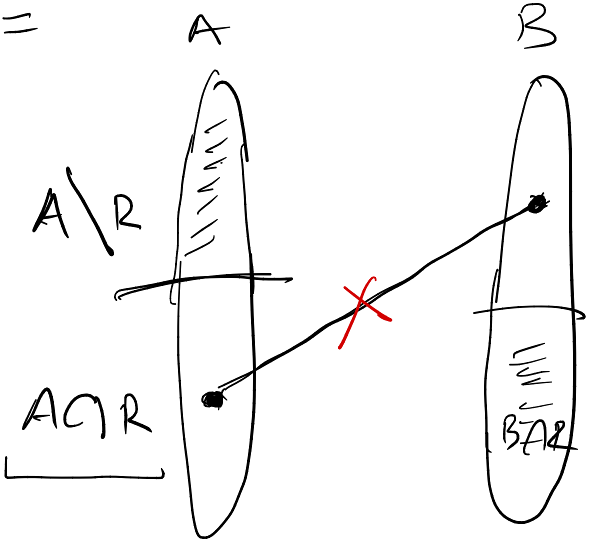
Königs Theorem:

- In a bipartite graph the size of the minimum vertex cover = size of max cardinality matching

$G = (V, E)$. $S \subseteq V$ is a V.C if $\forall\, e \in E$, one endpoint of e is in S.

$\Rightarrow$ min vertex cover $\geq |M|$;

$\Leftarrow$



Fix a max cardinality matching;

$S \subseteq A$, be the set of unmatched vertices;

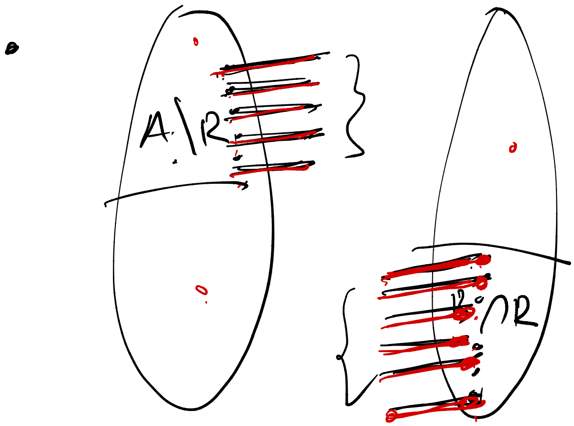
R = set of vertices in G reachable by an alt path from S.

$(A \setminus R \cup B \cap R)$ is a vertex cover;

$$\frac{\overset{A}{\downarrow}\;\overset{B}{\downarrow}}{(a\ b) \in M.} \quad \checkmark$$

$$\underline{(a, b) \notin M.}$$

$\therefore A \setminus R \cup B \cap R$ is a V.C.



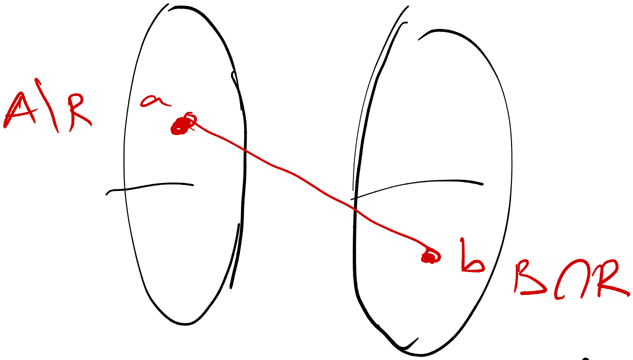
(a) All vertices in $A \setminus R$ are matched;
$\because S \subseteq R$, ~~are~~ all unmatched vertices;

(b) $B \cap R$ are all matched vertices;

A path going from S to an unmatched vertex in B, gives an augmenting path!

