Probabilistic Graphical Models

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- Can we strive for something in between?
  - "Local" dependencies between some variables

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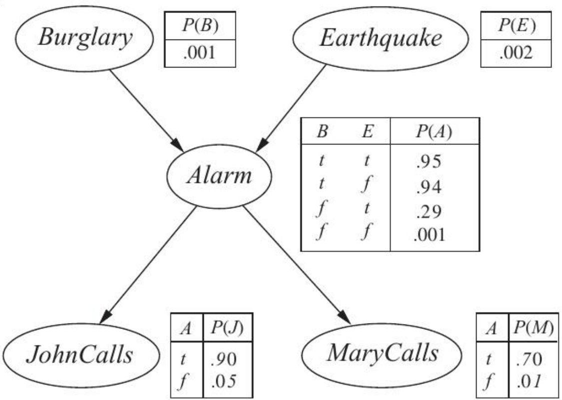
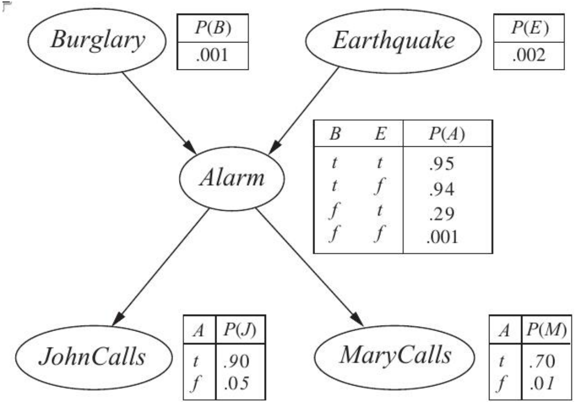
- Example: Burglar alarm
  - Pearl's house has a burglar alarm
  - Neighbours John and Mary call if they hear the alarm
  - John is prone to mistaking ambulances etc for the alarm
  - Mary listens to loud music and sometimes fails to hear the alarm
  - The alarm may also be triggered by an earthquake (California!)



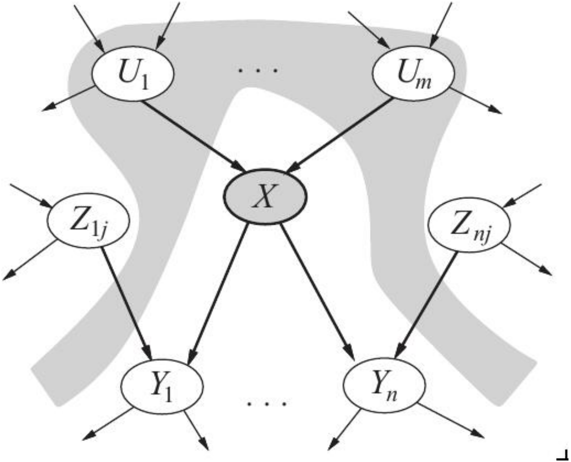
Probabilistic graphical models

- Each node has a local (conditional) probability table



Probabilistic graphical models

- Each node has a local (conditional) probability table
- Fundamental assumption:
  A node is conditionally independent of non-descendants, given its parents



