Number System

- Finding the successor of a natural number written in base b

$\Longrightarrow$ $x_{n-1}x_{n-2}\ldots x_0$ represents

$x_{n-1}b^{n-1} + x_{n-2}b^{n-2} + \cdots + x_0$.

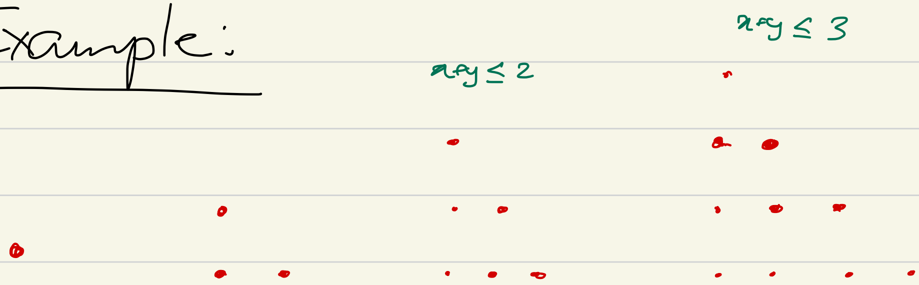
PRINCIPLE of INDUCTION:

Let $P(n)$ be a propositition about natural numbers, n. Suppose $P(1)$ is true.
Suppose if $P(n)$ is true, then $P(n+1)$ is true.

THEN $P(n)$ is true $\forall n$;

Example:

1)



Covering the above points on a plane using lines.

2) $n > 1$ an integer. $\mathcal{I}$ a set of closed intervals

Natural numbers

The endpoints u, v of each interval

s.t. $1 \le u < v \le n$;

$I_1, I_2 \in \mathcal{I}$, then $I_1 \cap I_2 = \phi$ or $I_1 \subset I_2$

or $I_2 \subset I_1$.

- Covering S_n:

Let $\mathcal{L}$ be a collection of lines covering S_n.

- If the line $l \stackrel{\Delta}{=} x+y=n \in \mathcal{L}$ - the remaining lines cover points in S_{n-1} - no point in S_{n-1} is covered by l; By induction $\mathcal{L} \setminus \{l\}$ has cardinality at least n $\therefore$ $\mathcal{L}$ has cardinality $n+1$

- If $l \notin \mathcal{L}$: Each line in $\mathcal{L}$ contains at most one point from $x+y=n$; There are $n+1$ points $\therefore$ at least $n+1$ lines.

QED

· Claim: $|I| \leq n-1$; In fact the largest number of intervals one can select is $n-1$.

Ex: $[1,n], [1,n-1], \ldots, [1,2]$.

w.l.o.g $[1,n] \in I$.

If there is no other closed interval $[1,j]$, in I, then I contains $[1,n]$ and intervals covering $[2,n]$. But by induction # intervals we can pick with $2 \leq u, v \leq n$, is at most $n-2$ $\therefore$ $|I| \leq n-2+1 = n-1$;

- On the other hand if I contains $[1,j]$ for some j; Among all such intervals pick the one where j is max s.t $j = k$;

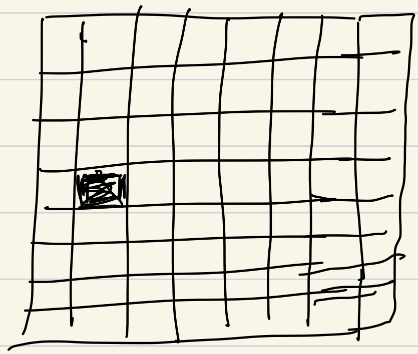
$\therefore$ $I = [1,n] \cup$ segments covering $[1,k] \cup$ segments covering $[k+1,n]$

By ind # intervals covering $[1,k]$ is at most $k-1$, and # " in $[k+1,n] \leq n-k-1$

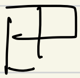
$\therefore$ # intervals in $I \leq 1 + k-1 + n-k-1 = n-1$;

QED

A $2^n \times 2^n$ chessboard with a hole;

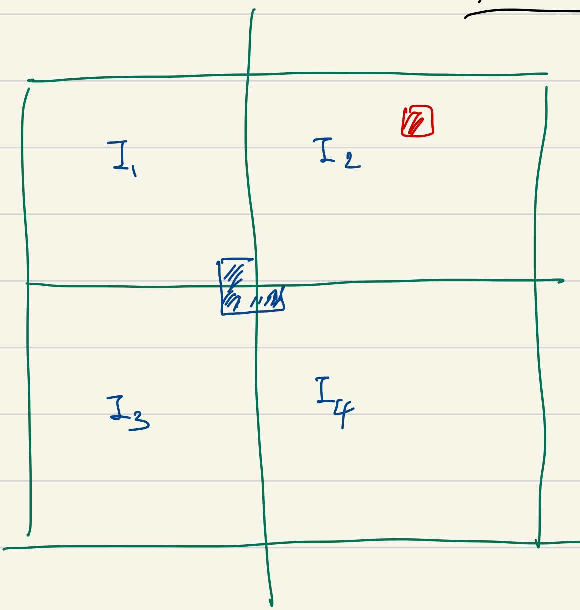


Claim:

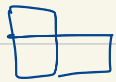


Allowed dominoes

Clear:



If the hole is in I_2, select the domino as shown; By induction each of I_1, I_2, I_3 & I_4 are

Square chessboards of size $2^{n-1} \times 2^{n-1}$ with a hole and so can be covered by dominoes of the type  .

