Concurrency Theory, 23 Aug 2019

M covers M' $(M \geq M')$ if $\forall p\ M(p) \geq M'(p)$

Coverability Problem

Does there exist $M' \in Reach(M_{in})$
such that $M' \geq M$

Similar to finite state automata, unfold $Reach(M_{in})$ as a tree and "cut off" paths

Difficulty: Infinite state space

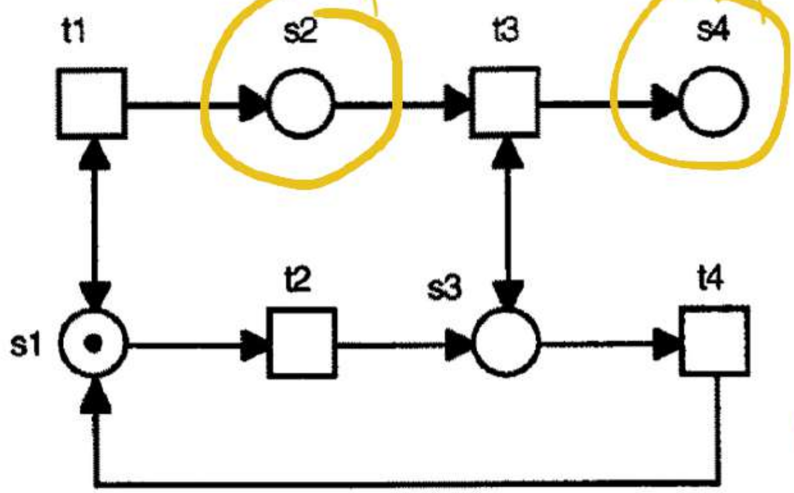
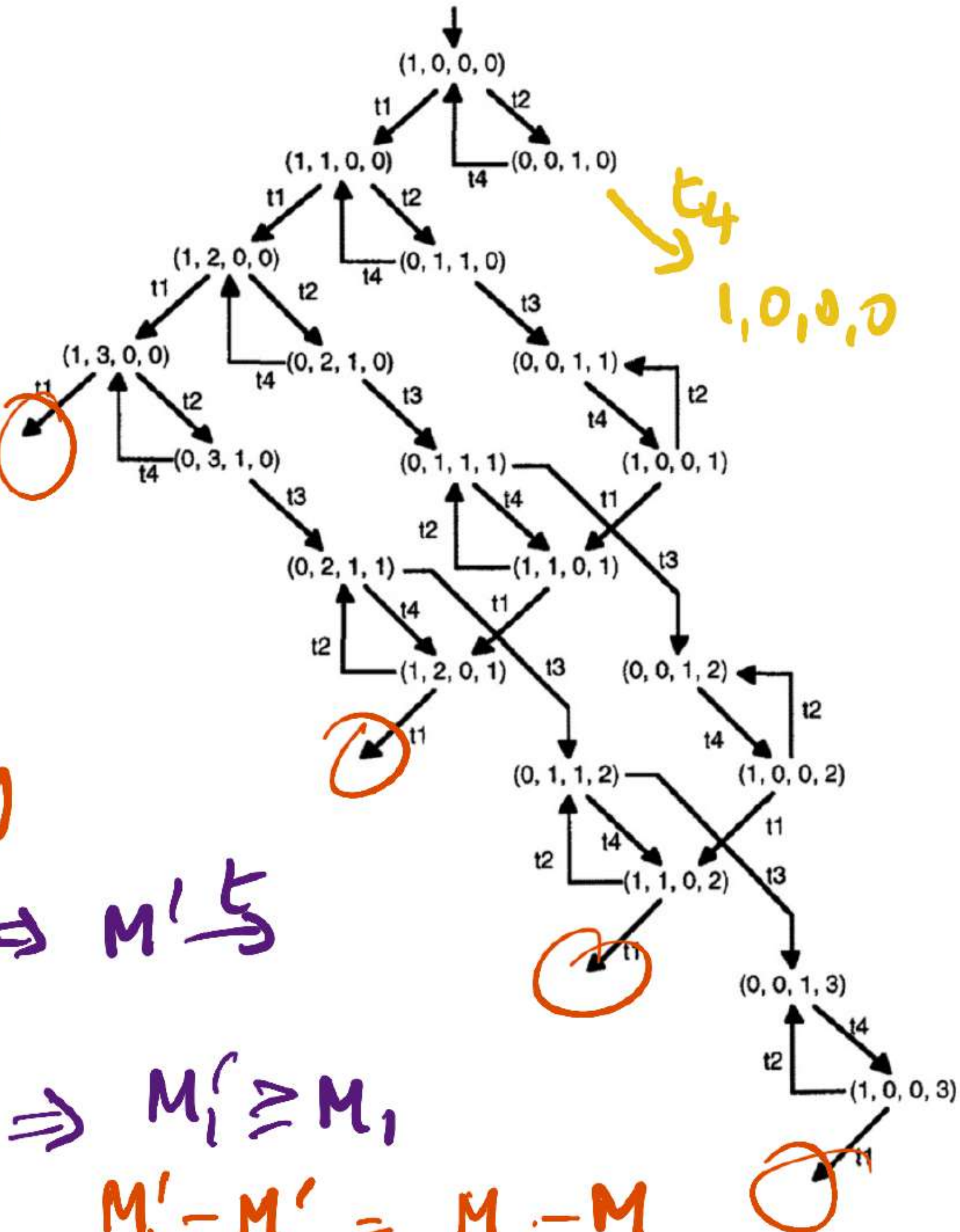


