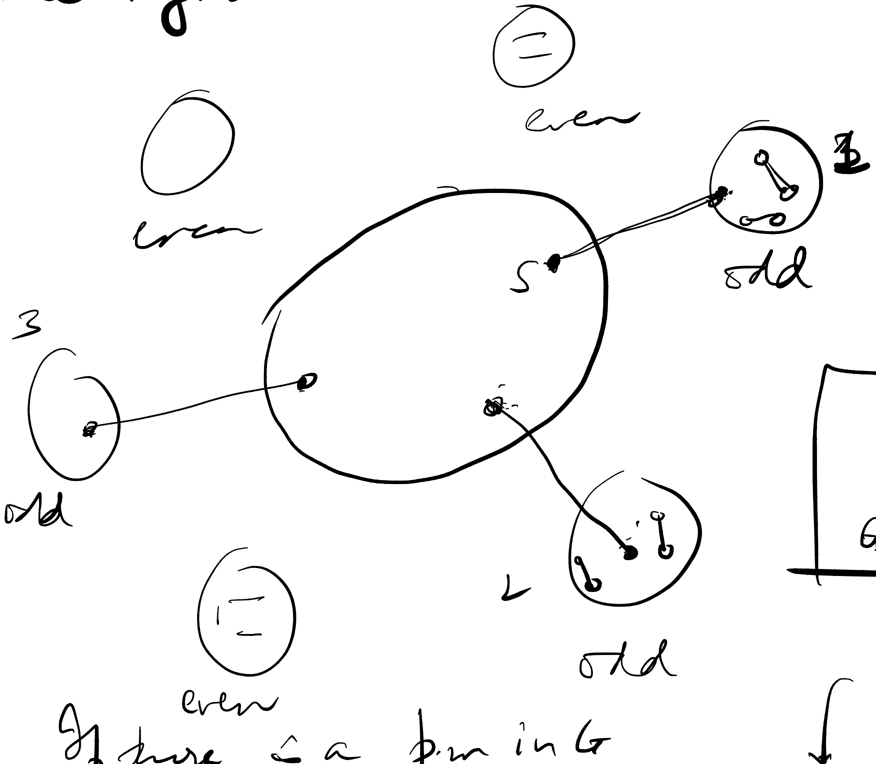
Tutte's theorem: -

PERFECT MATCHINGS.

~~Matchings~~ Perfect Matchings in general graphs:

Algorithm: Edmonds - 1965 - polynomial time algorithm.



$S \subseteq V$;

$G - S$.

If there is a p.m in G

$\forall S \subseteq V$ $\boxed{\#\text{ odd components in } G-S, \le |S|.}$ ✓

✓

TUTTE: Let G be connected. Then

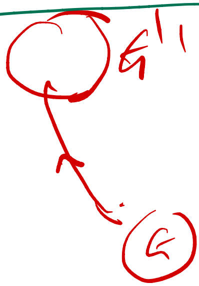
G has a pm iff
$\forall S \subseteq V$:
#odd components in $G-S$, $\le |S|$.

$\Longrightarrow$:

$\Longleftarrow$ $\boxed{\forall S \subseteq V, \ |C_0(G-S)| \le |S|,}$ ✓

Let $e \notin E(G)$, suppose $G' = G \cup \{e\}$ also has no pm,

$$\underbrace{c_0(G'-S) \le c_0(G-S) \le |S|}$$

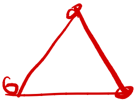


$e_1, e_2, \text{- - - -}, e_{|E|}$
$\downarrow \quad \downarrow$
?
✓ × × ✓ ✓

G' is maximal, with no perfect matching.