$[A \setminus R \cup B \cap R]$

$\therefore$ If there is no edge from M going from $A \setminus R$ to $B \cap R$, $\underline{|M| \geq (A \setminus R \cup B \cap R)}$



$\because$ $b \in B \cap R$,

$b \notin L$,

$\therefore$ alternating path from S to b,

But that can be then extended to reach a via an alternating path; $\Rightarrow a \in R$, contradiction;

— x — x — x — x — x —

- **Brooks theorem;**

- If G is a graph which not an odd cycle nor a complete graph, then G can be coloured with $\Delta(G)$ colours;

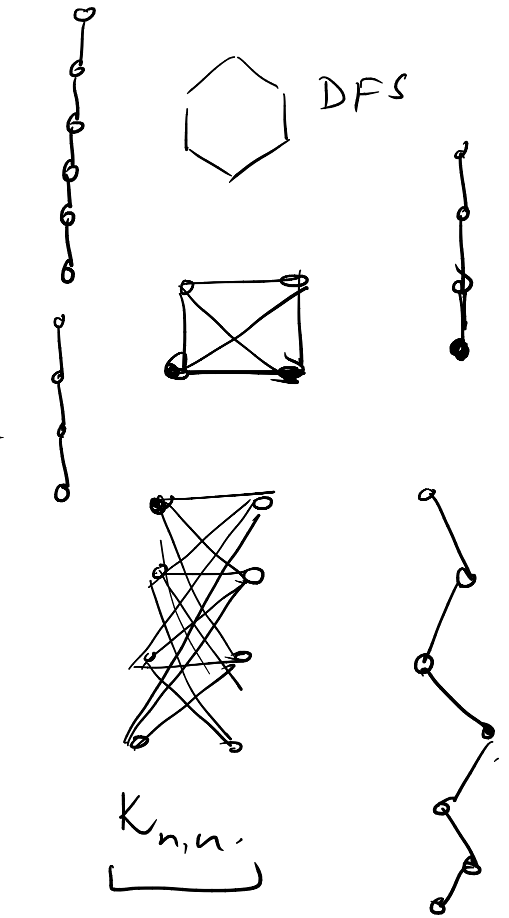
G can always be colored $1+\Delta(G)$ colours;

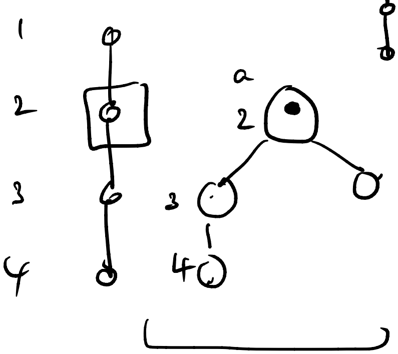
Turan's theorem:

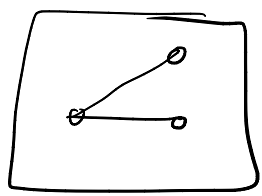
$T(n,k)$. $\underbrace{K_{k+1}}$;

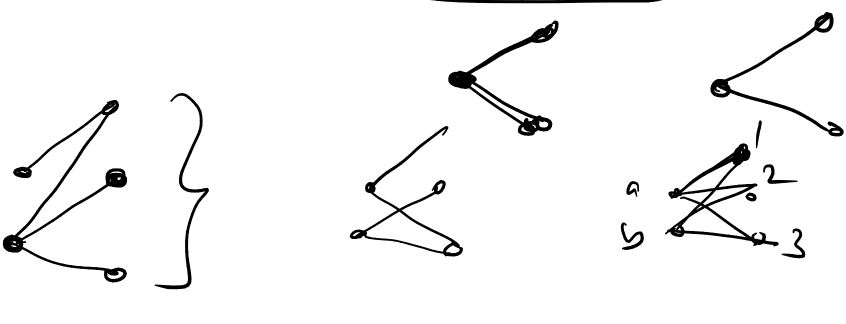
- **Thm:**

If every "DFS tree" of a graph G is a H.P., then G is a cycle, a complete graph or a $K_{n,n}$.