Fig. 12. An unbounded marked Petri net



Monotonicity property

$$M \xrightarrow{t}, \quad M' \geq M \Rightarrow M' \xrightarrow{t}$$

$$M \xrightarrow{t} M_1, \quad M' \xrightarrow{t} M_1' \Rightarrow M_1' \geq M_1$$

$$M_1' - M' = M_1 - M$$

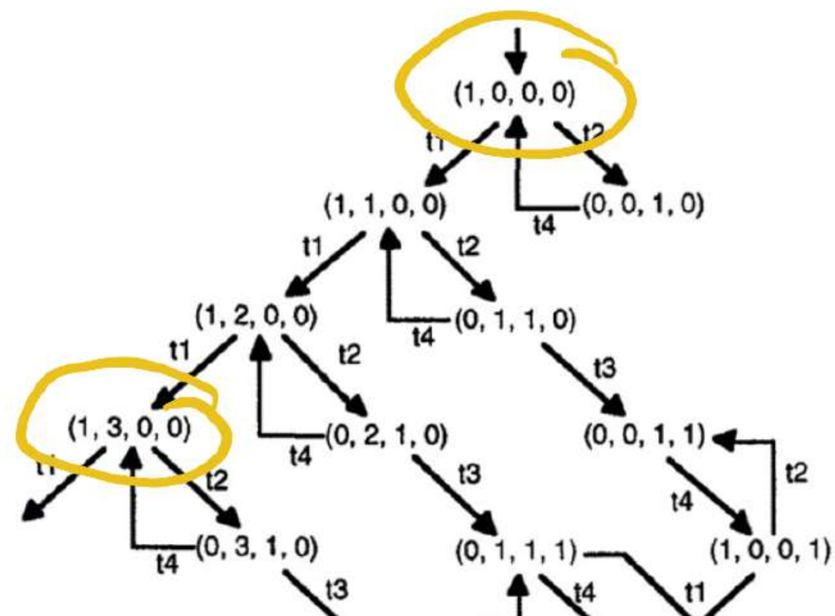
$M \xrightarrow[t_1 t_2 \cdots t_k]{\sigma} \boxed{M}$

Stop

$$M \xrightarrow[\text{diff}]{\sigma} M' \xrightarrow[\text{diff}]{\sigma} M''$$

M	M'	M''
=	=	=
	>	>

$M' > M$



Extend markings to allow symbol ω

ω-marking

ω represents an unbounded number of tokens

Modified firing rule

- Finite places, as before
- ω places, retain ω

ω

$\vdots$

3

2

1

0

How do we create ω-markings?

