- Last class - Matrix tree theorem;

- Recall - Laplacian (called unnormalized Laplacian).

- Adjacency matrix; $\nearrow$ # Spanning trees $= \frac{1}{n} \lambda_1 \cdots \lambda_{n-1}$

$L = D - A$, D - diagonal, $D(i,i) = \deg v_i$

- Clearly $(1, 1, \cdots 1)^T$ is an eigen vector of L with e.v. 0;

- L is positive semidefinite; So $\exists$ a basis of e.v in which L is similar to a diagonal matrix. Because e.v are real, this diagonal matrix has entries $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$,

Clearly $\lambda_n = 0$;

MT theorem: # spanning trees $= \frac{1}{n} \lambda_1 \cdots \lambda_{n-1}$

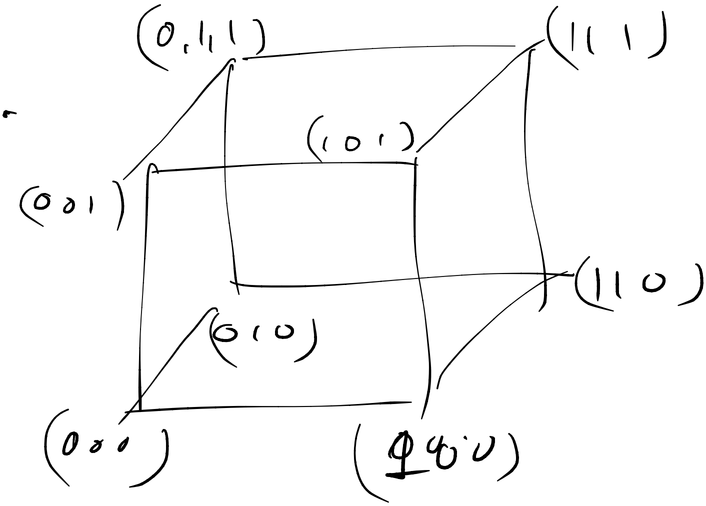
- Adjacency matrix:
- $A^k(i,j)$?
- e.v of A_G - Spectrum of a graph;
- Spec(K_n)? · Spec(Hypercube).
- Spec(K_{mn})?

$L = D - A$; D_{ii} = degree of vertex i.

If G is regular, $D = \begin{pmatrix} d & & \\ & \ddots & \\ & & d \end{pmatrix}$

If e.v of A $(\lambda_1 \dots \lambda_n)$

ev of $D - A = (d-\lambda_1, \dots, d-\lambda_n)$



Fix v,

$$f_v = \begin{array}{rl} w \longrightarrow & \left[\begin{matrix} \\ \\ (-1)^{v\cdot w} \\ \\ \\ \end{matrix}\right]\begin{matrix} w_1 \\ \vdots \\ \vdots \\ \vdots \\ w_{2^n} \end{matrix} \end{array}$$

Claim) $\{f_v \mid v \in \{0,1\}^n\}$ are linearly independent,

Claim:

f_v an eigen vector of $\mathcal{L}$;

$$\mathcal{L}\cdot f_v = \lambda_v \cdot f_v .$$

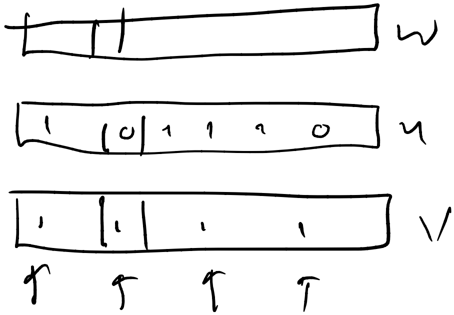
$v = [00000]$, $f_v = \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$
$[0000\cdot]$

and e-v of $f_{[0\ldots0]} = 0$.

