Markov Decision Processes

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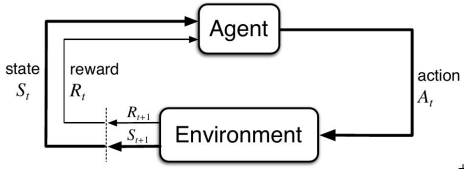
  Trajectory $S_0, A_0, R_1, S_1, A_1, R_2, S_2, \ldots$



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- Probabilistic transition function: $p(s', r \mid s, a)$
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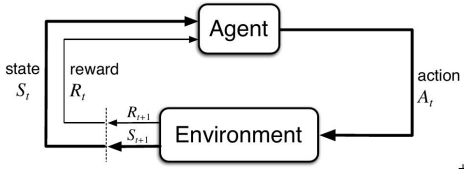
  - For each (s, a), $\sum_{s'} \sum_{r} p(s', r \mid s, a) = 1$



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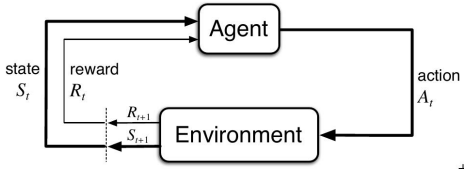

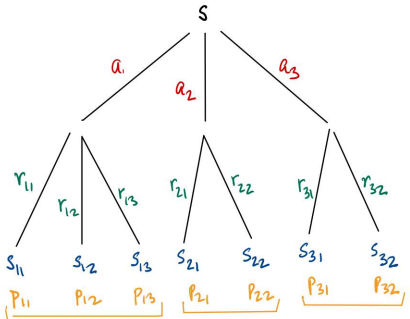

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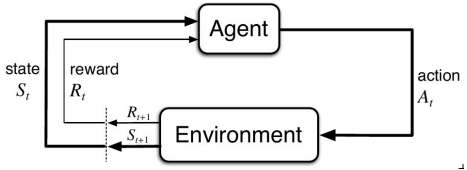
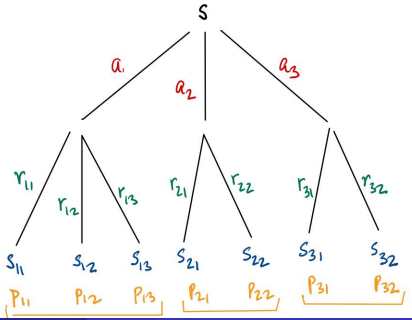
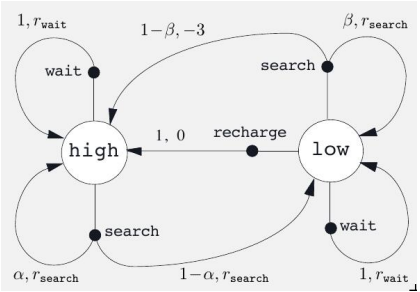
- Typically assume finite MDPs — S, A and R are finite





MDP Example: Robot that collects empty cans

- State — battery charge: high, low
- Actions: search for a can, wait for someone to bring can, recharge battery
  - No recharge when high



s	a	s'	$p(s' \mid s, a)$	$r(s, a, s')$
high	search	high	α	r_{search}
high	search	low	$1-\alpha$	r_{search}
low	search	high	$1-\beta$	-3
low	search	low	β	r_{search}
high	wait	high	1	r_{wait}
high	wait	low	0	-
low	wait	high	0	-
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low	recharge	high	1	0
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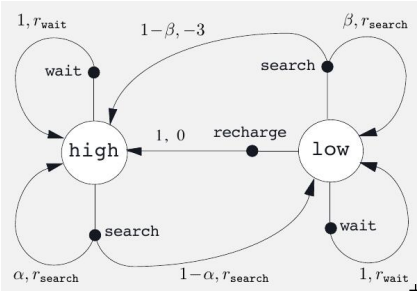

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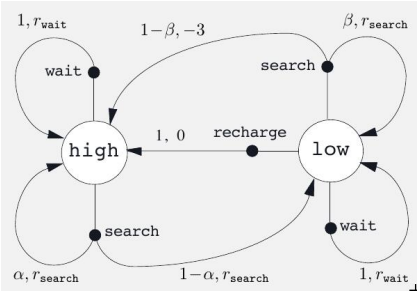
- α, β, probabilities associated with change of battery state while searching
- 1 unit of reward per can collected
- $r_{\text{search}} > r_{\text{wait}}$ — cans collected while searching, waiting
- Negative reward for requiring rescue (low to high while searching)



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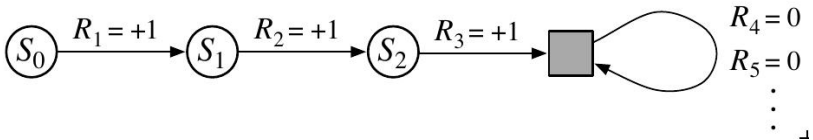
- Can make all episodes infinite by adding a self-loop with reward 0



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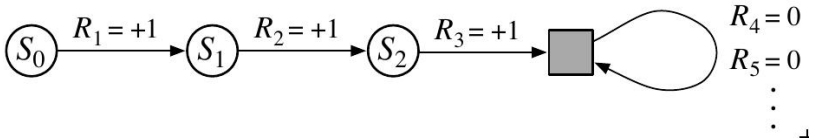

