EXTREMAL

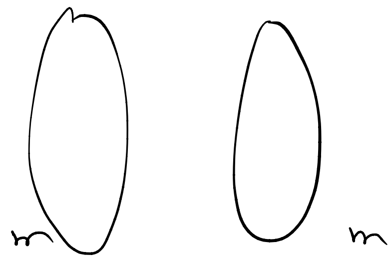
· GRAPH THEORY:

G is bipartite graph iff it has no odd cycle;

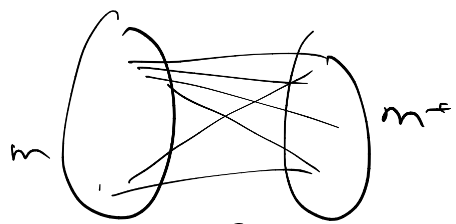
· G be a graph on n vertices - bipartite

max number of edges G can have is $\lfloor n^2/4 \rfloor$

$n = 2m$



$n = 2m+1$



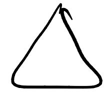
· MANTEL'S thm: If G has $\geq \lfloor \frac{n^2}{4} \rfloor + 1$ edges then it has a $\triangle$;

Note! It is not bipartite. ∴ odd cycle;

But we assert, it is a $\triangle$

Proof: ①:

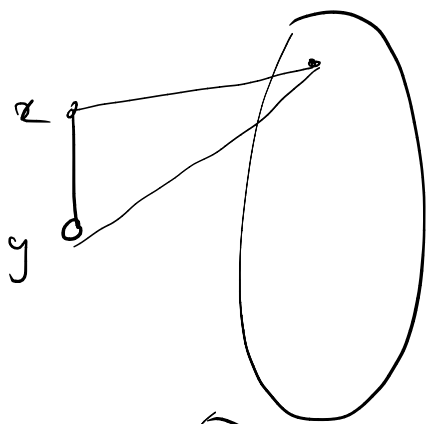
By induction on n: $(n \geq 3)$;

$n = 3$: $\left\lfloor \frac{n^2}{4} \right\rfloor + 1 = \underline{3}$. 

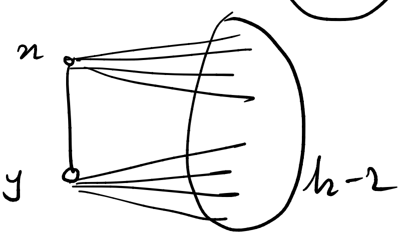
- Assume it is true when $\underline{n \leq k-1}$.

Prove it for k:

$\therefore$ # edges $= \left\lfloor \frac{k^2}{4} \right\rfloor + 1$



If x-y have a common neighbour done



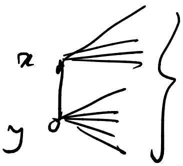
o.w.

$\underbrace{d(x) + d(y) \leq k.}$

Remove x & y from G;

$(k-2)$ vertices;

$\leq k-2$



edges removed: $\leq k-1$

edges left: $\left\lfloor \frac{k^2}{4} \right\rfloor + 1 - k + 1$

$$= \left\lfloor \frac{k^2}{4} \right\rfloor - k + 1 + 1$$

$$= \left\lfloor \frac{k^2}{4} \right\rfloor + \left\lfloor \frac{-4k+4}{4} \right\rfloor + 1$$

$$= \left\lfloor \frac{(k-2)^2}{4} \right\rfloor + 1$$

$\therefore$ a Δ in $G - x - y$.

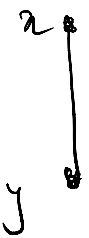
Proof: (2):

Cauchy Schwartz: $x_1, \dots x_n$
$y_1 \dots x_n$

$$\underbrace{\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2}$$

In particular: $\left(\sum x_i\right)^2 \le \left(\sum x_i^2\right) \cdot n$.

$$\therefore \quad \boxed{\sum x_i^2 \ge \frac{1}{n}\left(\sum x_i\right)^2}$$



Let x - y be connected,
If they have no common
nbr, $\underbrace{d(x) + d(y) \le n}$.

$$\underbrace{\sum_{(x,y)\in E} d(x) + d(y) \le n|E|}$$

$$\frac{1}{n}\left(\sum d(x)\right)^2 \le \sum_{x} d(x)^2 \le n|E|.$$

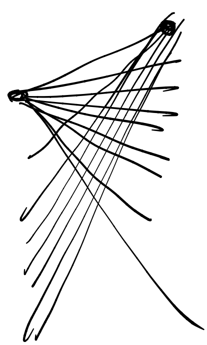
$\therefore$ If G has no Δ,

$$n|E| \geq \frac{1}{n} 4E^2$$

$$\therefore |E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

Q: Describe all graph on n vertices having $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges and not containing a Δ.

$=$ Then Cauchy Schwartz is tight, $|E| = \lfloor n^2/4 \rfloor$



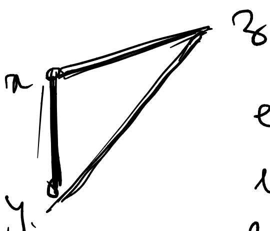
$\Longrightarrow$ one extremal graph.

