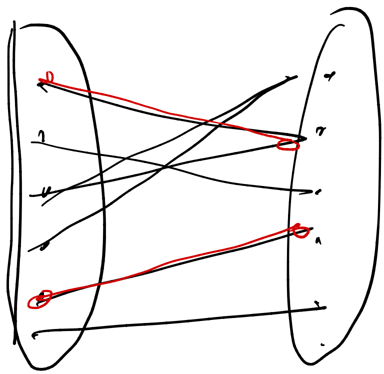
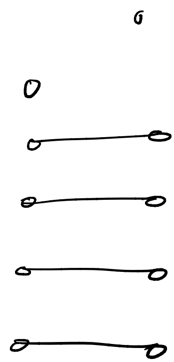
31|3|20

Matchings:

- In bipartite graphs:



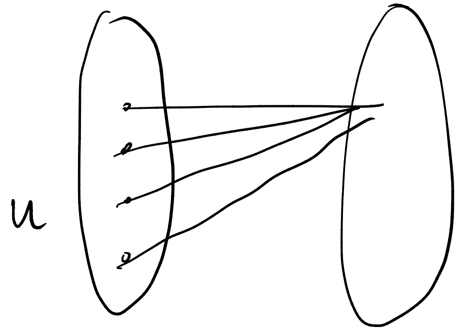


Matching: $M \subseteq E$,
In the subgraph induced by M every vertex has degree 0 or degree 1.

- Perfect matching: If every vertex has an edge of M incident on it.

Hall's theorem:

Let $G = (A \cup B, E)$ be a bipartite graph. Then G has a perfect matching iff $\forall\, U \subseteq A$, $N(U) \geq |U|$.

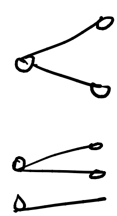


- **Necessary:** If M is a p.m. then every vertex in U reaches a different vertex in B. $\therefore N(U) \geq |U|$;

- $\Leftarrow$ $\forall U$, $N(U) \geq |U| \Rightarrow G$ has a p.m.;

Case:

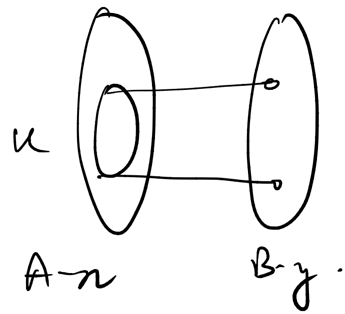
- ① $\forall\, U$, $N_G(U) > |U|$.



∘ Let $x \in A$, x —— y.

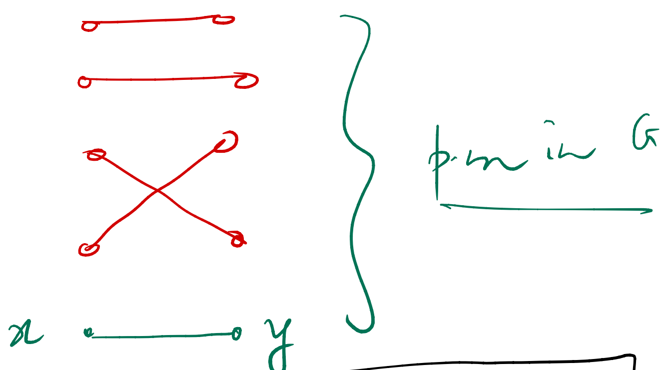
Remove from G, $\underbrace{x, y}$.

∘ $\underline{\tilde{G}:}$



$\underline{N_{\tilde{G}}(u) \; ?}$ $\quad\quad$ $\underline{N_{\tilde{G}}(u) \geq u.}$

∴ By induction $\tilde{G}$ has a P. matching;



<u>Case 2:</u> $\exists\, u$, $\boxed{N_G(u) = u.}$

G_1: u $\quad$ $N(u)$ $\quad$ $\tilde{u}$

$N_{G_1}(\tilde{u}) = N_G(\tilde{u}) \geq |\tilde{u}|$;

G_2: $A \setminus u$ $\quad$ w $\quad$ $B \setminus N(u)$;

Q: Does G_2 satisfy Hall's condition?

