April 14 - 2020

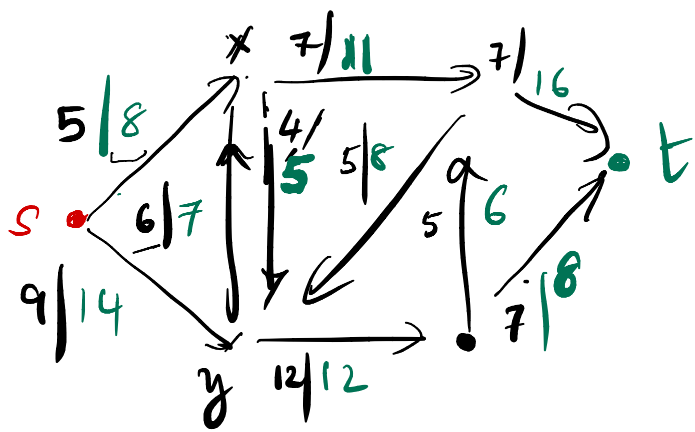
- Network flows:

Single source single sink.

$G = (V, E)$ - directed.

$c: E \to \mathbb{N}$ - capacity function,

$s, t \in V$, s - source & t - sink;



$5+9=14$.

$7+7=14$

- Looking for a flow from s to t

A flow is a function on edges;

$f : E \to \mathbb{Z}$

VE

(a) $u \to v \in E$,

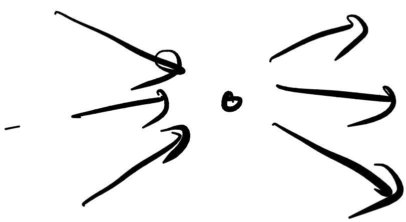
$$f(u,v) \le c(u,v) \to$$

(b) $\boxed{f(u,v) = -f(v,u)}$; ANTISYM

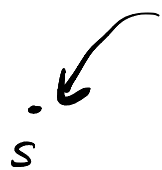
(c) Conservation of flow;

$u \in V \setminus \{s,t\}$,

$$\boxed{\sum_{v} f(u,v) = 0.}$$



$$\boxed{|f| = \sum_{v \in V} f(s,v)}$$



MAX FLOW:

Given G find max flow;

- there is a poly time alg for this;

$X, Y \subseteq V$, $\quad f(X, Y) \triangleq \sum_{\substack{x \in X \\ y \in Y}} f(x, y)$

- $|f| = f(\{s\}, V)$. $\rightarrow$

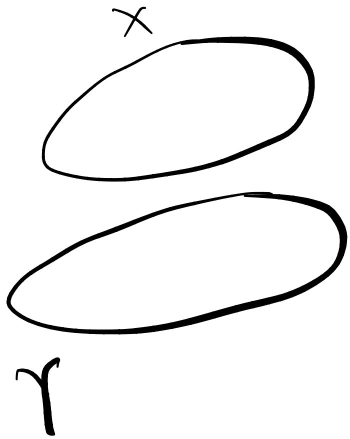
- **Obs:** (a) $\forall u \in V$, $f(u,u) = 0$;

(b) $\underline{f(X, X)}$, $X \subseteq V$,

$\qquad \| \; 0$

$$= \sum_{u \neq v} \Big[\underbrace{f(u,v) + f(v,u)}_{=0} \Big] + \underbrace{\sum_{u \in X} f(u,u)}_{=0}$$

$= 0$



$X \cap Y = \emptyset$,

$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$

$X \cap Y = \phi$.