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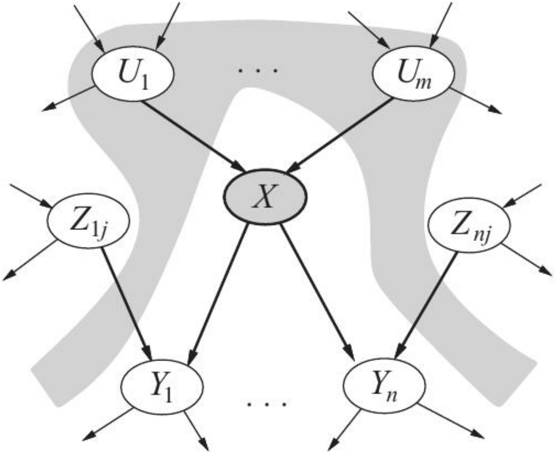
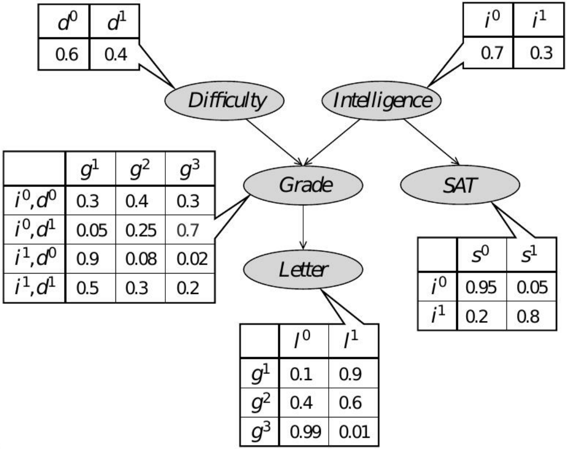
- Graph is a DAG, no cyclic dependencies



Student example

- Example due to Nir Friedman and Daphne Koller
- Student asks teacher for a reference letter
- Teacher has forgotten the student, so letter is entirely based on student's grade in the course



d^0	d^1
0.6	0.4

i^0	i^1
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	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
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	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

Evaluating a network

- John and Mary call Pearl. What is the probability that there has been a burglary?
- $P(b,m,j)$, where b: burglary, j: John calls, m: Mary calls
- $P(b,m,j) = \sum_{a=0}^{1}\sum_{e=0}^{1} P(b,j,m,a,e)$, where a: alarm rings, e: earthquake
- Bayes Rule: $P(A,B) = P(A \mid B)P(B)$
- $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2, x_3, \ldots, x_n)$
- Recursively:
  $P(x_1, x_2, \ldots, x_n) = P(x_1 \mid x_2, \ldots, x_n)P(x_2 \mid x_3, \ldots, x_n) \cdots P(x_{n-1} \mid x_n)P(x_n)$

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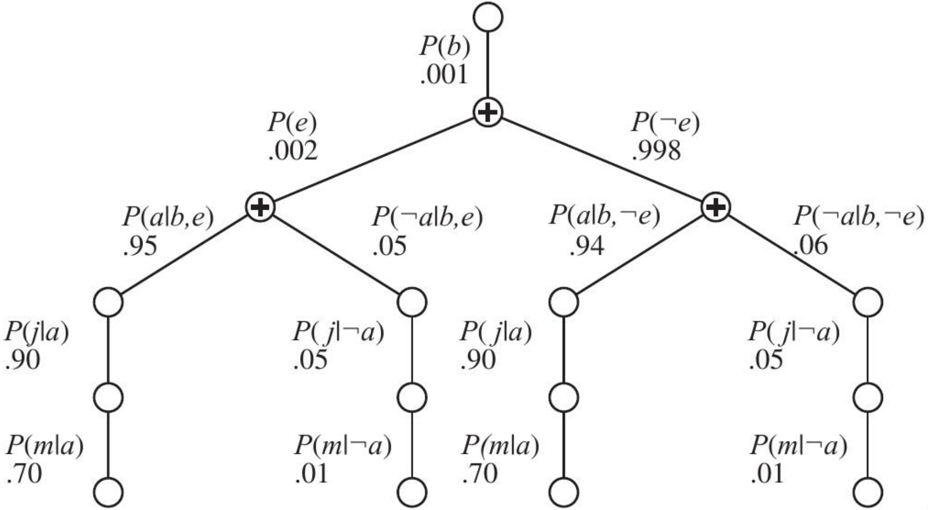
P(b)
.001
P(e)
.002
P(¬e)
.998
P(a|b,e)
.95
P(¬a|b,e)
.05
P(a|b,¬e)
.94
P(¬a|b,¬e)
.06
P(j|a)
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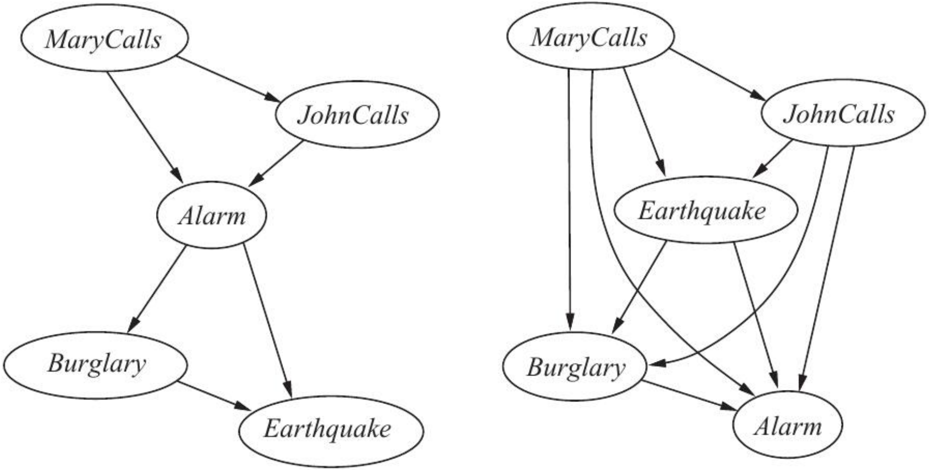
Alternative networks