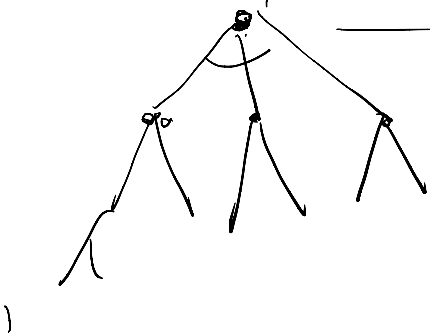








smallest degree.



Colouring with $\Delta(G)$ colours.

Suppose G is not regular; $\delta(G) \neq \Delta(G)$.

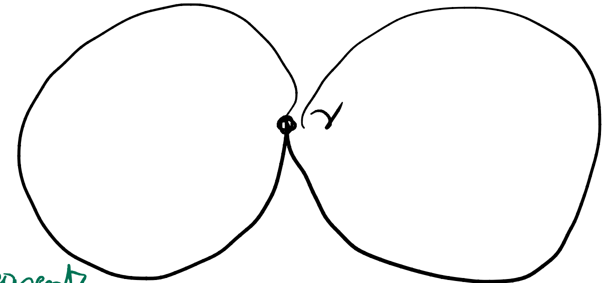
DFS tree starting at vertex with degree $\delta(G)$;

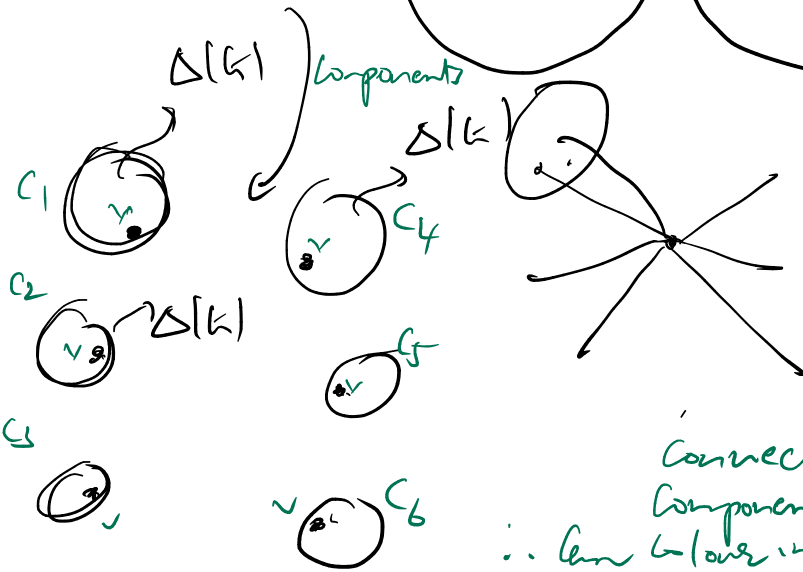
= Color inductively starting at leaf; Using the smallest colour available to colour a leaf.

- Take the subtree consisting of vertices which are uncoloured. And pick a leaf v that is connected to its parent in the tree, we have only seen $\delta(G)-1$ neighbours of v; so we can colour v with a color in $\{1,2,\dots,\Delta(G)\}$

Now assume:

• G is regular: Case (1): It has a cut vertex





For all components put back v, and connect it to vertices in that component originally connected to v, Components are non regular ∴ Can colour in Δ(G) colours, Make sure v has same color in each C_i

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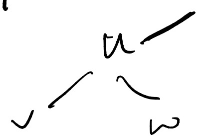
O.W. take a DFS tree. If every DFS tree is a Hamiltonian path, G is a $K_{n,n}$ (regular) a cycle or a K_n; our assumption then implies G is an even cycle or a $K_{n,n}$. But these are 2 colourable.