A: Yes,

$$N_{G_2}(w) \geq |w|.$$

$$\underbrace{N_G(u \cup w)} = \underbrace{N_G(u)} \cup \underbrace{N_{G_2}(w)}$$

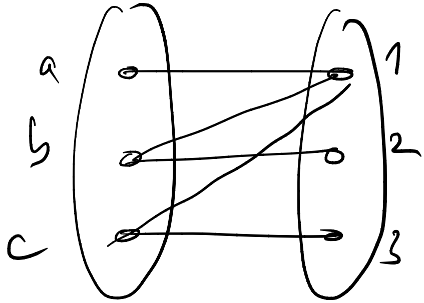
$$= \cancel{N_G(u)} + N_{G_2}(w) \geq \cancel{|u|} + |w|$$

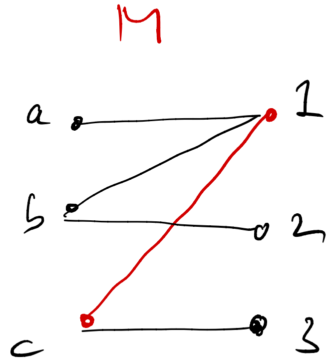
$$\therefore N_{G_2}(w) \geq |w| \quad \forall w \subseteq A \setminus u;$$

$\therefore$ G_2 has a p.m

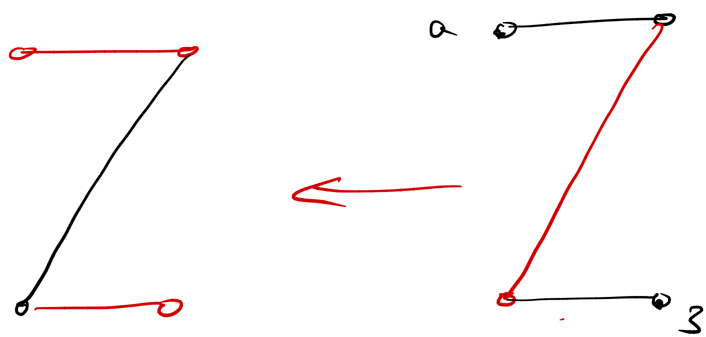
$\therefore$ G has a p.m.

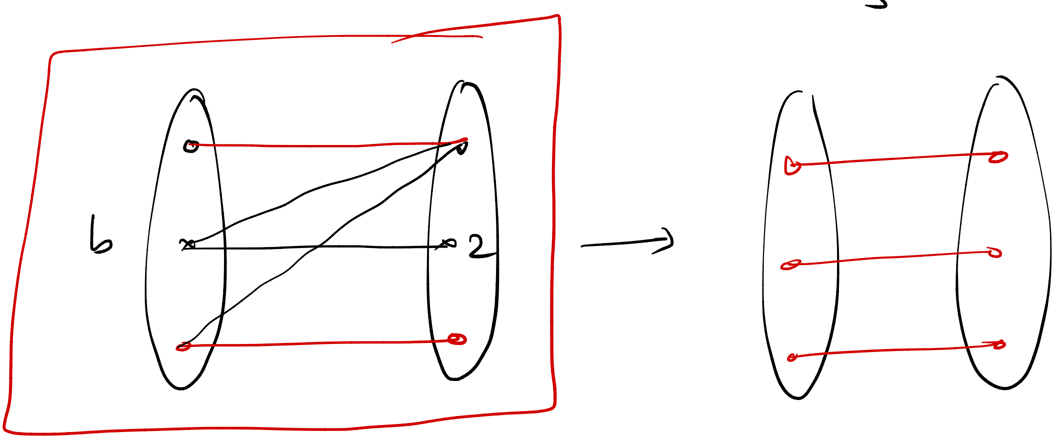
- Hungarian Method:





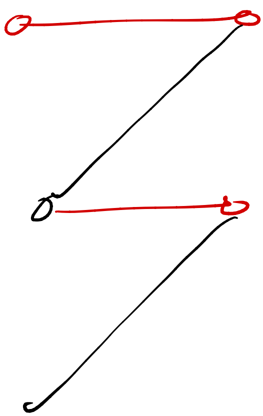
there is a path from a to 3, consisting alternately of edges in M and edges not in M;



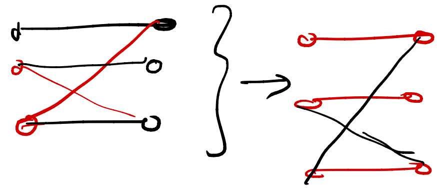


Alternating path w.r.t M:

- p (path) is said to be alternating if edges in p alternate from those in M and those not in M;