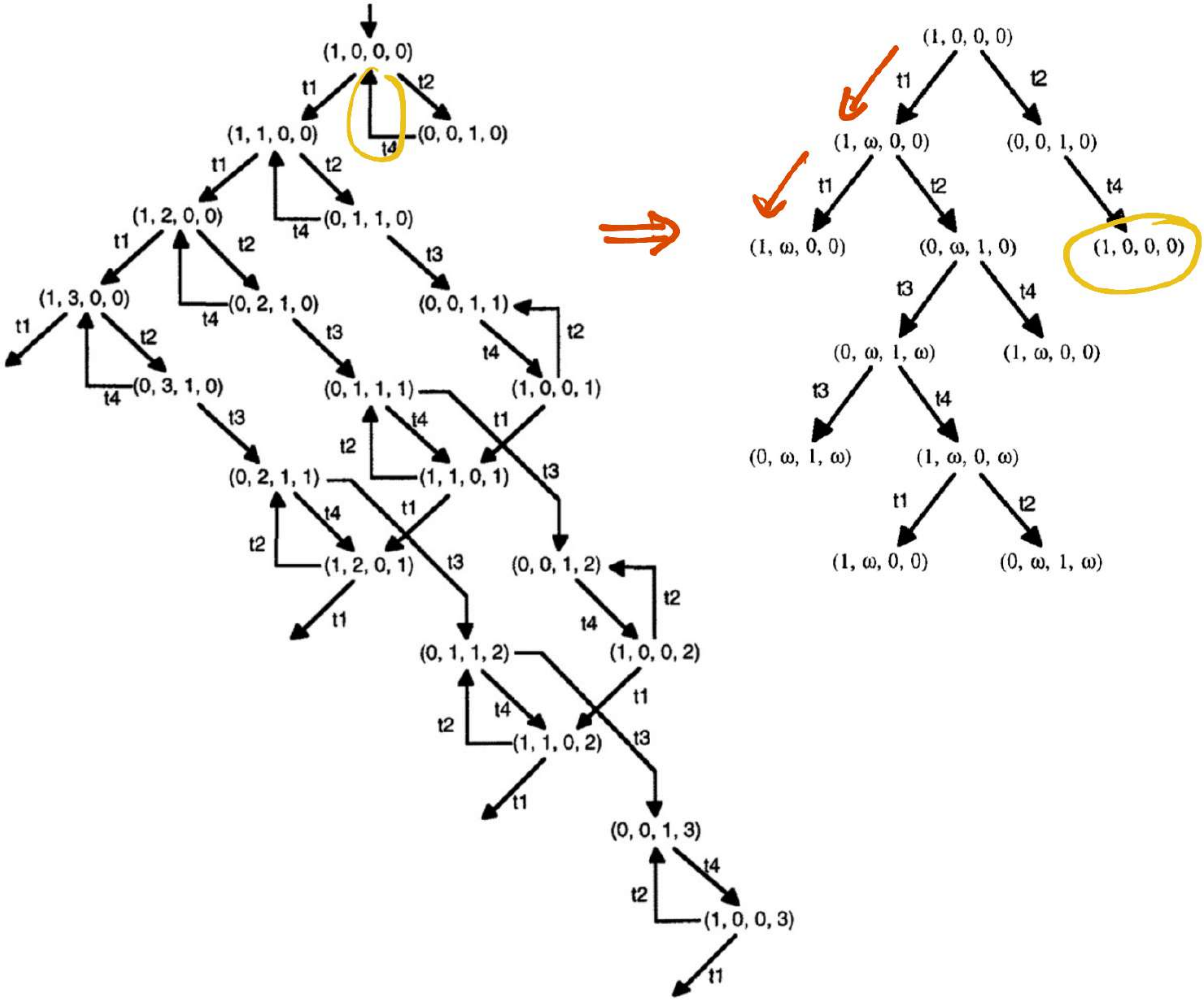
$\omega > j$ for all $j \in \mathbb{N}_0$

$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2$ - - - .

if $M_j > M_i$, $i < j$ — promote all the components that have increased to ω

if $M_j = M_i$, $i < j$ Stop .

Use this principle to construct coverability tree



(1, 0, 0, 0)
t1
t2
t4
(1, 1, 0, 0)
(0, 0, 1, 0)
(1, 2, 0, 0)
(0, 1, 1, 0)
t3
(1, 3, 0, 0)
(0, 2, 1, 0)
(0, 0, 1, 1)
(0, 3, 1, 0)
(0, 1, 1, 1)
(1, 0, 0, 1)
(0, 2, 1, 1)
(1, 1, 0, 1)
(1, 2, 0, 1)
(0, 0, 1, 2)
(0, 1, 1, 2)
(1, 0, 0, 2)
(1, 1, 0, 2)
(0, 0, 1, 3)
(1, 0, 0, 3)
(1, ω, 0, 0)
(0, ω, 1, 0)
(0, ω, 1, ω)
(1, ω, 0, ω)

Theorem The coverability tree is always finite.

Proof If infinite $\Rightarrow \exists$ infinite path (König's lemma)

Can we construct an infinite path that is not cut-off by our rule?

Infinite sequence of mutually incomparable ω-markings

Dickson's lemma Any infinite sequence over $\mathbb{N}_0^k$ must have a non-decreasing infinite subsequence

$\underline{k=1}$ Infinite sequence over $\mathbb{N}_0$

$$\underbrace{n_0 \quad n_1 \quad n_2 \quad n_3}_{\leq} \; \underbrace{\cdots}_{\leq} \; \underbrace{n_i}_{\leq} \; \underbrace{n_{i+1} \cdots}_{\leq}$$

Cannot have infinite decreasing sequence.
Well ordering

Base case of inductive proof.