QED

· On an infinite sheet of white paper, n squares are coloured black. At time $t = 1, \ldots$ squares are recoloured using:
each square gets the colour occurring at least twice in the triple formed by the square, its top neighbour and its right neighbour.
Show that all squares are white after $t = n$.

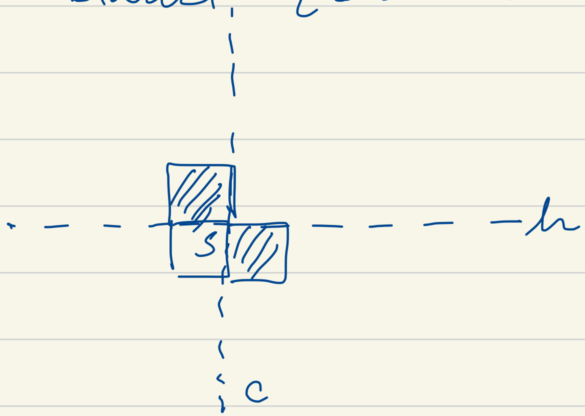
Proof:

Let h be a horizontal line extending to ∞ on either side with the # black squares on or above l being k;
Let c be a vertical line extending to ∞ on both sides with # black squares on c or to the right of c being k;
Claim: For time step k and beyond, all squares to the right of h and above c are white (please check!)

• We prove the claim by induction on k.

To complete the proof

Let S be the square s.t. all the given black squares are north/east of S;



Case ①: The two squares to the right of S & above S are black.

Then at $t=1$, S goes black;

But # black squares to the right and on c $\le n-1$. So too on h & above.

$\wedge$ By time $t = n-1$ all above h & to the right of c are white and remain white beyond $n-1$;

But then at n, s goes white & we are done;

(Please check ↑)

Variation:

$$\forall n \left[\forall m < n,\ P(m) \Longrightarrow P(n) \right] \Longrightarrow \forall n\ P(n);$$

Proof:? $Q(n)$ be the statement "$P(m)$ holds $\forall\ m < n$";

= This is equivalent to the first version of PI; (Please make sure you understand this)

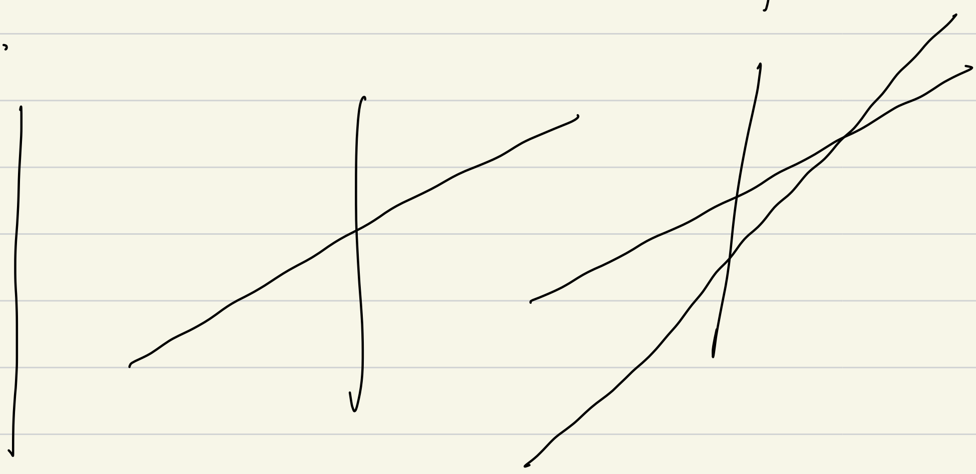
Induction in definition.

$$F_0 = 0, \quad F_1 = 1$$

$$F_{n+1} = F_n + F_{n-1}, \qquad n \geq 1,$$

$$F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

· # regions n lines can divide the plane into.



1 2   2 4   3 7   4 11

5 16   - - - -

Pattern is clear!

$$f(m) = f(m-1) + m; \qquad f(1) = 2$$

$\therefore \quad \dfrac{m(m+1)}{2} + 1$

= The above is clear; QED

Extension to planes in $\mathbb{R}^3$;

Let the max # regions with m planes be $p(m)$;

- w.l.o.g the max z - co-ordinate of a point of intersection in the configuration is say $z = a$;
- W.l.o.g we can rotate the given configuration so that one of the planes is the plane $z = a$.

Now assume we remove the plane $z = a$;

The remaining $m-1$ planes give us at most $p(m-1)$ regions.

We claim that every new region introduced by $z=a$ is in bijection with regions formed on $z=a$ by the lines we get considering the intersections $p_1 \cap \{z=a\}$, $p_2 \cap \{z=a\}$, ..., $p_{m-1} \cap \{z=a\}$. (Check)

$\therefore$ we have

$$\boxed{p(m) \leq p(m-1) + \frac{(m-1)(m)}{2} + 1}$$

Max we can get to $p(m-1) + \frac{m(m-1)}{2} + 1$

$p(1) = 1$

$$\Longrightarrow p(m) = \frac{m^3 + 5m + 6}{6}.$$

QED

PITFALLS: All horses are brown.

- $l_1, l_2, \ldots, l_n$ be $n \geq 2$ distinct lines in the plane, no two of which are parallel. Then all these lines have a point in common.