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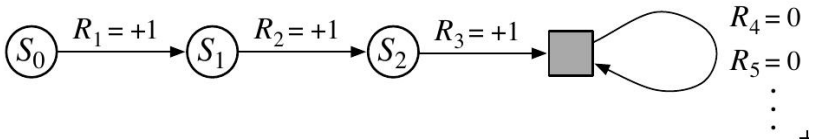

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- Alternatively, $G_t \stackrel{\triangle}{=} \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$,

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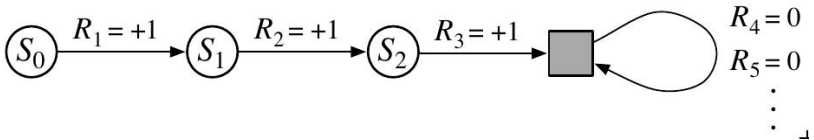

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- If $T = \infty$, $R_k = +1$ for each k, $\gamma < 1$, then $G_t = \dfrac{1}{1-\gamma}$

Policies and value functions

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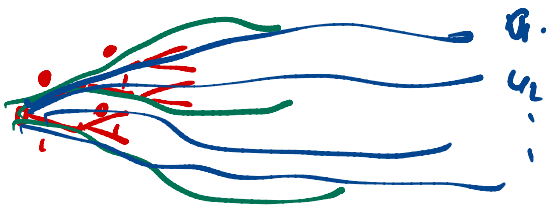
$$v_\pi(s) \triangleq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s\right]$$



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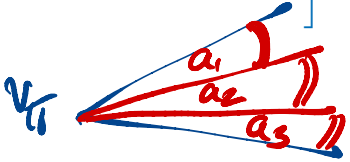
$$q_\pi(s, a) \triangleq \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a\right]$$

  - Note that $v_\pi(s) = \sum_a \pi(a \mid s) q_\pi(s, a)$



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- Goal is to find an optimal policy, that maximizes state/action value at every state

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$$= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s]$$

$$= \sum_a \pi(a \mid s) \sum_{s'} \sum_r p(s', r \mid s, a)[r + \gamma \mathbb{E}_\pi[G_{t+1} \mid S_{t+1} = s']]$$

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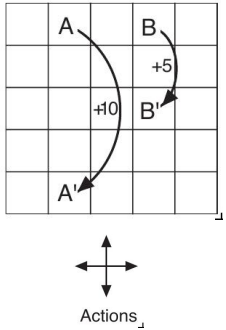
- Bellman equation relates state value at s to state values at successors of s
- Value function v_π is unique solution to the equation

Alternatively – sample & estimate

Gridworld Example

- Actions in each cell are {N,S,E,W}, with usual interpretation



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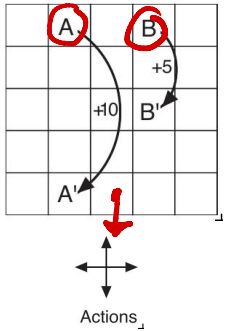

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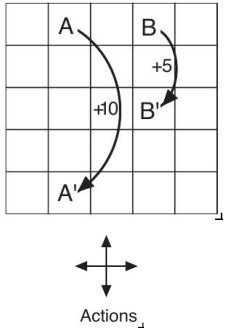

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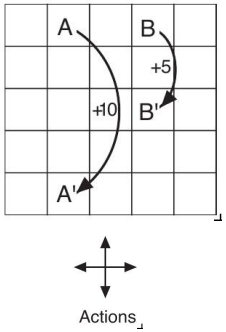

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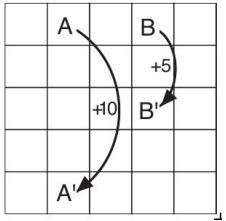

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0.1	0.7	0.7	0.4	-0.4
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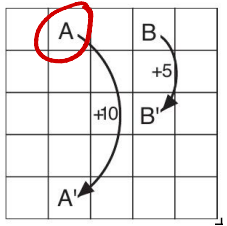
- Values at boundary are negative



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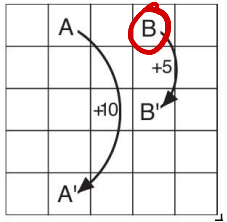
- Value at A is less than 10 because next move takes agent to boundary square with negative value



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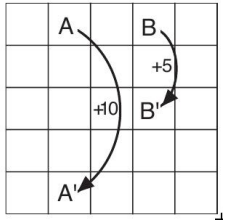
- Values at boundary are negative
- Value at A is less than 10 because next move takes agent to boundary square with negative value
- Value at B is more than 5 because next move is to a square with positive value



Actions

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

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- The new policy π' is strictly better

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Policy Improvement Theorem

For deterministic policies π, π':

- If $q_\pi(s, \pi'(s)) \geq v_\pi(s)$ for all s, then $\pi' \geq \pi$,
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- Nested iteration — each policy evaluation is itself an iteration
  - Speed up by using v_{π_i} as initial state to compute $v_{\pi_{i+1}}$

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- Requires knowledge of the model — $p(s', r \mid s, a)$
- How to combine policy evaluation and policy improvement is flexible
  - Value iteration is policy iteration with policy evaluation truncated to a single step
  - Generalized policy iteration — simultaneously maintain and update approximations of π_* and v_*