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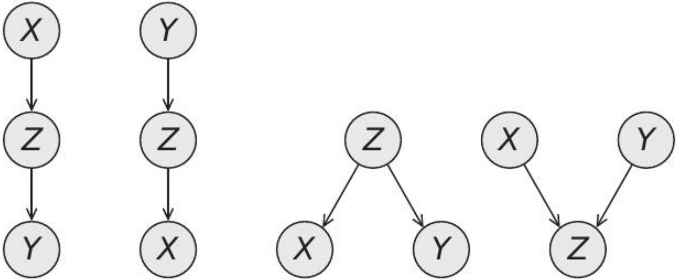
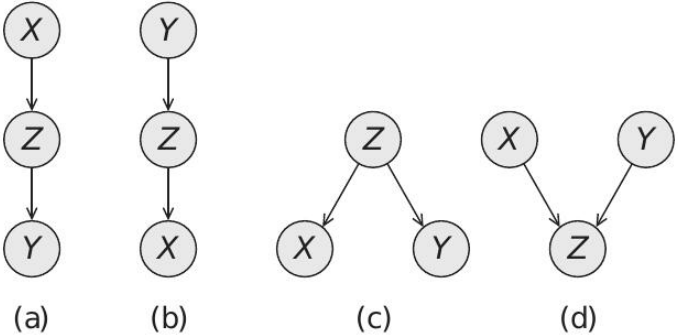
- Construct trails between nodes
  - Path in the underlying undirected graph

Basic trails



(a) (b) (c) (d)

Basic trails



- V-structure in (d) allows influence to flow
- In all other cases, Z blocks flow between X and Y

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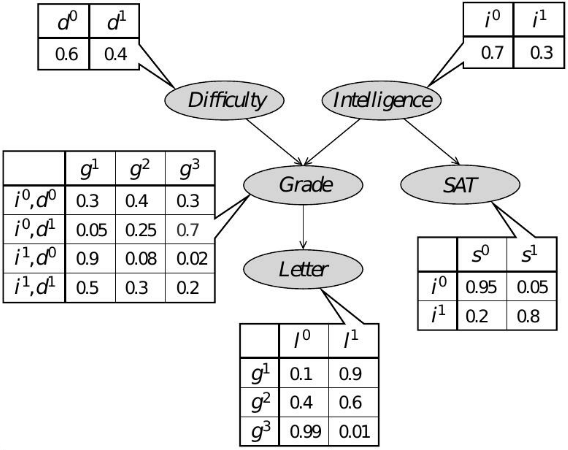
- How does dependence "flow" through a network?
- Construct trails between nodes
  - Path in the underlying undirected graph
- X and Y are conditionally independent given Z if Z blocks every trail between X and Y
  - Adapt breadth-first search to check this

Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure)



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