- We introduced partially ordered sets

Let X be a set and R a relation on X

write $x \le y$ if xRy.

- We say R is a partial order iff:

xRx $\qquad$ $x \le x$

If xRy then $\neg(yRx)$ $\quad$ (Irr) $\quad$ $x \le y \Rightarrow y \nleq x$

xRy & $yRz \Longrightarrow xRz$ $\qquad$ $x \le y,\ y \le z \Longrightarrow x \le z$.

- **Ex:** 1) $X = 2^{[n]}$, set of subsets of $\{1, \ldots n\}$;

Say: $s_1 R s_2$ $(s_1 \le s_2)$ iff $s_1 \subseteq s_2$

Check this is a partial order.

2) Fix a +ve integer n and let X be the divisors of n;

If $y_1, y_2 \in X$ say $y_1 \le y_2$ iff $y_1 | y_2$

Check this is a partial order;

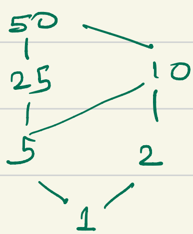
- Hasse diagram of a partial order.

We put an edge b/w x & y if $x \le y$, and $\nexists z$, $x \le z \le y$, $z \neq x, y$;

- Check: This can be done; Ex: Start with maximal elements & write them. Then below that write elements whose immediate successor under $\le$ are maximal elements, and so on;

- Check: Can recover $\le$ from the Hasse diagram.

Ex $n=50$: divisors, $1, 2, 5, 10, 25, 50$



- Games on posets.

- Incidence algebra.

$$I(P) = \{ f : P \times P \to \mathbb{R} \mid f(x,y) = 0 \text{ when } x \not\leq y \}.$$

- Examples:

- zero function; Kronecker delta δ; zeta function ζ

- Algebra Structure? A vector space over $\mathbb{R}$

- $f * g$.

Example: $f * \delta$; $f * 0$, $f * \zeta$

- Identity element;

- Is $f \in I(P)$ invertible; When?

THM:

on n elements,

Lemma: Given any poset P, $\exists f$, a bijective function from $P \xrightarrow{f} [1, 2, \ldots, n]$

s.t.

if $x \le y$ in P then $f(x) \le f(y)$

Proof: By induction.

- $\exists$ a maximal element (we are dealing with finite posets, so no problem) x.
- Send $x \xrightarrow{f} n$,
- Remove x from P and all edges into x. Get P'.
- By induction $\exists f'$, $P' \xrightarrow{f'} [1, \ldots, n-1]$
- Let $f(z) = f'(z) \ \forall z \in P \setminus \{x\}$.

- Fix a linear extension f.

- order elements in P

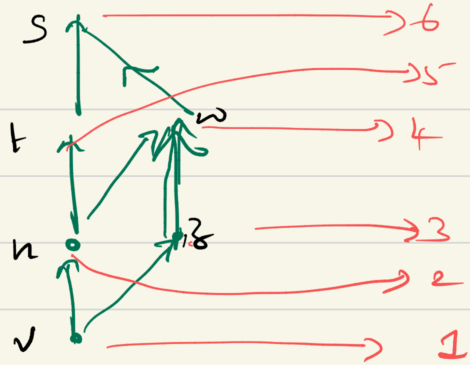
$$\begin{array}{c|ccc} & f^{-1}(1) & \cdots & f^{-1}(n) \\ \hline f^{-1}(1) & & & \\ f^{-1}(2) & & & \\ \vdots & & & \\ f^{-1}(n) & & & \end{array}$$

Let $f \in I(P)$. Write $f(x,y)$ in the above matrix;

Observe: If $\overset{3}{f(x)} < \overset{7}{f(y)}$ (say), then x appears in a row, column earlier than y.

- Entry $(7,3)$ of the matrix? Is zero.
- In fact all entries below diagonal are zero.

Say $P =$



Any f in $I(P)$ in matrix form

$$\begin{array}{c|cccccc} & v & u & z & t & w & s \\ \hline v & * & * & * & * & * & * \\ u & 0 & * & 0 & * & * & * \\ z & 0 & 0 & * & 0 & * & * \\ t & 0 & 0 & 0 & * & 0 & * \\ w & 0 & 0 & 0 & 0 & * & * \\ s & 0 & 0 & 0 & 0 & 0 & * \end{array}$$

- $\dim(I(P))$ = # edges in the graph + # vertices

  check;

- What if we write a different linear order?

- How many linear extensions does a poset have.
- Write a program to calculate the above

Again we don't know a polynomial time algorithm!

- Inverse of ζ function? denote by μ

for the divisor lattice?

$\underbrace{}_{\text{Möbius function}}$

<u>THM:</u> Let P be any poset, and $f : P \to \mathbb{R}$; Suppose P has a $!_0$ minimal element.

Set $$g(y) = \sum_{x \leq y} f(x)$$

Then $$f(y) = \sum \mu(x, y)\, g(x)$$

- Möbius function of the Boolean lattice;
- Möbius " of the divisor lattice;