Otherwise there is a DFS tree of G with a vertex of degree ≥ 3;



Now some vertices in the subtree under v are connected to an ancestor of u; so too in the subtree rooted under w; this is because v, w are not cut vertices, and $G-v$ is connected & $G-w$ is connected; so $G-\{v,w\}$ is also connected

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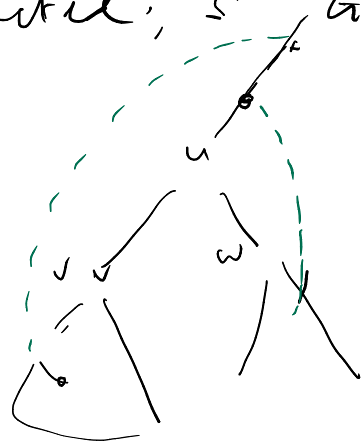
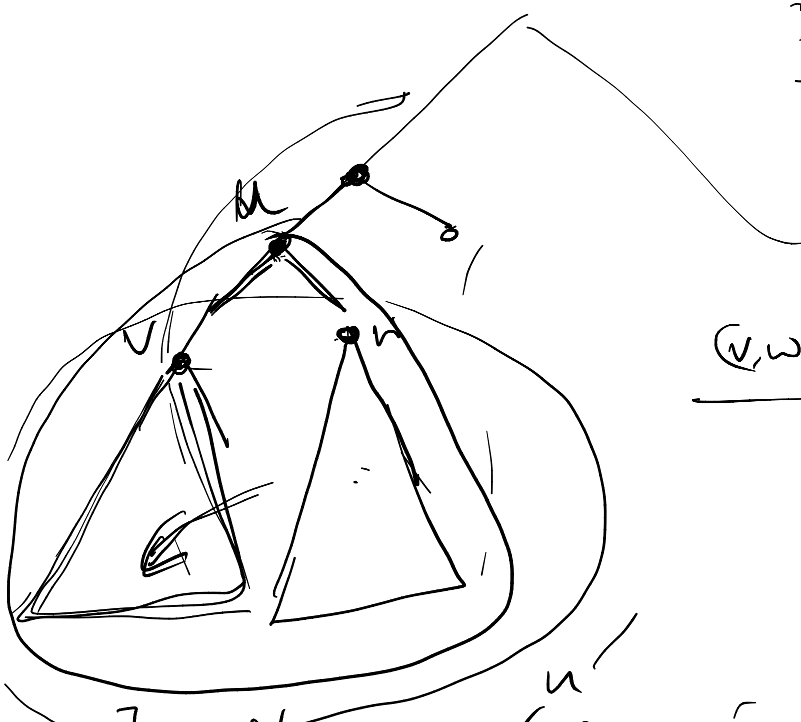
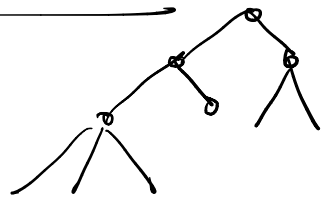


DFS tree:

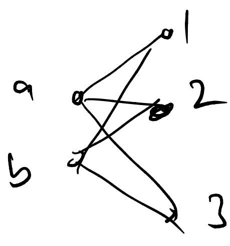


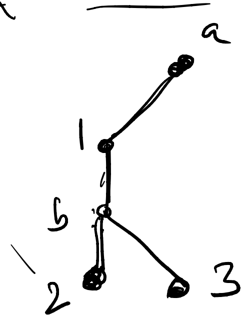
$(v,w) \overset{?}{\in} E$



• $\exists$ u st 

s in the DFS tree;





Now take $G-\{v,w\}$ and a DFS tree rooted at u;

First colour v, w both with 1;
Run the colouring algorithm on the tree;
As before when colouring a leaf in some iteration, the leaf is connected to its parent in $G-\{v,w\}$, and so we can colour it with an available color; This holds at all stages.
Finally when we see u, we have only used $\Delta(G)-1$ colours for the neighbours of u; $\therefore$ can colour u;