Reinforcement Learning

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Advanced Machine Learning
September–December 2021

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- Examples
  - Playing games — AlphaGo, reward is result of the game
  - Motion planning — robot searching for an optimal path with obstacles
  - Feedback control — balancing an object

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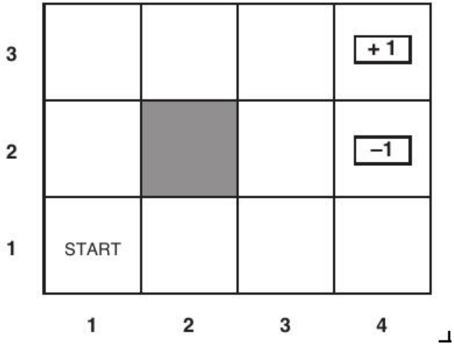
- Environment Model How the environment will behave
  - Given a state and action, what is the next state, reward?
  - Probabilistic, in general
  - Use models for *planning*
  - Can also use RL without models, trial-and-error learners

Motion planning example

- 4×3 grid
- Rewards are attached to states
  - Two terminal states with rewards $+1$, -1
  - All other states have reward -0.04
  - Move till you reach a terminal state
  - Maximize the sum of the rewards seen



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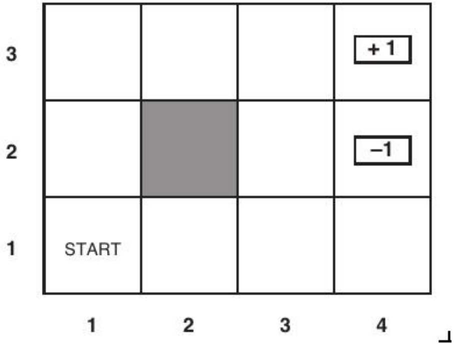

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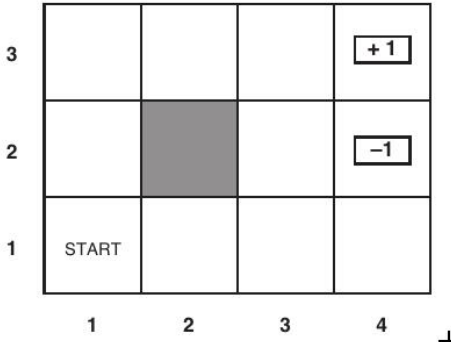
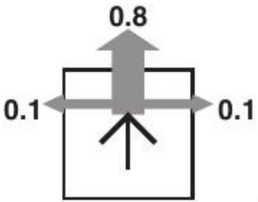
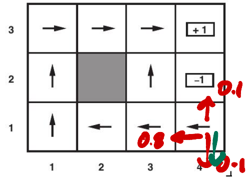
- Policy — which direction to move from a given square in the grid
- Outcome of action is nondeterministic
  - With probability 0.8, go in intended direction
  - With probability 0.2, deflect at right angles
  - Collision with boundary keeps you stationary





Motion planning example

- Optimal policy learned by repeatedly moving on the board
  - From bottom right, conservatively follow the long route around the obstacle to avoid -1



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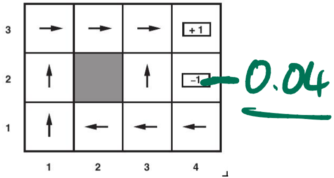
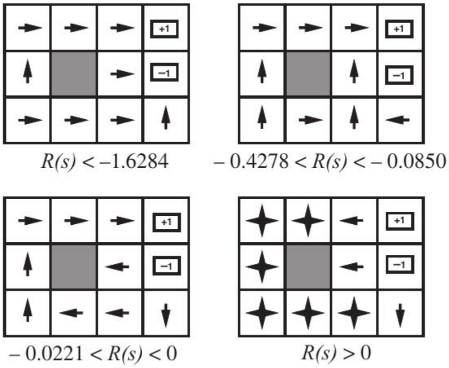
- Optimal policies for different value of $R(s)$, reward for non-final states
  - If $R(s) < -1.6284$, terminate as fast as possible
  - If $-0.4278 < R(s) < -0.0850$, risk going past -1 to reach $+1$ quickly
  - If $-0.0221 < R(s) < 0$, take no risks, avoid -1 at all cost
  - If $R(s) > 0$ avoid terminating





