Probabilistic Graphical Models

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

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Conditional probabilities

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- Can we strive for something in between?
  - “Local” dependencies between some variables

Probabilistic graphical models

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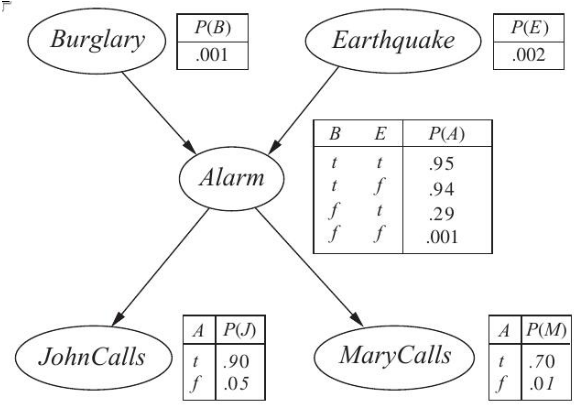
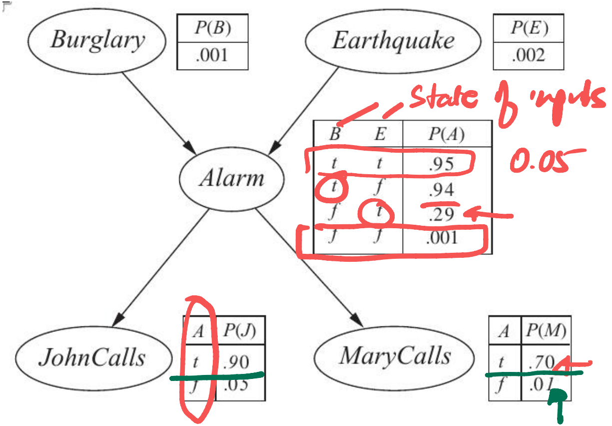
- Example: Burglar alarm
  - Pearl's house has a burglar alarm
  - Neighbours John and Mary call if they hear the alarm
  - John is prone to mistaking ambulances etc for the alarm
  - Mary listens to loud music and sometimes fails to hear the alarm
  - The alarm may also be triggered by an earthquake (California!)



- Each node has a local (conditional) probability table

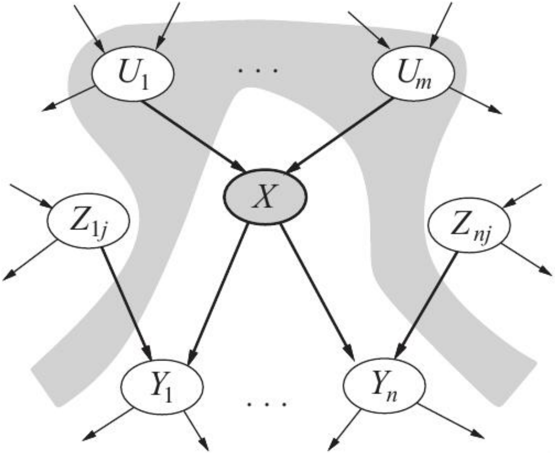
Suppose J & M both call

What is P(Burglary)?



Probabilistic graphical models

- Each node has a local (conditional) probability table
- Fundamental assumption:
  A node is conditionally independent of non-descendants, given its parents



- Each node has a local (conditional) probability table
- Fundamental assumption:
  A node is conditionally independent of non-descendants, given its parents
- Graph is a DAG, no cyclic dependencies