- $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$
- $\boxed{u \neq s, t}$, $f(u, V) = 0$. $\quad \sum_{v \in V} f(u, v)$
- $\boxed{|f| = \sum f(s, V)}$ ✓

Claim: $\boxed{|f| = f(V, t)}$ ←

Proof:

$$
\begin{aligned}
|f| &= f(s, V) = f\big(V - (V - s), V\big) \\
&= \underbrace{f(V, V)}_{=\,0} - f(V - s, V) \\
&= f(V, V - s) \\
&= f(V, t) + f\big(V, V - \{s,t\}\big) \\
&= f(V, t) + \underbrace{\sum_{u \neq \{s,t\}} f(V, u)}_{=\,0} \\
&= f(V, t).
\end{aligned}
$$

<u>**Residual graph**</u>: Given f, $\underbrace{G_f}$ – is the residual graph

• residual capacity of an edge;

$(u,v) \in E$

$$f(u,v) \le c(u,v).$$

$$\boxed{c_f(u,v) = c(u,v) - f(u,v)}$$

↑ residual capacity of edge (u,v).

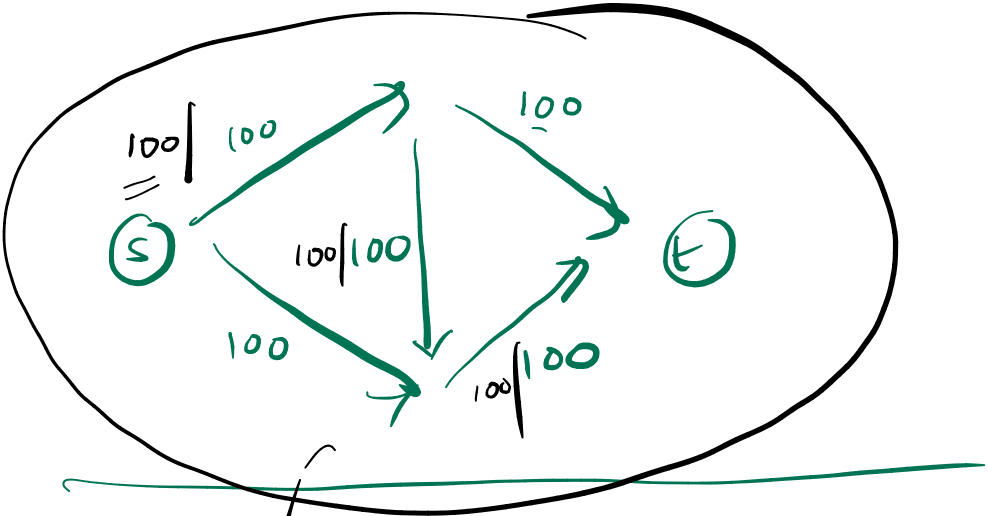
$\forall u \in V, v \in V;$

$$u \xrightarrow{5/8} v$$

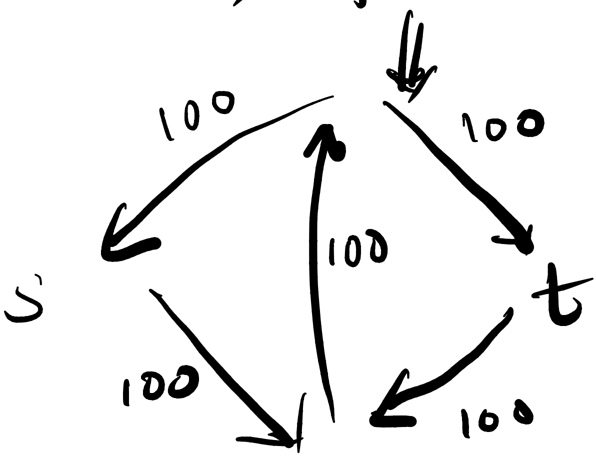
Residual network:

$$\boxed{E_f = \{(u,v) \mid c_f(u,v) > 0\}}.$$

$$c_f(u,v) = 3 \quad u \xrightarrow{3} v \qquad c_f(v,u) = 5 \quad v \xrightarrow{5} u$$



G_f



$\boxed{\text{Given } f \text{ get } G_f}$

augmenting flow $f \longrightarrow f+h$

$$\underline{|f+h| = |f| + |h| > |f|}.$$

Max flow

If there is no path from s–t in G_f, then $|f|$ is maximum flow

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• Max flow min cut theorem:

• Kleinberg Tardos

or C.L.R. Cormen, Leiserson, Rivest;