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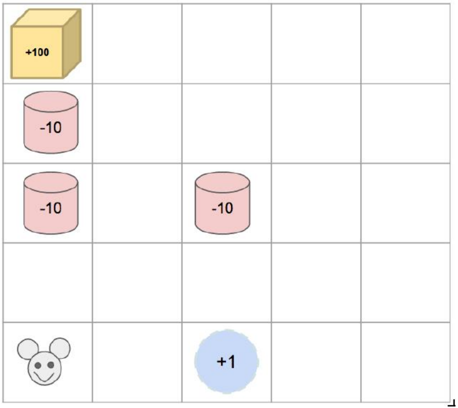

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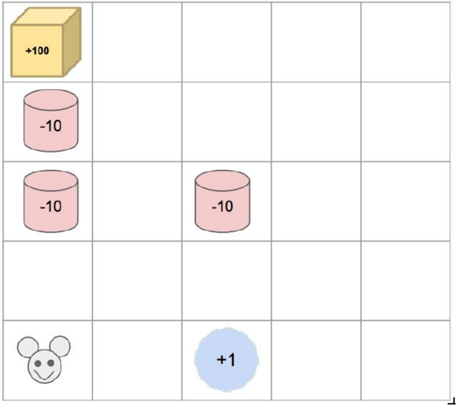

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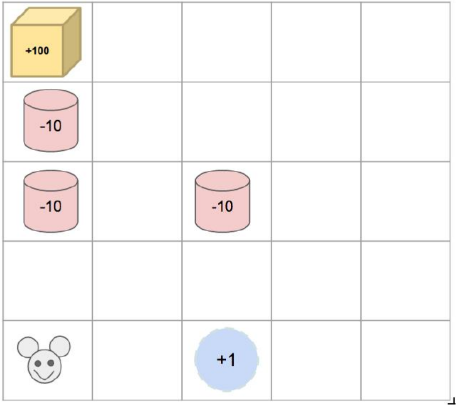
- Formalize these ideas using Markov Decision Processes



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- One-armed bandit — slang for a slot machine in a casino
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  - If we knew $q_*(a)$ we would always choose $A_t = \arg\max_a q_*(a)$
  - Assume $q_*(a)$ is unknown — build an estimate $Q_t(a)$ of $q_*(a)$ at time t

Exploration and exploitation

- Build $Q_t(a)$, estimate of $q_*(a)$ at time t, from past observations (sample average)

$$\frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{1}_{A_t=a}}{\sum_{i=1}^{t-1} \mathbb{1}_{A_t=a}}$$

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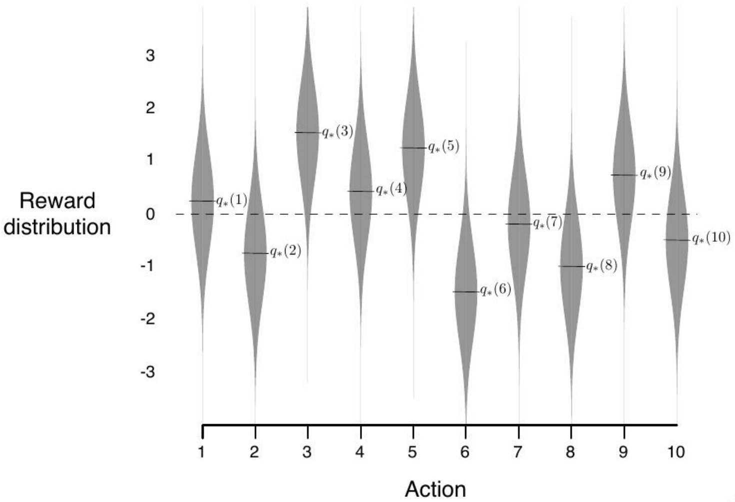
- ε-greedy is a simple way to balance exploitation with exploration
  - Theoretically, explores all actions infinitely often
  - Practical effectiveness depends