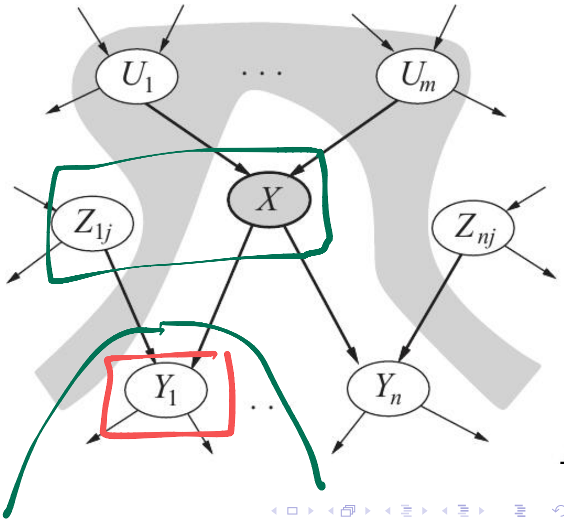
$X \perp Y$

$X \perp Y \mid Z$

Nondescendant

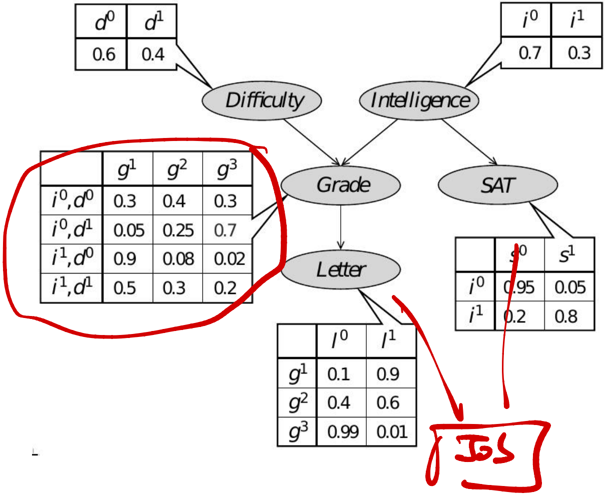
Parents

"Markov Blanket"



Student example

- Example due to Nir Friedman and Daphne Koller
- Student asks teacher for a reference letter
- Teacher has forgotten the student, so letter is entirely based on student's grade in the course



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

Evaluating a network

- John and Mary call Pearl. What is the probability that there has been a burglary?
- $P(b, m, j)$, where b: burglary, j: John calls, m: Mary calls
- $P(b, m, j) = \sum_{a=0}^{1} \sum_{e=0}^{1} P(b, j, m, a, e)$, where a: alarm rings, e: earthquake
- Bayes Rule: $P(A, B) = P(A \mid B)P(B)$
- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2, x_3, \ldots, x_n)$
- Recursively:
  $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n)P(x_n)$

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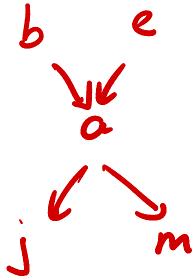

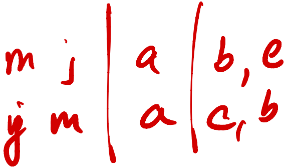

Evaluating a network

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- $P(m, j, a, b, e) = P(m \mid a) P(j \mid a) P(a \mid b, e) P(b) P(e)$

Evaluating a network

$P(m \mid j,a,b,e) \cdot P(j \mid a,b,e)\ P(a \mid b,e)\ P(b \mid e)\ P(e)$

- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n) P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n) P(x_n)$
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$P(m \mid j,a,b,e) = P(m)$ given a, j, b, e

