perhaps be more convenient for certain purposes, or suggest new applications. On the other hand, it might lead to a representation theorem, to the effect that every model of our axioms for a "category of domains" can in fact be embedded in one of the concrete categories we have been studying in this Chapter.

Thus far, only a limited amount of progress has been made on this programme. One step that can be made relatively cheaply is to generalize from concrete categories of domains to categories enriched over some suitable subcategory of $\mathbf{DCPO}$. Much of the force of Domain Theory carries over directly to this more general setting [SP82, Fre92]. Moreover, this additional generality is not spurious. A recent development in the semantics of computation has been towards a refinement of the traditional denotational paradigm, to reflect more intensional aspects of computational behaviour. This has led to considering as semantic universes certain categories in which the morphisms are not functions but sequential algorithms [Cur93], information flows [AJ94b], game-theoretic strategies [AJ94a], or concurrent processes [Abr94]. These are quite different from the "concrete" categories of domains we have been considering, in which the morphisms are always functions. Nevertheless, they have many of the relevant properties of categories of domains, notably the existence of fixpoints and of canonical solutions of recursive domain equations. The promise of axiomatic domain theory is to allow the rich theory we have developed in this Chapter to be transposed to such settings with a minimum of effort.

The most impressive step towards Axiomatic Domain Theory to date has been Peter Freyd's work on algebraically compact categories [Fre91, Fre92]. This goes considerably beyond what we covered in Section 5. The work by Crole and Pitts on FIX-categories should also be mentioned [CP92].

In another direction, there are limitative results which show that certain kinds of structures *cannot* serve as categories of domains. One such result appeared as Exercise 5.4.11(3). For another, see [HM93].

8.5 Synthetic Domain Theory

A more radical conceptual step is to try to absorb all the structure of convergence and approximation, indeed of computability itself, into the ambient universe of sets, by working inside a suitable constructive set theory or topos. The slogan is: "Domains are Sets". This leads to a programme of "Synthetic Domain Theory", by analogy with Synthetic Differential Geometry [Koc81], in which smoothness rather than effectivity is the structure absorbed into the ambient topos.

The programme of Synthetic Domain Theory was first adumbrated by Dana Scott around 1980. First substantial steps on this programme were taken by Rosolini [Ros86], and subsequently by Phoa [Pho91], and Freyd, Mulry, Rosolini and Scott [FMRS90]. Axioms for Synthetic Domain Theory have been investigated by Hyland [Hyl91] and Taylor [Tay91], and the subject is currently under active development.

9 Guide to the literature

As mentioned in the Introduction, there is no book on Domain Theory. For systematic accounts by the two leading contributors to the subject, we refer to the lecture notes of Scott [Sco81] and Plotkin [Plo81]. There is also an introductory exposition by Gunter and Scott in [GS90]. An exhaustive account of the theory of continuous lattices can be found in [GHK+80]; a superb account of Stone duality, with a good chapter on continuous lattices, is given in [Joh82]; while [DP90] is an excellent and quite gentle introduction to the theory of partial orders.

Some further reading on the material covered in this Chapter:

Section 2: [DP90, Joh82];

Section 3: [Plo81, Gun92b, Win93];

Section 4: [Jun89, Jun90];

Section 5: [SP82, Fre91, Fre92, Pit93b, Pit93a];

Section 6: [Plo76, Smy78, Win83, Hec91, Sch93];

Section 7: [Abr90c, Abr91a, AO93, Ong93, Hen93, Bou94, Jen92, Jen91, Smy83b].

Applications of Domain Theory

There is by now an enormous literature on the semantics of programming languages, much of it using substantial amounts of Domain Theory. We will simply list a number of useful textbooks: [Sch86, Ten91, Gun92b, Win93].

In addition, a number of other applications of Domain Theory have arisen: in Abstract Interpretation and static program analysis [Abr90a, BHA86, AJ91] (see also the article on Abstract Interpretation in this Handbook); databases [BDW88, BJO91]; computational linguistics [PS84, PM90]; artificial intelligence [RZ94]; fractal image generation [Eda93b]; and foundations of analysis [Eda93a].

Finally, the central importance of Domain Theory is well indicated by the number of other chapters of this Handbook which make substantial reference to Domain-theoretic ideas: Topology, Algebraic Semantics, Semantics of Types, Correspondence between Operational and Denotational Semantics, Abstract Interpretation, Effective Structures.

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Index