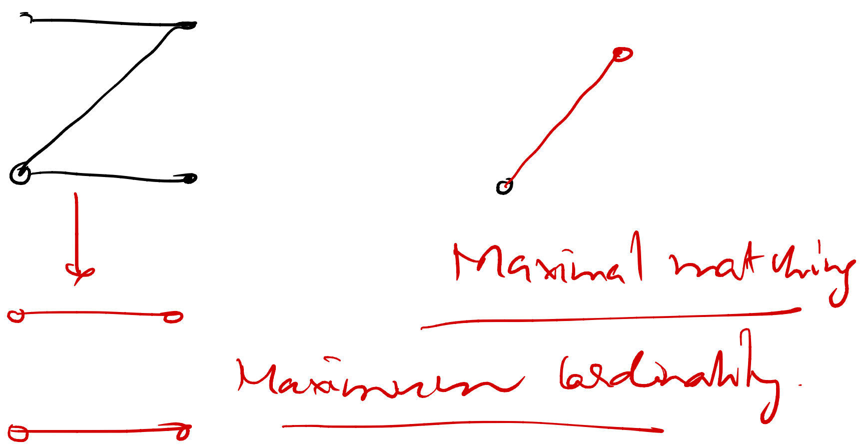
- **Augmenting path:** Is an alternating path starting at an unmatched vertex & ending at an unmatched vertex.



Thm: Let $G = (A \cup B, E)$ be a bip graph.

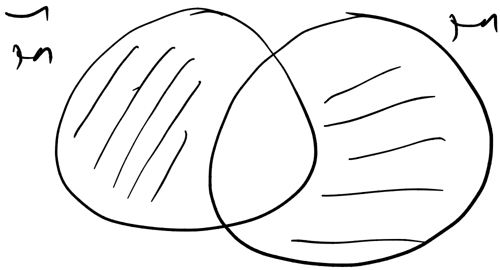
Let M be a matching in G; then M is a maximum cardinality matching iff there is no augmenting path w.r.t M in G;

Proof: If there is no augmenting path then M is a largest cardinality matching.



- If not — let us M be a matching in G, and suppose $|\hat{M}| > |M|$;

$$\tilde{M} \triangle M := \{\text{edges in } \tilde{M} \text{ not in } M\} \cup \{\text{edges in } M \text{ not in } \tilde{M}\}$$



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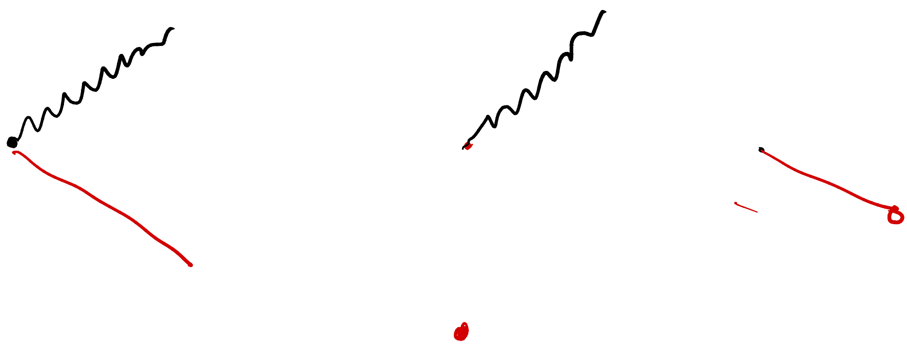
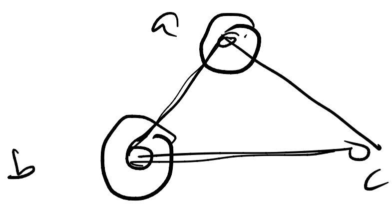


- Every vertex has degree $\underbrace{0, 1 \text{ or } 2}$

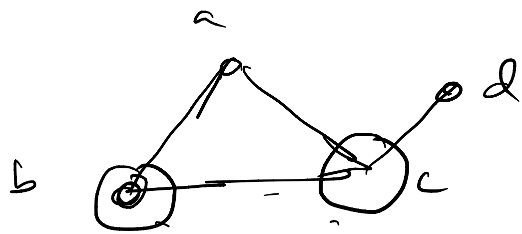
∴ the graph induced by $M \triangle \tilde{M}$ has isolated vertices, paths or even cycles;

• Vertex cover in G:

$S \subseteq V$ is a vertex cover if every edge of G, has at least one endpoint in S;

