Network flows:

- **flow**: - antisymmetric; $f(u,v) = -f(v,u)$
  - $f(u,v) \leq c(u,v)$
  - Kirchoff's law:

Given a flow in a network G, we define residual capacity:

$$c_f(u,v) = c(u,v) - f(u,v)$$

- $u \xrightarrow{f|c} v$ $\qquad \xrightarrow{c-f}$

- Assume that $(v,u) \notin E$,

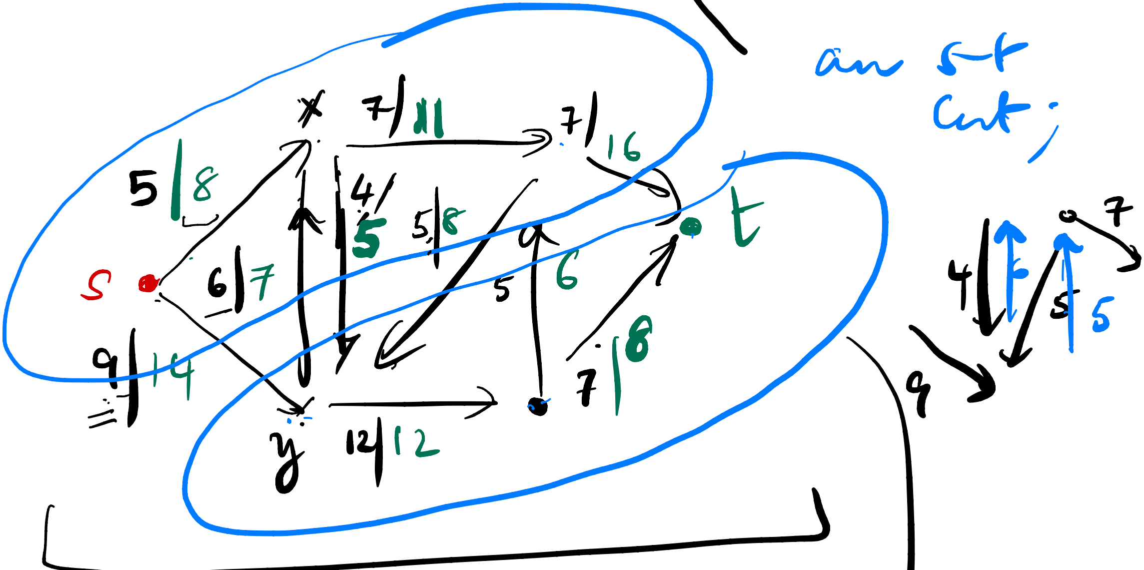
then $u \xrightarrow{f|c} v$

$$\begin{tikzcd} u \arrow[r, bend left, "c-f"] & v \arrow[l, bend left, "f"] \end{tikzcd}$$

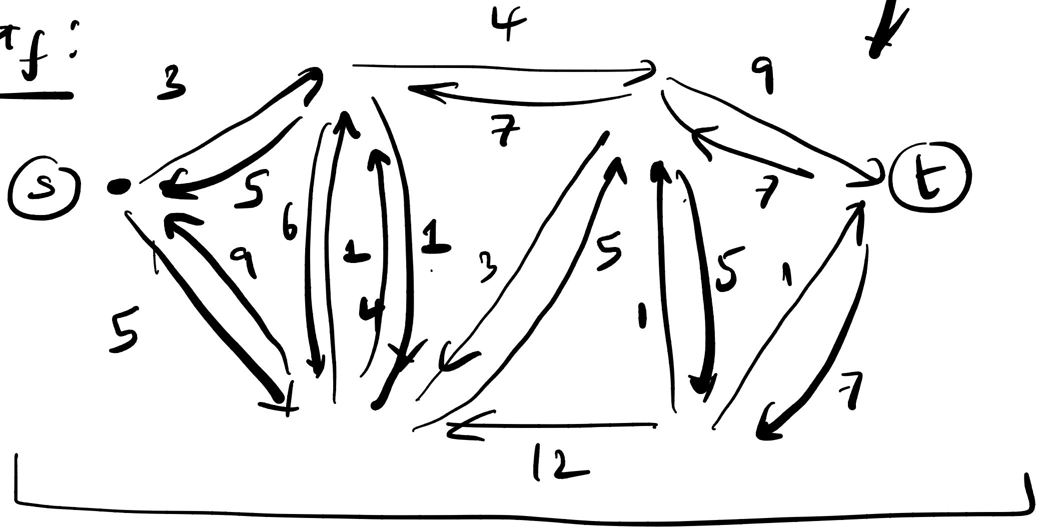
$$c_f(v,u) = c(v,u) - f(v,u) = 0 - (-f(u,v))$$

