- Linear extensions of posets.
- $I(P) \Leftrightarrow |P|\times|P|$ lower triangular matrices.
- Invertibility of $f \in I(P)$
- Inverse of ζ – Divisor lattice; Boolean lattice?
- Suppose P has a minimal element, $\underline{min}$
  $f : P \to \mathbb{R}$
- How to regard f as an element of $I(P)$?

Define: $$g(\underline{min}, x) = \sum_{y \le x} f(\underline{min}, y)$$

$f \overset{?}{=}$ In terms of g?

- Inclusion exclusion:

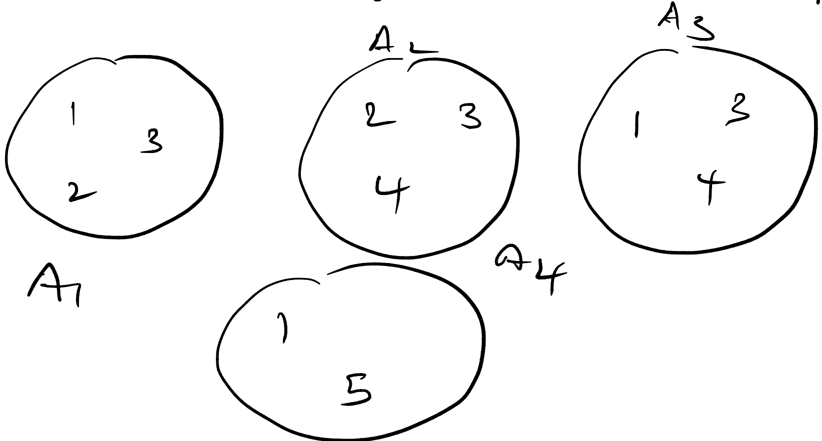
$A_1, A_2, \ldots, A_m \subseteq X$;

$$|X \setminus A_1 \cup \cdots A_m| = |X| - \sum_{i=1}^{n} |A_i| + \sum_{1\le i<j\le m} |A_i \cap A_j| - \cdots + (-1)^m \sum |A_1 \cap A_2 \cdots \cap A_m|$$

$\forall\, I \subseteq [1, \dots, m]$ define:

$$f(I) = \left| \{ x \in X \mid x \text{ is in exactly the sets } A_i, i \in I \} \right|$$

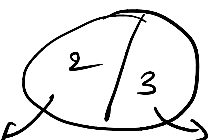
For example



Then: $|f(\{1,2\})| = 1$ $\because$ 2 is exactly in A_1 & A_2

$|f(\{1,2,3\})| = 1$, the only element exactly in A_1, A_2, A_3 is $\{3\}$.

Now: $A_1 \cap A_2$:



contributes to $f(\{1,2\})$

contributes to $f(\{1,2,3\})$

$$g(I) = \left| \{ x \in X \mid x \in A_i \ \forall\, i \in I \} \right|$$

Clearly $g(I) = \left| \bigcap_{i \in I} A_i \right|$

Observation: $\displaystyle |g(I)| = \sum_{I \subseteq J} |f(J)|$

$\therefore$ By Möbius,

$$|f(I)| = \sum_{I \subseteq J} \mu(I,J)\,|g(J)|$$

We are interested in $|f(\phi)|$

$$\|$$

$$X \setminus A_1 \cup A_2 \cdots \cup A_m$$

$$\therefore |X \setminus A_1 \cup \cdots A_m| = \sum_{\phi \subseteq J} \mu(\phi, J)\,|g(J)|$$

$$= \sum_{\phi \subseteq J} (-1)^{|J|} \left|\bigcap_{i \in J} A_i\right|$$

$$= |X| - \sum |A_i| + \sum |A_i \cap A_j| - \cdots$$

Tools

- To calculate Möbius function – find an isomorphic poset where it is easier

- Example:

If $n = p_1 \cdots p_k$, $p_i \neq p_j$, $i \neq j$, p_i, prime

$div(s) \cong B_{[k]}$ - the Boolean poset on k elements.

$\therefore \mu(1, n) = (-1)^k$.

- Check: if $x | y$, then

  - $\mu(x, y) = \mu(1, y/x)$ (Use isomorphism of the posets)

  $\mu(A, B) = \mu(\emptyset, B \setminus A)$, (Use isomorphism of the posets)

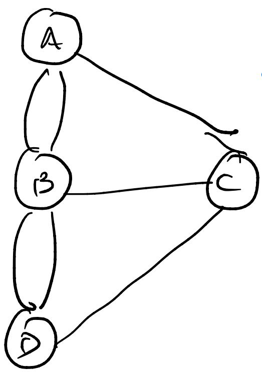
  $= (-1)^{|B \setminus A|}$.

-

Graph theory:

- $G = (V, E)$;
- edges, loops, simple graph; neighbours of v
-

ORIGINS: Königsberg bridge problem.



A Multigraph

- Thm: $G=(V,E)$, non empty, nontrivial graph. Then $\exists$ two vertices in G having equal degree.

Thm: $$2|E| = \sum_{v \in V} \deg(v)$$

- Graphic sequence.