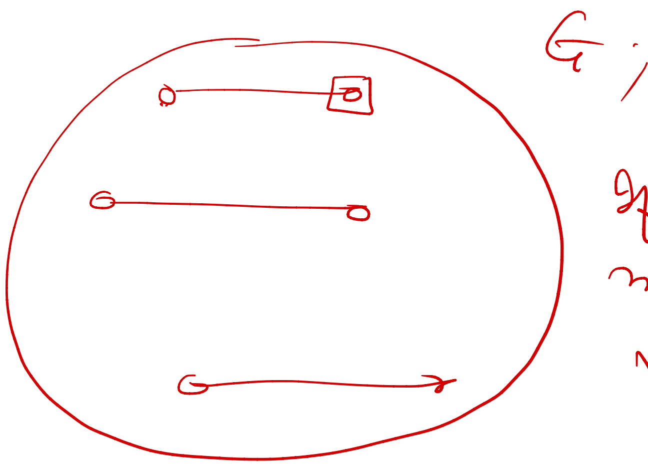


$\{a, b\}$ is a vertex Cover



$\{b, c\}$ is a vertex cover;

• Q: Find the smallest vertex cover in G;



If M is a matching $v.c(G) \geq |M|$;

In any vertex cover, one of the endpoints of M must be present;

· **König theorem:**

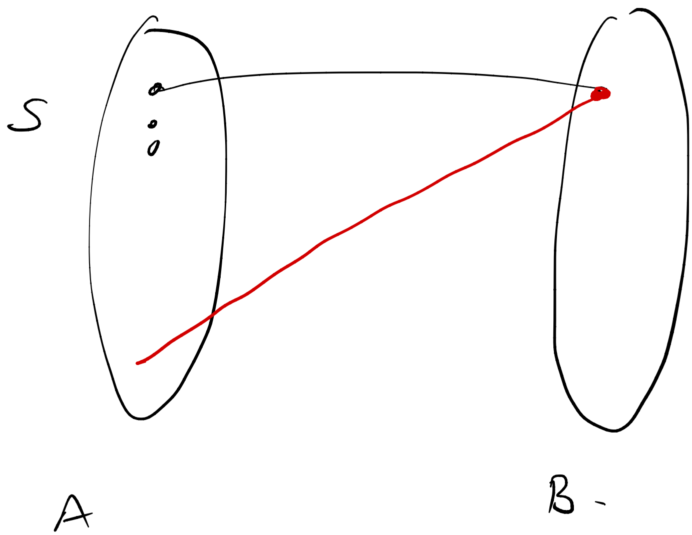
Let G be a bipartite graph. The size of the smallest vertex cover in G is equal to the cardinality of the largest matching in G;

= **Proof:** Start with a max cardinality matching M obtained using Hungarian;

Using Hungarian method:

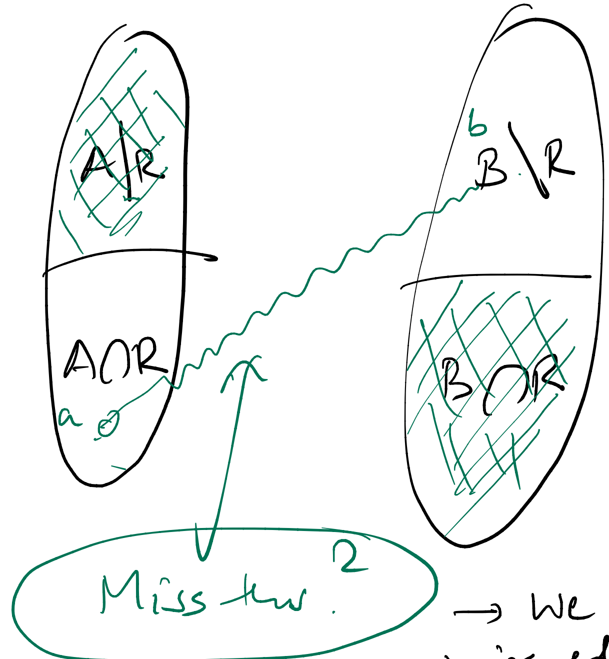
$\subseteq A$

S – the subset of unmatched vertices.



Let R be the set of vertices reachable from S using alternating paths;

By def:ⁿ $S \subseteq R$.



Claim:

$C = A \setminus R \cup B \cap R$

Claim:

C is a vertex cover:

→ we could potentially only miss edges from $A \cap R$ to $B \setminus R$

$a \in A \cap R$

- Is $(a\,b)$ in M?

a is reachable from S via an alternating path; But then, if $(a\,b) \in M$, $(b\,a)$ must have been used; But then $b \in R$; X

- If $(a,b) \notin M$ ⇒ again we reach a via alternating path from S, But then we could continue to b, and then

again $b \in R$, a contradiction.

$\therefore$ we don't have edges from $A \cap R$ to
$B \setminus R$,
$\therefore$ $A \setminus R \cup B \cap R$ is a vertex cover,