m indep of non-desc. given parent

Evaluating a network

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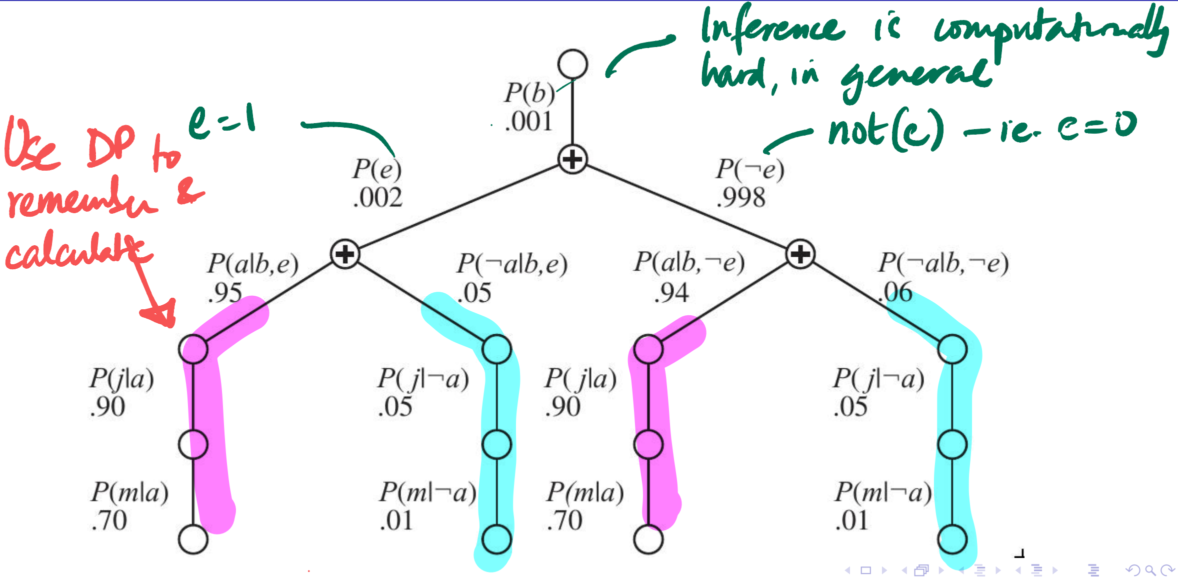
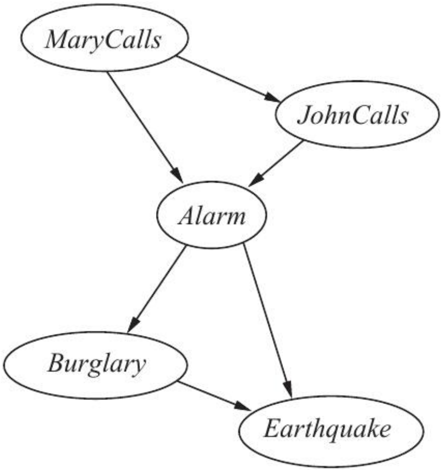
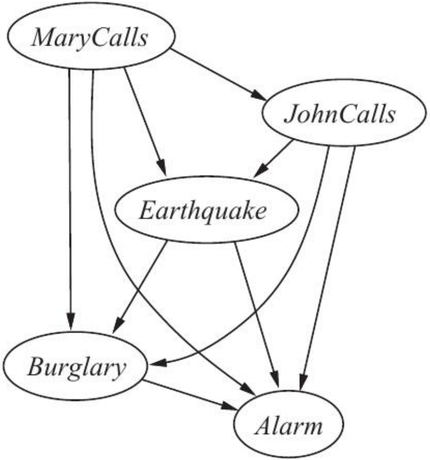
Evaluation tree



Alternative networks





Conditional independence

- $X \perp Y$ — X and Y are independent
  - $P(X \cup Y) = P(X)P(Y)$

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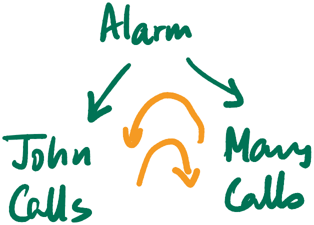
- $X \perp Y \mid Z$ — X and Y are independent if Z is known