* * *

SATURATED NON FACTORIZABLE GRAPH (G'')



Structure of saturated non factorizable graph: (G)

ⓐ G has an odd # vertices and is a complete. ✗

ⓑ G has an ~~even~~ # vertices and has

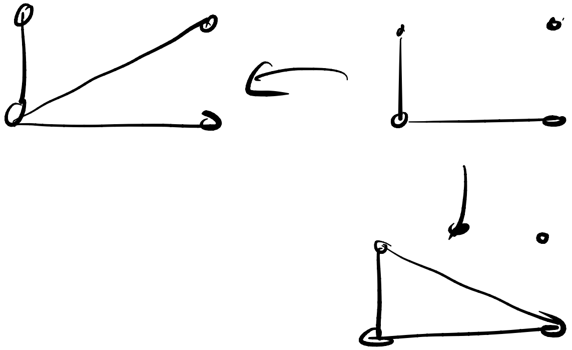
$k = |S_0| + 2$

G_i has odd # of vertices,

G_i is complete,

S_0 is complete

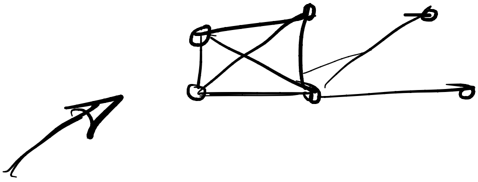
- # odd$(G' - S)$

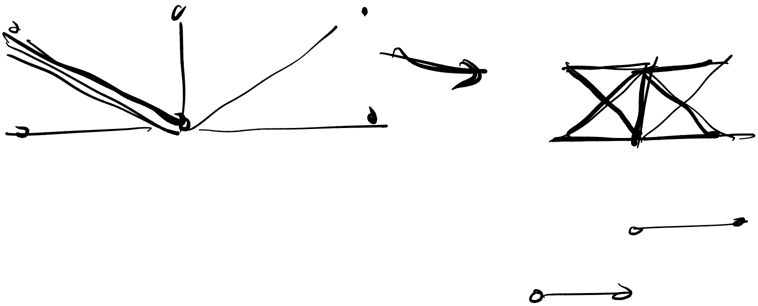


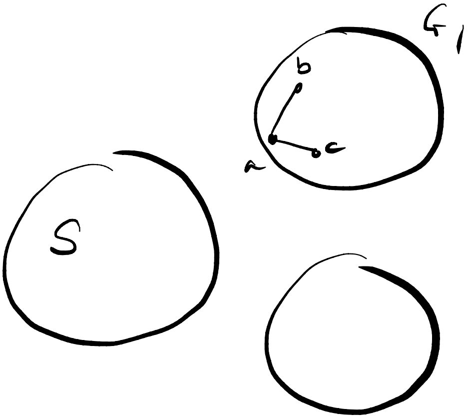
- **Lemma:** Suppose G is saturated & non factorizable. Let $S \subseteq V$, be be vertices of degree $n-1$ in G; then the components of $G-S$ are complete graphs;

* * *

Here is an example where maximal graphs have diff # edges.





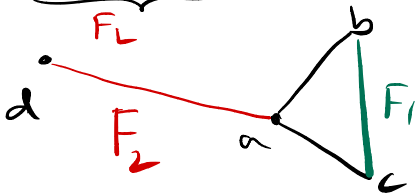


Let $a\,\begin{array}{l}/\,b\\ \backslash\,c\end{array}$ in component G_1

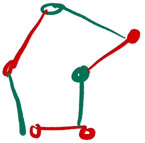
that $b - c$ is an edge.

$a \notin S$, Why?
$\because$ $a - d$ is missing for some d;

$\underbrace{G \cup \{bc\}}_{F_1}$ $\underbrace{G \cup \{ad\}}_{F_2}$



$\boxed{F_2 \oplus F_1}$: degree $\underline{0, 1, 2}$.

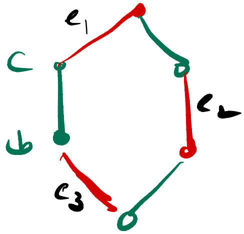


A cycle containing  C_1

A cycle containing  C_2

① $\underline{C_1 \neq C_2}$:

Take $\underbrace{\boxed{F_1 \oplus C_1}}_{\cancel{\phi}}$



bc but e_1, e_2, e_3 are in $F_1 \oplus C_1$.

But then $F_1 \oplus C_1$ is a matching in G. ✗

② $\boxed{C_1 = C_2}$:

every vertex in S_0 is connected to every vertex in each t_i.