Let $p(k)$ be a polynomial of degree d

Let $q(n) = \sum_{k=1}^{n} p(k)$

Show that q is a polynomial in n of degree $d+1$

Further $q(0) = 0$

<u>Proof:</u> By strong induction on d.

If $d=0$: $p(k) = c$

$\therefore \sum_{k=1}^{n} p(k) = \underbrace{c+c+\cdots+c}_{n} = n \cdot c$ is a

poly in n of degree $\underline{1}$.

. Notice that if the statement is true for $p(k) = k^d$, then the statement is true for $a_d \cdot k^d$; if $\sum_{i=1}^{n} p(i) = b_{d+1} n^{d+1} + b_d \cdot n^d + \cdots + b_1 n$ then

$$\sum_{i=1}^{n} a_d \cdot i^d = a_d \sum_{i=1}^{n} i^d = a_d \cdot b_{d+1} n^{d+1} + a_d \cdot b_d \cdot n + \cdots + a_d \cdot b_1 n$$

· If $p(k) = a_d \cdot k^d + a_{d-1} k^{d-1} + \cdots + a_d$

Then

$r(k) \triangleq p(k) - a_d \cdot k^d$ is a poly of degree $d-1$

$\therefore \sum_{i=1}^{n} r(i) = \sum_{i=1}^{n} p(i) - \sum_{i=1}^{n} a_d \cdot i^d$

is a poly of degree $\leq d$

$\therefore \sum_{i=1}^{n} p(i) = \sum_{i=1}^{n} r(i) + \sum_{i=1}^{n} a_d \cdot i^d$ → If we show it for poly $a_d \cdot k^d$ → is a poly of degree $d+1$ in n.

poly of degree $d+1$ in n. as required

$\therefore$ E.T.S for $p(k) = k^d$.

We will show by explicit calculation that $\exists$ a poly $z(n)$ of degree $d+1$ in n s.t. $z(n+1) - z(n) = n^d$.

Further $z(0) = 0$.

Then

$$1^d + 2^d + \cdots + n^d = \big(z(1) - z(0)\big) + \big(z(2) - z(1)\big) + \cdots + z(n+1) - z(n)$$

$$= z(n+1) - z(0)$$

$$= z(n+1)$$

=

To show this:

Say $z(n) = a_{d+1} n^{d+1} + a_d n^d + \cdots + a_1 n$.

$$z(n+1) - z(n) = a_{d+1}\left[(n+1)^{d+1} - n^{d+1}\right] + a_d\left[(n+1)^d - n^d\right] + \cdots$$

Want this to be $$+ a_1\left[n+1 - n\right]$$

$$f(n) = a_{d+1} n^{d+1} + a_d n^d + \cdots + a_0$$

$$f(n+1) = a_{d+1}(n+1)^{d+1} + a_d (n+1)^d + \cdots + a_0$$

$$f(n) = a_{d+1} n^{d+1} + a_d n^d + \cdots \cdots + a_0$$

$$f(n+1) - f(n) = a_{d+1}\left[(n+1)^{d+1} - n^{d+1}\right] + a_d\left[(n+1)^d - n^d\right] + \cdots + a_1(n+1-n)$$

$$n^d = a_{d+1}\left[\cancel{n^{d+1}} + \binom{d+1}{1} n^d + \cdots + 1 - \cancel{n^{d+1}}\right]$$

$$+ a_d\left[\cancel{n^d} + \binom{d}{1} n^{d-1} + \cdots + 1 - \cancel{n^d}\right]$$

$$+ \cdots$$

Want:

$$a_{d+1} \binom{d+1}{1} = 1 \quad \therefore a_{d+1} = 1/(d+1)$$

$$a_{d+1} \binom{d+1}{2} n^{d-1} + a_d \cdot d \left(n^{d-1}\right) = 0 \cdot n^{d-1}$$

$\therefore$ can solve for a_d;

$$\frac{1}{\cancel{d+1}} \cdot \frac{(\cancel{d+1})(d)}{2} + a_d \cdot d = 0 \qquad \therefore a_d = -1/2.$$

*The above facts applications in algebraic geometry.

- The Hilbert function of the coordinate ring of a variety & the Hilbert polynomial.

Let $n_1+n_2+\dots+n_k = n$.

s t

$$n_1^2+\dots+n_k^2 \le (n-k+1)^2 + k-1.$$

Pf:

- SHOW THAT:

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}$$

· PIGEON HOLE :

- More than kn marbles are distributed over n jars. Then one jar contains $\geq k+1$ marbles.

- Ex: Bishops on a 8×8 chessboard

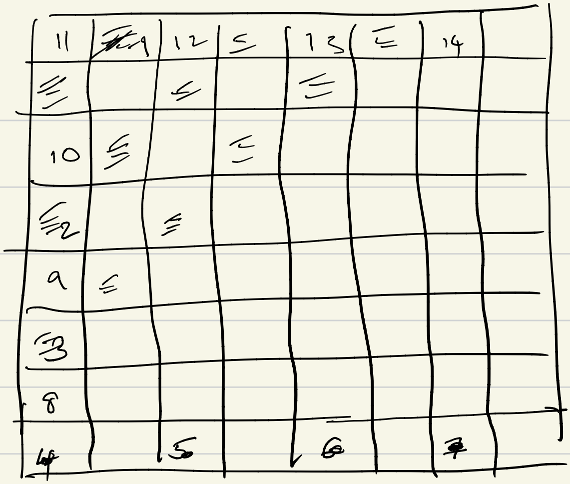
- 9, 99, 999, ... $\exists$ an element divisible by 2019

- $S \subseteq \{1, .. 2n\}$, How large can S be so that $\forall a, b \in S$, $a \nmid b$

- S a set of n numbers, non negative. There is a subset $T \subseteq S$, s.t $\sum_{x \in T} x \equiv 0 \bmod n$.

- $mn+1$ numbers. $\exists$ a $\uparrow$ subsequence of size $m+1$ or a decreasing " of size $n+1$..

Bishops:



- For each odd number $2k-1$, $k = 1, 2, \ldots, n$,

  Consider

  $$(2k-1, (2k-1)\cdot 2, \ldots, (2k-1)2^l) \cap [1, \ldots 2n].$$

  These are a partition of $[1, \ldots 2n]$. Totally n parts. We can select at most one from each $\therefore$ $\underline{\underline{n}}$.

- $\uparrow$ and $\downarrow$ subsequence:

  $\forall i$, Consider the length l_i largest $\uparrow$ subsequence ending in a_i & the length of the largest $\downarrow$ subsequence d_i, ending in a_i (i.e.) $i \to (l_i, d_i)$.

  Observe that if $i \neq j$, $(l_i, d_i) \neq (l_j, d_j)$

- $\therefore$ If $a_i > a_j$, then d_j is at least $d_i + 1$

  If $a_i < a_j$ then l_j is at least $l_i + 1$

Now if $\forall i$, $l_i \leq m$, $d_i \leq n$, the total number of pairs (l_i, d_i) is at most <u>mn</u>.
But we have $mn+1$.