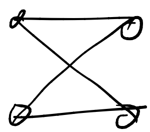
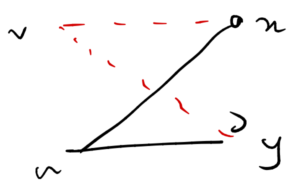
Today's class:

- **THM:** Let G be a graph with no $K_{2,2}$. Then $\# E$ (edges) $= O(n^{3/2})$.



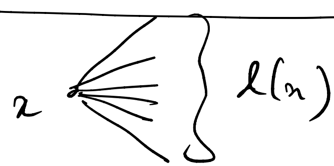
don't want;



two vertices can share at most 1 common nbr;

$\#$ paths of length 2; $\overset{\text{upper}}{\leq} \binom{n}{2}$;

- $d(x)$; –



$\binom{d(x)}{2}$

$$\# \text{ paths of length } 2 = \underbrace{\sum_x \binom{d(x)}{2} \leq \binom{n}{2}}$$

$$\binom{n}{2} \geq \sum_x \frac{d(x)(d(x)-1)}{2} \geq \sum_x \frac{d(x)^2}{2} - \sum_x \frac{d(x)}{2}$$

$$\binom{n}{2} \geq \sum \frac{d(n)^2}{2} - \frac{\sum d(x)}{2}$$

$$\geq \frac{1}{n}\left[\frac{\sum d(x)}{2}\right]^2 - \frac{\sum d(x)}{2}$$

$$\sum d(x)^2 \geq \left[\sum d(x)\right]^2 \cdot n$$

$$\left(\sum x_i y_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum y_i^2\right)$$

$$\binom{n}{2} \geq \frac{2E^2}{n} - E$$

$$\geq \frac{E^2}{n} \qquad \therefore\ n^2 \geq \frac{E^2}{n}$$

$$\therefore\ E \leq \left(n^{3/2}\right)$$

$$= O\left(n^{3/2}\right).$$

$=$

Pick a prime p;

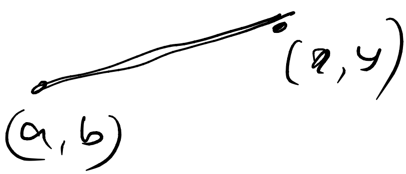
$$\left.\underbrace{0, 1, \ldots, (p-1)}\right\}$$

Finite field;

$\mathbb{F}_p \times \mathbb{F}_p$; (a, b)

PLEASE SEE CORRECTION on next page

Vertex set of G — $\underbrace{\mathbb{F}_p \times \mathbb{F}_p}$;



(a, b) — (x, y) if

$$\underbrace{ax + by = 1}.$$

$$\boxed{b^{-1}(1 - ax)}$$

• if $\underbrace{(a,b) \neq (0,0)}$:

degree of all vertices $(a,b) \neq (0,0)$ is $\geq p - 1$

$$\underline{\#E \geq \frac{1}{2}(p^2 - 1)(p - 1)}$$

No $K_{2,2}$

has no solution

$$\boxed{\left.\begin{aligned} ax + by &= 1 \\ a'x + b'y &= 1. \end{aligned}\right\}}$$

$n = p$

CORRECTION to video recording:

We don't want loops;

$\therefore$ $(a,b) - (a,b)$ not allowed;

for example $(1,0) - (1,0)$

$= 1\cdot 1 + 0\cdot 0 = 1.$

$\therefore$ we can only say the degree is at most $\underline{p-1.}$

$\#$ edges: $\frac{1}{2}(p^2-1)(p-1) = \frac{1}{2}p^3 - \frac{1}{2}p^2 - p + 1$

$$\geq \underbrace{\frac{1}{2}p^3 - p^2}.$$

$\therefore$ $\#$ vertices: p^2.

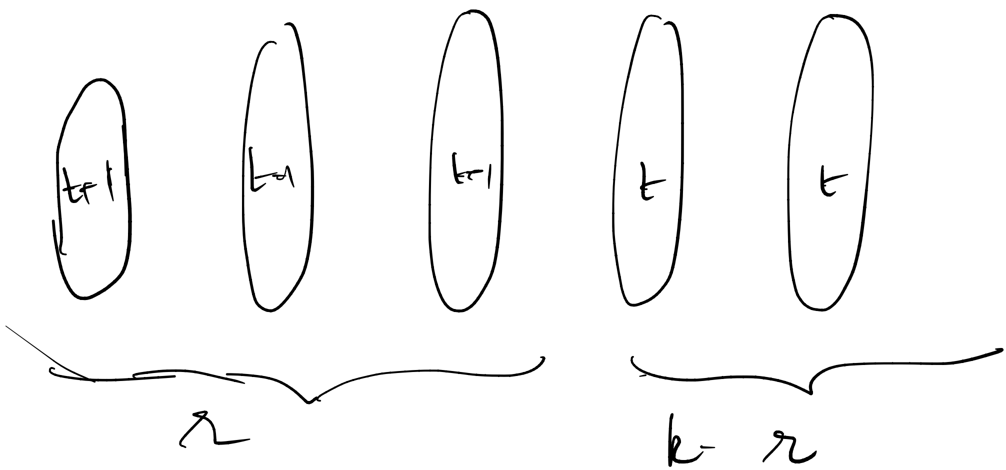
$$\geq \underbrace{\frac{1}{2}n^{3/2} - n}.$$

