- In how many ways can you pick $S \subseteq [15]$ so we do not have a, b in S with $a+b \equiv 0 \bmod 3$;

- How many permutations $a_1\, a_2 \ldots, a_n$ of $1..n$ do we have with no $\underbrace{a_{j-1} < a_j > a_{j+1}}_{=}$ $j=2,\ldots,n-1$

- Binomial theorem.

- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

- $\binom{n+1}{k+1} = \binom{k}{n} + \binom{k+1}{k} + \ldots + \binom{n}{k}$

- $\sum k \binom{n}{k} = n \cdot 2^{n-1}$ [choosing a team of some size & a captain]

- $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$

$$\binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}$$

Proof: Subset of size $k+1$

LHS: $\overbrace{\boxed{0 \mid 1 \mid 0 \mid 1 \mid 1 \mid 0 \ - \ - \ -}}^{n+1}$

$\#$ zeros $= k+1$; $\#$ 1's $= n-k$;

Look at the position of the $k+1$ st zero.
And partition the selected based on that.

- If $k+1$'st zero is at position $k+1$, the first k must all be zero $\therefore \binom{k}{k}$ ways.
- If $k+1$'st zero is in pos $k+2$, then among the first $k+1$, we select k zeros $\therefore \binom{k+1}{k}$

$\vdots$

$$\therefore \binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}$$

a) $\{3, 6, 9, 12, 15\}$ S_0

$\{1, 4, 7, 10, 13\}$ S_1

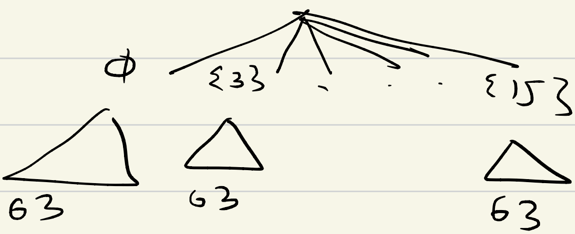
$\{2, 5, 8, 11, 14\}$ S_2

- Can pick none from S_0, or at most 1

$\therefore$ 6 choices.

- Can pick none from S_1, S_2
- Non empty subset of S_1 & None from S_2
- " " " of S_2 & None from S_1

$\rightarrow$ $1+31+31=63$ possibilities.

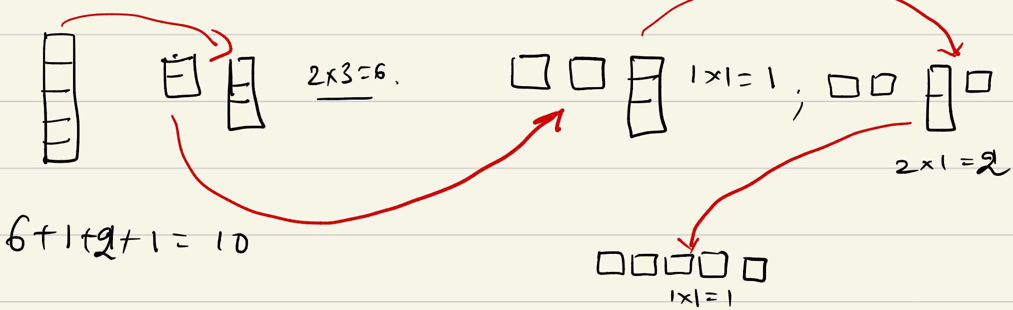


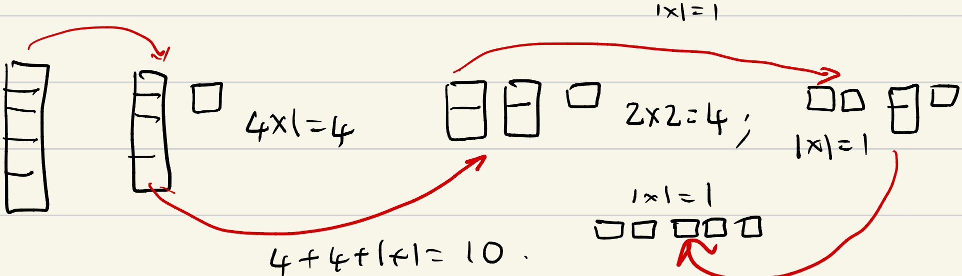
$\therefore$ 63×6 possibilities.

- for $k \le \frac{n-1}{2}$, $\binom{n}{k} \le \binom{n}{k+1}$ – equality iff $n = 2k+1$
- for $k \geq \frac{n-1}{2}$ $\binom{n}{k} \geq \binom{n}{k+1}$.

UNIMODAL sequences.

- $\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}$
- $\sum_{j=0}^{k} \binom{n}{j} = \sum_{j=0}^{k} \binom{n-1-j}{k-j} \cdot 2^j$ $(0 \le k \le n-1)$
- $\sum \binom{n}{i}\binom{i}{k} = 2^{n-k}\binom{n}{k}$.

- 



- Binomial theorem when exponent is not a +ve integer.

$\sum k^2 \binom{n}{k}$:

= Choose a subset and a captain & a vicecaptain

Can choose a captain & vicecaptain first & then build a subset around them.

If the captain & vicecaptain are same.
left with $(n-1)$ $\therefore$ a subset can be chosen in 2^{n-1} ways. $\therefore$ total $n(2^{n-1})$

If the captain & vicecaptain are different.
left with $(n-2)$ $\therefore$ a subset in 2^{n-2} ways.

$\therefore$ # possibilities $= 2^{n-1}n + n(n-1)2^{n-2}$

$$= 2^{n-2}\left[2n + n^2 - n\right]$$

$$= 2^{n-2}(n^2+n) = 2^{n-2}n(n+1).$$

Def:

For any real number m, & $k > 0$, integer,
set $\binom{m}{0} = 1$

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

Taylor Series of $(1+x)^m$ about $x=0$.

$\Rightarrow$

THM: $(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n$

k-th derivative of $(1+x)^m = m(m-1)\cdots(m-k+1)(1+x)^{m-k}$

infinite power series.

If $m>0$ and +ve integer

Ex: Power series expansion of $\sqrt{1-4x}$

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n}(-4x)^n$$

$\binom{1/2}{0} = 1$, $\binom{1/2}{1} = 1/2$

$$\binom{1/2}{n} = \frac{(1/2)(1/2-1)\cdots(\frac{1}{2}-n+1)}{n!} \quad (n\geq 2)$$

- Multinomial theorem:

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{a_1 \ldots a_k \\ \sum a_i = n}} \binom{n}{a_1, a_2, \ldots a_k} x_1^{a_1} \cdots x_k^{a_k}$$

$$\binom{n}{a_1 \ldots a_k} = \binom{n}{a_1}\binom{n-a_1}{a_2} \cdots \binom{n-a_1-a_2 \cdots - a_{k-1}}{a_k}$$

- Compositions:

weak composition of n. $(n \geq 0)$
$(a_1, \ldots, a_k)$, $a_i \geq 0$, $\sum_{i=1}^{k} a_i = n$
integers.

If $a_i \geq 1$, called a composition.

\# weak Compositions of n into k parts?

\# Composition of n into k parts?

Compositions of n: $\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$

subsets of $[n-1]$.

<u>Inductive proof?</u>

· Partitions of the set $\{1, \ldots, n\}$ into non empty subsets. Each nonempty subset is called a block.

partitions into k blocks? $S(n,k)$

Thm: $S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

COR

$\#$ surjective functions. $f : [n] \to [k]$

COR: $\forall$ real numbers x, and $n \geq 0$, integer,

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k.$$