THM:

- If G is simple, with $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges then G has at least $\left\lfloor \frac{n}{2} \right\rfloor$ Δ's;

Proof:

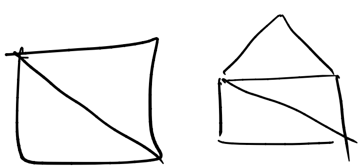
Suppose every edge is in a Δ;



By our assumption every edge is on a Δ; $\therefore$
each edge contributes to a Δ;
But the same Δ could be contributed by at most 3 edges
$\therefore$ # Δ's at least $|E|/3$

$$\therefore \# \Delta\text{'s} \geq \frac{|E|}{3} \geq \frac{1}{3}\left(\left\lfloor \frac{n^2}{4} \right\rfloor + 1\right) \geq \frac{1}{3}\left(n^2/4\right) \geq \frac{n^2}{12} \geq \frac{n}{2}$$

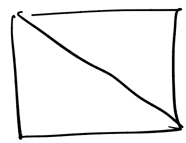
for $n \geq 6$



So if we assume that each edge is on a $\triangle$ then the theorem is true when $n \geq 6$.

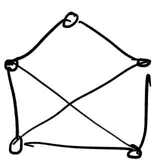
- We check for $n=4, 5$.

<u>$n=4$</u>: $\left\lfloor \frac{4^2}{4} \right\rfloor + 1 = 5$ $\therefore$



and we see 2 $\triangle$ as required $\left(\left\lfloor \frac{4}{2} \right\rfloor \right)$

<u>$n=5$</u>: $\left\lfloor \frac{5^2}{4} \right\rfloor + 1 = \underline{\underline{7}}$



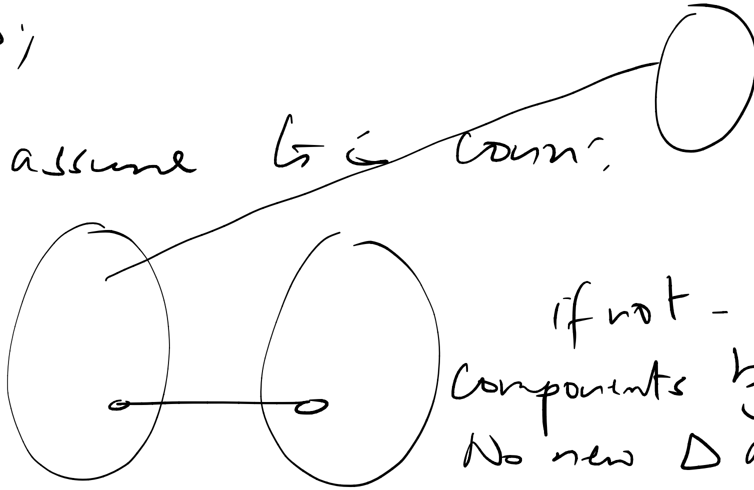
See 2 $\triangle$'s.

check other cases.

If our assumption does not hold:

$\exists$ an edge which is not on a Δ;

Can assume G is conn;



if not - connect the components by edges. No new Δ are created.

- we do not create any new Δ

- Hypothesis continues to hold;

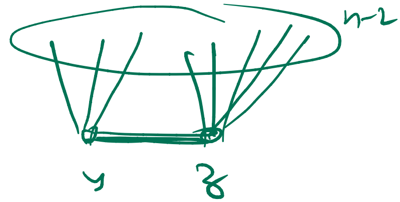
Let

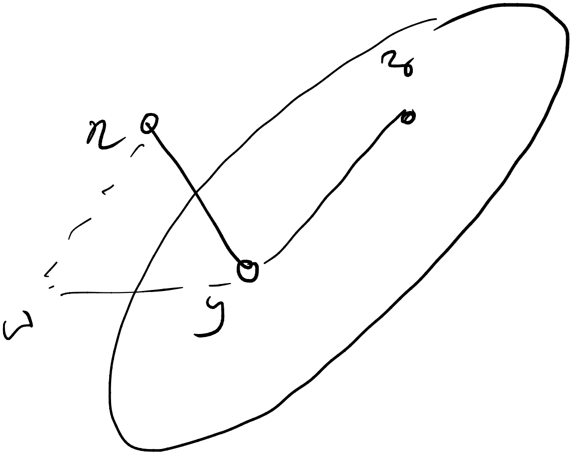
$E = E_1 \cup E_2$

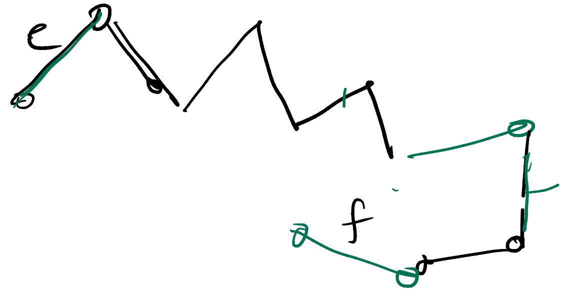
E_1: edges on some Δ

E_2: edges not on any Δ;

Take a walk from $e \in E_1$ to $f \in E_2$







Remove $y-z$ and look at $G - y - z$

$d(y) + d(z) \le n$

<u>\# edges left</u>: $\ge \left\lfloor \frac{n^2}{4} \right\rfloor + 1 - (n-1)$

$$\ge \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$$

<u>By ind</u>: $G - x - y \longrightarrow \ge \left\lfloor \frac{n-2}{2} \right\rfloor$ Δ's in this;

$\therefore \ge \left\lfloor \frac{n}{2} - 1 \right\rfloor$ Δ's;

Adding $x-y-w$; $\left\lfloor \frac{n}{2} - 1 \right\rfloor + 1 \ge \lfloor n/2 \rfloor$ Δ's.

- kvsatcmi@gmail.com

by 5 p.m.