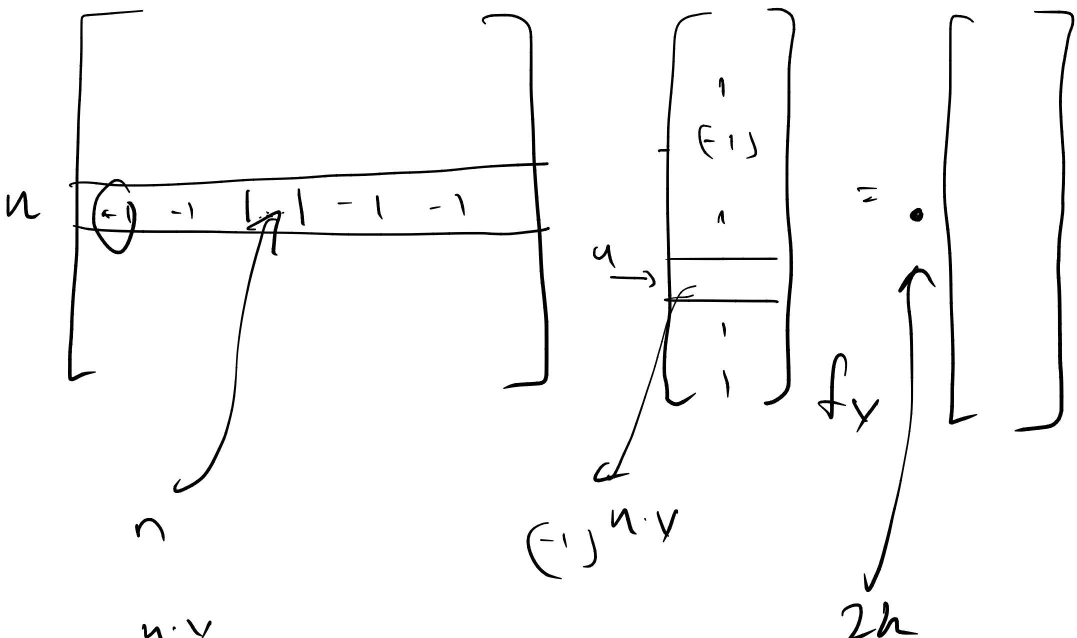
$\therefore$ # spanning trees $= \prod_{v \neq [0000]} \lambda_v$

Eigen value λ_v:

Completing this calculation



k ones.



$n\,(-1)^{u\cdot v}$

Compare $u\cdot v$ with $w\cdot v$. If w obtained from u by flipping a bit

$(-1)^{w\cdot v} = -(-1)^{u\cdot v}$ $\longrightarrow$ where v is 1

If w is a neighbour of u obtained by flipping a bit j of u where $v_j = 0$, $(-1)^{w \cdot v} = (-1)^{w \cdot u}$

$\therefore$ for k neighbours of u, $(-1)^{w \cdot v} = -(-1)^{w \cdot u}$

for $n-k$ " of u, $(-1)^{w \cdot v} = (-1)^{w \cdot u}$.

$\therefore$

$$u \cdot f_v = n(-1)^{u \cdot v} - \left[k \cdot -(-1)^{u \cdot v} + (n-k)(-1)^{u \cdot v}\right]$$
$$= 2k(-1)^{u \cdot v}$$

This is true for all u;

$\therefore$ $\mathcal{L} f_v = 2k(-1)^{u \cdot v}$ $\therefore$ f_v is an eigen vector of e-value $2k$;

vectors with e.v. $2k$, $\binom{n}{k}$;

Now use matrix trace theorem:

$\#$ spanning trees:

$$\frac{1}{2^n}\prod_{k=1}^{n}(2k)^{\binom{n}{k}} = \#\text{ spanning trees.}$$

$$\frac{1}{2^n}\, 2^{2^n-1}\prod (k)^{\binom{n}{k}}$$

$$\boxed{2^{2^n-n-1}\prod (k)^{\binom{n}{k}}}$$

once again:

for k nbrs w of u, $(-1)^{w\cdot v} = -(-1)^{u\cdot v}$

for $n-k$ nbrs w of u, $(-1)^{w\cdot v} = (-1)^{u\cdot v}$

$\mathcal{L}_u \cdot f_v =$

$$n(-1)^{u\cdot v} - \Big[\underbrace{-(-1)^{u\cdot v} - (-1)^{u\cdot v} \cdots}_{k} + (n-k)(-1)^{u\cdot v} \Big]$$

$$= n(-1)^{u\cdot v} + k(-1)^{u\cdot v} - (n-k)(-1)^{u\cdot v}$$

$$= 2k(-1)^{u\cdot v}$$

$=$