$f(u,v)$ ||

$$\boxed{G_f: \{(u,v) \mid c_f(u,v) > 0\}}$$



G_f:



$$\# E(G_f) \le 2|E|$$

Let f' be a flow in the residual graph G_f;

$$f : V \times V \to \mathbb{Z}.$$
$$(E^T \cup E) \to \mathbb{Z}$$

Define:

$$(f + f')(u,v) = f(u,v) + f'(u,v).$$

Claim: $f + f'$ is a flow on G.

$$(f+f')(u,v) = f(u,v) + f'(u,v)$$
$$= -f(v,u) - f'(v,u)$$
$$= -(f+f')(v,u).$$

$$(f+f')(u,v) = f(u,v) + \underbrace{f'(u,v)}$$
$$\leq f(u,v) + \underbrace{c(u,v) - f(u,v)}_{c_f(u,v)}$$
$$\leq c(u,v)$$

$\forall\, u \in V - \{s,t\}$;

$$\sum_{v} (f+f')(u,v) = \underbrace{\sum_{v} f(u,v)}_{=\,0} + \underbrace{\sum_{v} f'(u,v)}_{0}$$

$$= 0$$

$$\boxed{|f'| = \sum_{v \in V} f'(s,v)}$$

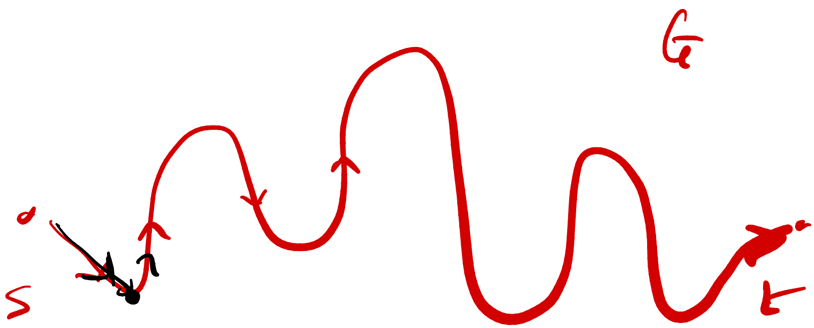
$$\underbrace{|f+f'|}_{} = \sum_{v} \underbrace{(f+f')(s,v)}_{}$$

$$= \sum_{v} f(s,v) + \sum_{v} f'(s,v)$$

$$= |f| + |f'|.$$

$\times$

- Augmenting alg:



Let p be a path in G_f, from $s \to t$;

Let $c_f(p) = \min \{ \underbrace{c_f(u,v)} : (u,v) \in p \}$.

$f_p : V \times V \to \mathbb{Z}f$.