$\underline{\underline{k+1}}$ $\mathbb{N}_0^k$ by induction has inf. nondec seq.
Consider $\mathbb{N}_0^{k+1}$ over this subsequence
Must be an infinite dec subseq. over $\mathbb{N}_0^{k+1}$

<u>Cannot</u> have an infinite set of pairwise incomparable ω-markings

Need to show a connection between reachable markings and the coverability tree

p_3 may be strictly decreasing

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{\sigma_1} \begin{bmatrix} a \\ b \\ \omega \\ d \end{bmatrix} \xrightarrow{\sigma_2} \begin{bmatrix} a \\ b \\ \omega \\ \omega \end{bmatrix}$$

<u>Theorem</u> Let M be an ω-marking in the tree with places $p_1, \ldots, p_k$ marked ω

For any choice of $n_1, \ldots, n_k \in \mathbb{N}_0$ it is possible to reach a marking $M' \in Reach(M_{in})$ s.t. $M'(p_i) \geq n_i$ for $i \in \{1, 2, \ldots, k\}$

[$p_1, \ldots p_k$ are simultaneously unbounded]

<u>Proof Sketch</u>: Pump the ω-marking sequence from root to M bottom up

Corollary 1 M is coverable iff there is a marking $M' \geq M$ in the coverability tree.

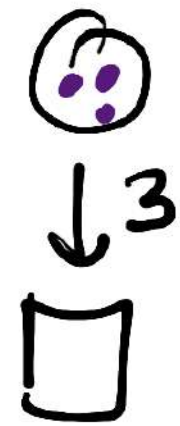
Corollary 2 (N, M_{in}) is bounded if no ω appears in the coverability tree.

Karp & Miller – Tree is bounded by the size of longest pairwise incomparable set over $\mathbb{N}_0^k$ – no good bound for this

Extensions

(P, T, F) $\qquad$ $F \subseteq (P \times T) \cup (T \times P)$

Generalize F to substract/add multiple tokens. — arc weights



(P, T, W) $\qquad$ $W : (P \times T) \cup (T \times P) \to \mathbb{N}_0$

Corresponding generalization of firing rule

Nothing much changes in whatever we have seen so far

Add capacities to places

$K : P \to \mathbb{N}_0$

$M(p) \leq K(p)$ always

- blocks transitions if K would be exceeded
- like "contact" for ENS

Construct complementary places with

$M_{in}(\bar{p}) = K(p) - M_{in}(p)$, $K(\bar{p}) = K(p)$

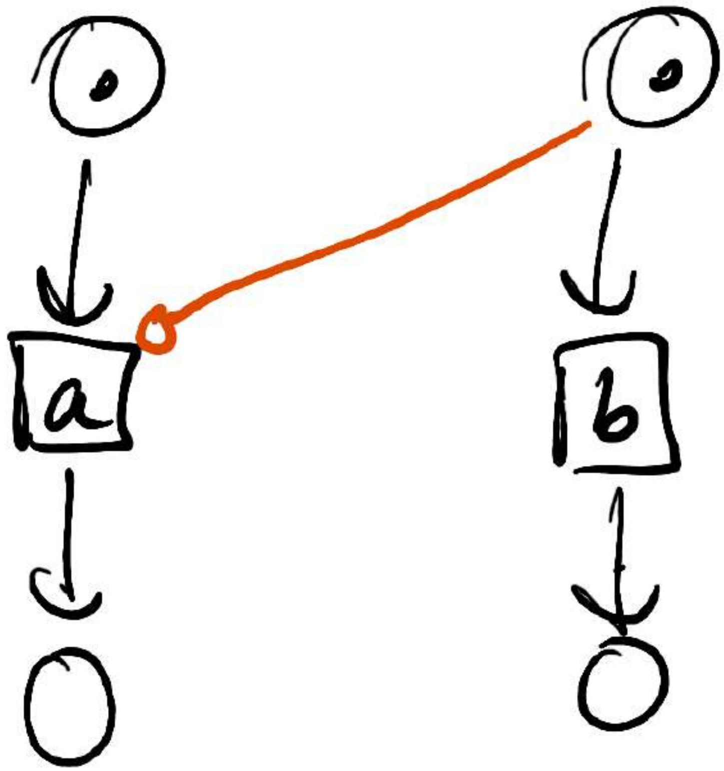
What if we affect monotonicity?

Adding a token can (actively) disable a transition

(P, T, F, I) $\quad I \subseteq P \times T$

I & F should not contradict each other

Encode priorities



If both are enabled, b happens before a

No longer have the monotonicity property

$$M \xrightarrow{t}$$

$$M' \geq M, \quad M' \stackrel{t}{\not\to}$$

<u>Theorem</u> Reachability is undecidable for nets with inhibitor arcs