Calculational defn

and



Is $J \perp M$? No

Is $J \perp M \mid A$? Yes

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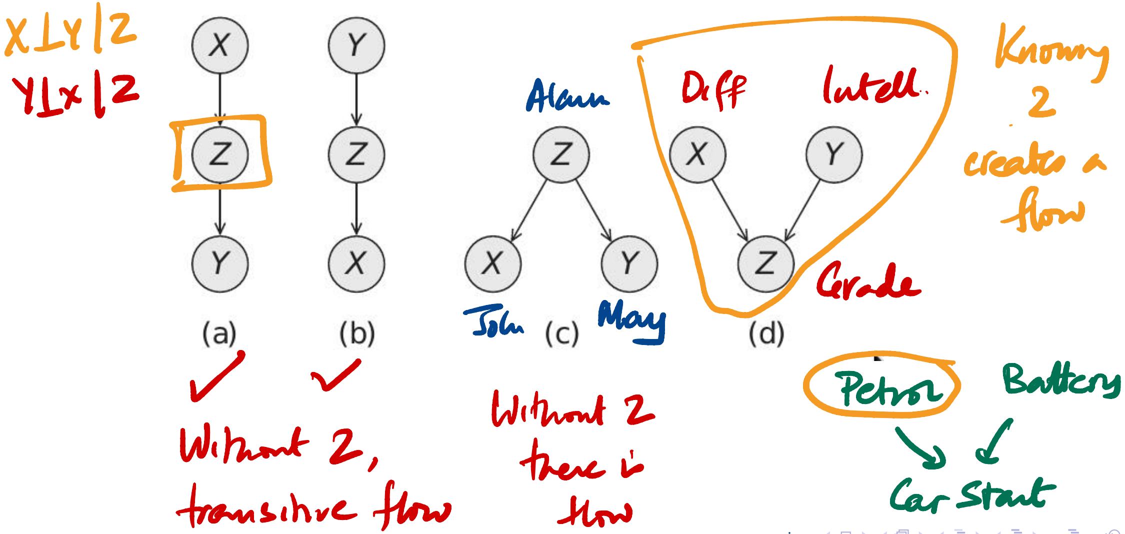
- How does dependence "flow" through a network?
- Construct trails between nodes
  - Path in the underlying undirected graph

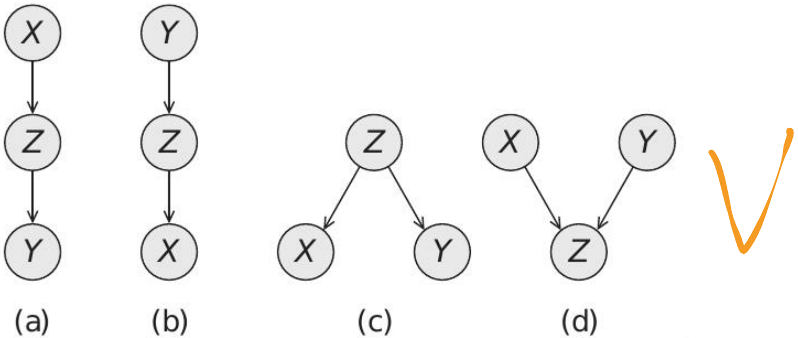
Technically X & Y are sets of variables

Every $x \in Y$ independent of every $y \in Y$

Basic trails



Basic trails



- V-structure in (d) allows influence to flow
- In all other cases, Z blocks flow between X and Y