Turan's theorem

Let G be a graph on n vertices having no K_{k+1}. Then # edges in G

$$\le \underbrace{\frac{(k-1)n^2}{2k} - \frac{r(k-r)}{2k}}_{= T(n,k)}$$

$\underline{n = tk + r.}$ $r \le k-1.$



$n = tk + r;$ $r \le k-1.$

$$\frac{1}{2}\Big[r(t+1)(k-r)t + (k-r)t\, r(t+1) + r(t+1)\big[(r-1)(t+1)\big] + (k-r)t\big[t(k-r-1)\big]$$

$$= \frac{(k-1)n^2}{2k} - \frac{r(k-r)}{2k}$$

$$\boxed{T(n,k) = \frac{(k-1)n^2}{2k} - \frac{r(k-r)}{2k}}$$

Proof: By induction on t.

$t = 0$

$\therefore n \le k-1$

$\therefore n = r$.

Can have $\frac{n(n-1)}{2}$ edges.

Putting $n = r$

$$\frac{(k-1)r^2}{2k} - \frac{r(k-r)}{2k}$$

$$= \frac{r^2}{2} - \cancel{\frac{r^2}{2k}} + \cancel{\frac{r^2}{2k}} - \frac{r}{2}$$

$$= \frac{r(r-1)}{2} = \frac{n(n-1)}{2}$$

Matches that Turan gives.

Assume we know the statement of the theorem for $t \le t_0 - 1$.

(I.H.) If $n' = tk + r$, $t \le t_0 - 1$, we know that a graph on n' vertices which has no K_{k+1} has at most $\frac{k-1}{2k}(n')^2 - \frac{r(k-r)}{2k}$ edges;

- We prove it for t_0:

Say $\underline{n = kt_0 + r}$ $\quad r \le k-1$.

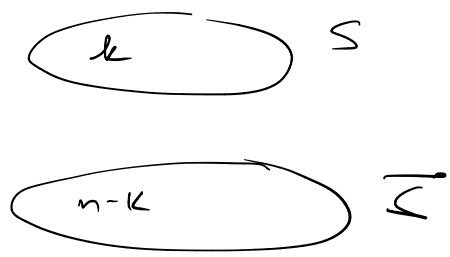
We have a maximal graph on n vertices (in terms of # edges) which has no K_{k+1}

• ∵ the graph is maximal, adding any edge will result in a K_{k+1}.

Say x - - - - y is not an edge.

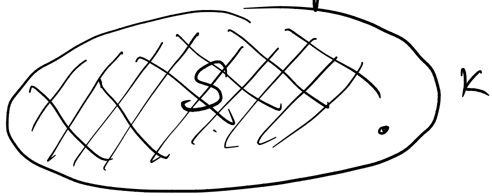
Adding it creates a K_{k+1} containing x, y; ∴ y must already be part of a K_k (the vertices in the K_{k+1} formed excluding x)

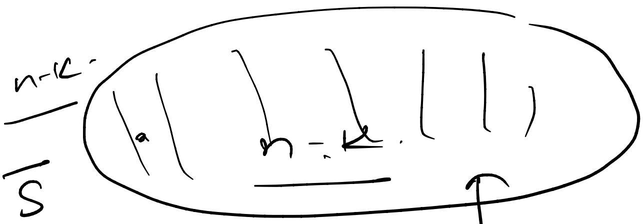
Take that K_k - call the set of vertices S.



Each vertex of $\bar{S}$ is connected to at most $(k-1)$ vertices in S. & within $\bar{S}$, no K_{k+1}.

S is a complete graph on k vertices;





No K_{k+1}.

$\therefore$ # edges in G is at most.

$$\boxed{\binom{k}{2} + (n-k)(k-1) + \underbrace{T(n-k, k)}}$$

We claim this is $T(n,k)$,

$n-k = (t_0-1)k + r$.

$\therefore$ # edges within $\bar{S}$

$= T(n-k, k)$

$= \frac{k-1}{2k}(n-k)^2 - \frac{r(k-r)}{2k}$

Check:

$$\binom{k}{2} + (n-k)(k-1) + \frac{k-1}{2k}(n-k)^2 - \frac{r(k-r)}{2k}$$

$$= \frac{k(k-1)}{2} + (n-k)(k-1) + \frac{k-1}{2k}(n-k)^2 - \frac{r(k-r)}{2k}$$

$$\frac{k(k-1)}{2} + (n-k)(k-1) + \frac{k-1}{2k}(n-k)^2 - \frac{r(k-r)}{2k}$$

$$= \frac{k(k-1)}{2} + (k-1)n - k(k-1) + \frac{k-1}{2k}\left(n^2 - 2kn + k^2\right) - \frac{r(k-r)}{2k}$$

$$= \left(\frac{k-1}{2k}n^2 - r\frac{(k-r)}{2k}\right) + \cancel{\frac{k(k-1)}{2}} + \cancel{(k-1)n} - \cancel{k(k-1)}$$

$$= \cancel{(k-1)n} + \cancel{\frac{k(k-1)}{2}}$$

$$= \frac{k-1}{2k}n^2 - \frac{r(k-r)}{2k}.$$

QED