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \in p \\ -c_f(p) & \text{if } (v,u) \in p \\ 0 & \text{o.w.} \end{cases}$$

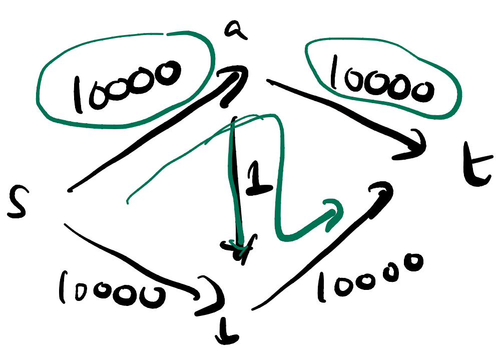
Claim: f_p is a valid flow in G_f;

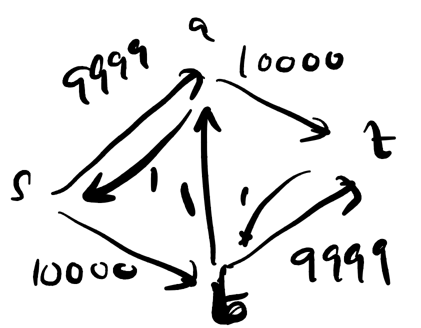
$|f_p| = c_f(p)$.

Ford Fulkerson alg.:

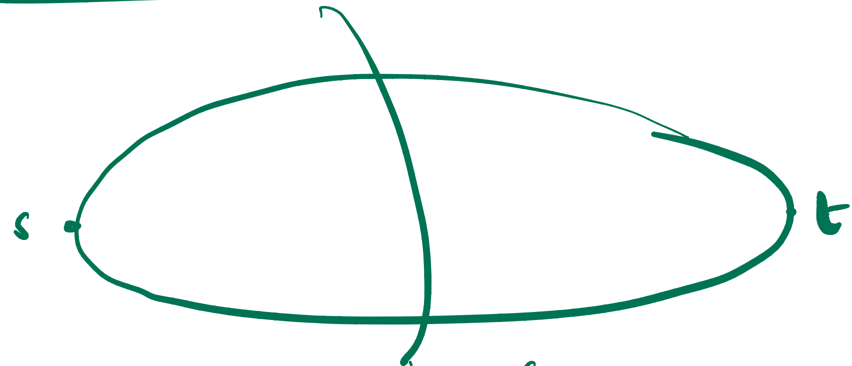
- **Algorithm:**

1) Input G;
2) Find flow f in G;
3) Construct G_f;
4) Find an s-t path p in G_f; construct flow f_p o.w. Declare f to be a maximum flow;
5) $f \leftarrow f + f_p$
6) Go to 3;





Capacity of a cut:



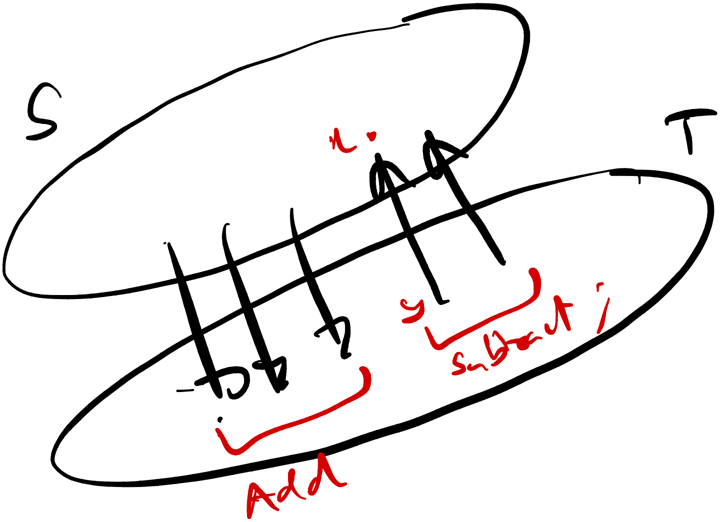
An (s,t) cut in G is a partition of V, $= S \cup T$, $s \in S$, $t \in T$.