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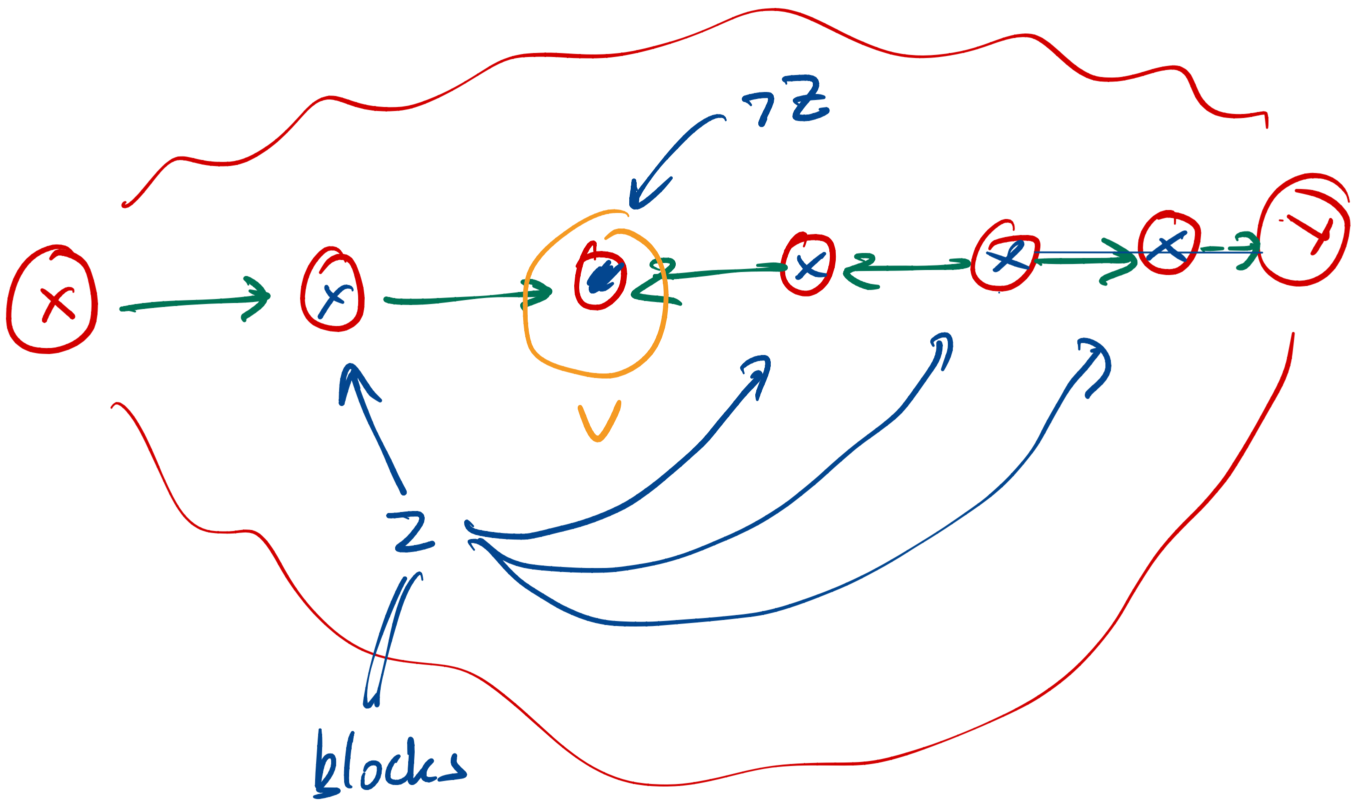
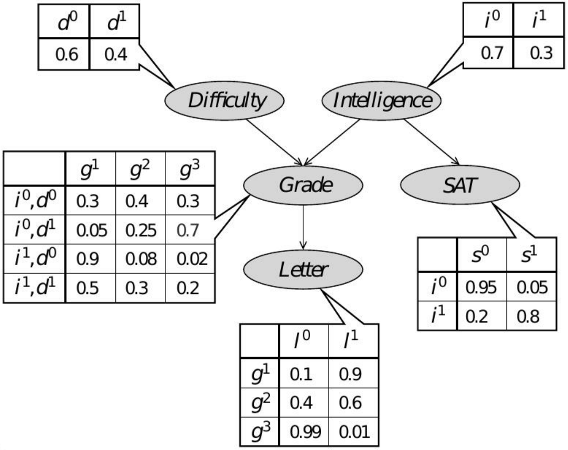
  - $P(X \cup Y) = P(X)P(Y)$
- $X \perp Y \mid Z$ — X and Y are independent if Z is known
- How does dependence "flow" through a network?
- Construct trails between nodes
  - Path in the underlying undirected graph
- X and Y are conditionally independent given Z if Z blocks every trail between X and Y
  - Adapt breadth-first search to check this



X
Y
¬Z
V
Z
blocks

- Is SAT independent of Difficulty given Intelligence?



Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?

- Is SAT independent of Difficulty given Grade?