- Euler tours.
- Euler trails. – G has an Euler trail iff exactly 2 vertices in G have odd degree and the rest have even degree.
- Example graphs.

  K_n.

  $K_{m,n}$

- Subgraphs.
- Induced subgraphs.

- Existence of large bipartite subgraphs.

Algorithmic proof   Probabilistic Proof.

- Hamiltonian Cycles.

Thm: Let G be a graph on $2n$ vertices and $|E|$ edges. Then $\exists$ a bipartite subgraph of G with at least $|E|/2$ edges;

- Algorithmic proof:

$G = (V, E)$

$|V| = n$.

$V = A \cup B$, arbitrarily;

$V = v_1, \ldots, v_n$.

flag = true

while flag **do**

flag = false;

for $i = 1, \ldots, n$ **do**

if v_i is connected to strictly more vertices in its side

move v_i to the other side.

flag = true;

end;

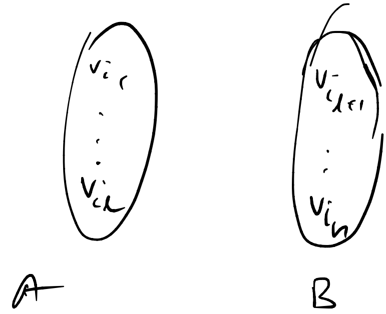
end

end

- Everytime we move a vertex to the other side # edges crossing from A to B increases

$\therefore$ This stops.

Finally:



$\forall i_j$: 2(# edges going from A to B)

$$\geq \sum_{j=1}^{n} \frac{deg(v_{i_j})}{2}$$

$\therefore$

$$2(\text{\# crossing edges}) \geq \frac{\cancel{2}|E|}{\cancel{2}}$$

$\therefore$ # crossing edges

$$\geq \frac{|E|}{2}$$

Probabilistic proof:

For each vertex toss a coin.
If heads put it in A.
o.w. put it in B.

Let $e_1, \ldots, e_{|E|}$ be the edges;

Let $X_{e_i} = \begin{cases} 1 & \text{if end points of } e_i \text{ are on diff sides.} \\ 0 & \text{o.w.} \end{cases}$

$$\mathbb{E}(X_{e_i}) = 1 \cdot \Pr(\text{end pts on opp sides}) + 0 \cdot (\qquad\qquad)$$

$$= 1 \cdot \text{Prob}(\text{end pts on opp side})$$

$$= 1/2$$

$$X = \sum_{i=1}^{|E|} X_{e_i} \qquad \mathbb{E}(X) = \frac{|E|}{2}$$

$\therefore \exists$ a sample point with $\geq \frac{|E|}{2}$ edges crossing (ie) the required bipartite subgraph