Exploration and exploitation

10 bandit experiment

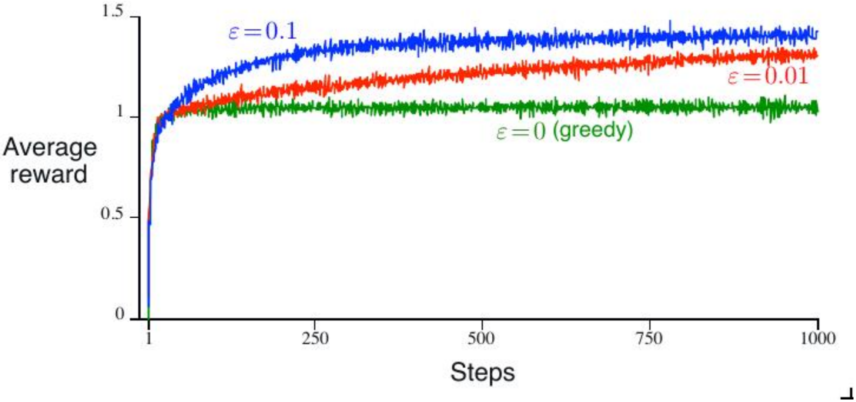
- Each bandit's reward follows Gaussian distribution
- Same variance, mean is chosen randomly



Exploration and exploitation

Performance of ε-greedy strategies

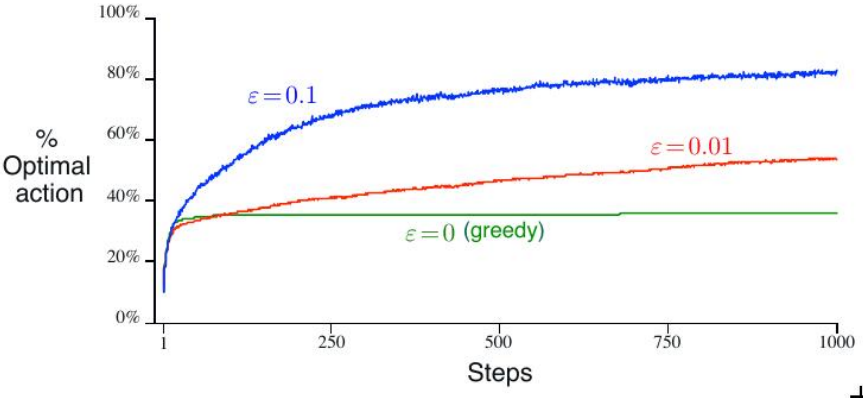
- Pure greedy strategy is sub-optimal
- Initial "learning rate" is more or less equal



Exploration and exploitation

Discovery of optimal actions

- Pure greedy strategy discovers optimal action only $1/3$ of the time



Incremental calculation

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upto n-1

at n

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- We will see this pattern often:

  NewEstimate = OldEstimate + Step [Target - OldEstimate]

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- Exponentially decaying weighted average of rewards

Stationary vs non-stationary

- Non-stationary: Reward probabilities change over time
- Use a constant step $\alpha \in (0, 1]$ — $Q_{n+1} = Q_n + \alpha[R_n - Q_n]$
- $$\begin{aligned} Q_{n+1} &= Q_n + \alpha[R_n - Q_n] \ = \alpha R_n + (1-\alpha)Q_n \\ &= \alpha R_n + (1-\alpha)[\alpha R_{n-1} + (1-\alpha)Q_{n-1}] \\ &= \alpha R_n + \alpha(1-\alpha)R_{n-1} + (1-\alpha)^2 Q_{n-1} \\ &= \alpha R_n + \alpha(1-\alpha)R_{n-1} + \alpha(1-\alpha)^2 R_{n-2} + \cdots + \alpha(1-\alpha)^{n-1}R_1 + (1-\alpha)^n Q_1 \\ &= (1-\alpha)^n Q_1 + \sum_{i=1}^{n} \alpha(1-\alpha)^{n-i} R_i \end{aligned}$$
- Exponentially decaying weighted average of rewards
- Initial value Q_1 affects the calculation — different heuristics possible

Summary

- k-armed bandit is the simplest interesting situation to analyze
- ε-greedy strategy balances exploration and exploitation
- Incremental update rule for estimates

  $$\text{NewEstimate} = \text{OldEstimate} + \text{Step [Target - OldEstimate]}$$

- Exponentially decaying weighted average when rewards change over time (non-stationary)
- UCB action selection — explore actions selectively