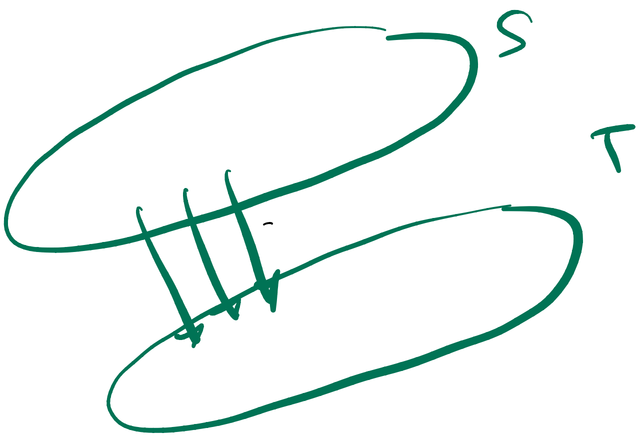
$S \cup \{V \setminus S\}$.

$f(S,T)$ is flow across cut S,T

$$= \sum_{u \in S} \sum_{v \in T} f(u,v).$$

Capacity of a cut: (S,T) ; $\sum_{u \in S, v \in T} c(u,v)$.





Lemma: $f(S,T) = |f|$.

$$f(S,\underset{V\setminus S}{\underset{\parallel}{T}}) = f(S,V) - \underbrace{f(S,S)}_{0}$$

$$= f(S,V)$$

$$= f(s, V) + f(S - \{s\}, V)$$

$$= |f| + \underline{\sum_{u \in S - \{s\}} f(u, V)}$$

$$= |f| + \sum_u 0$$

$$= |f|;$$

<u>Lemma</u>: Value of the flow ($|f|$) is upper bounded by the capacity of any cut in G;

$$|f| = f(S, T) = \sum_{u \in S} \sum_{v \in T} \underline{f(u,v)}$$

$$\leq \sum_{u \in S} \sum_{v \in T} c(u,v) = c(S, T)$$

Max flow min cut theorem:

Let G be a network with s, t;
The max flow in G from s to t
is equal to

$$\min_{\substack{V = S \cup T, \\ s \in S,\, t \in T}} c(S,T)$$

Proof: Let f be a flow in $G = (V,E)$;

TFAE:

1. f is a max flow
2. G_f has no aug path;
3. $|f| = c(S,T)$ for some (S,T) cut;

$1 \Rightarrow 2$ ✓ o.w. $|f + f_p|$ is bigger than $|f|$;

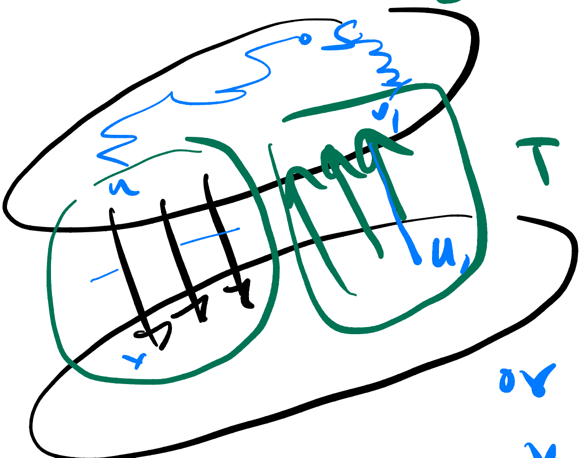
$2 \Rightarrow 3$:

Look at G_f:

$S = \{v \mid \exists$ a path in G_f from s to $v\}$

$s \in S$; $t \notin S$; $T = V \setminus S$; $t \in T$;

Claim: $\underline{|f| = c(S,T)}$.



$|f| = f(S,T)$

$\underline{f(u,v) = c(u,v)}$.

$< c(u,v)$

or $u \to v \in G_f$; $v \in S$; ⊗

$u_1 \xrightarrow{0/u} v_1$ if flow is not zero, in

G_f: $v_1 \to u_1$ so $u_1 \in S$ ⊗

In G_f . p: $s \longrightarrow b \to a \to t$;

$\therefore$ $c(f_p) = 1$;