Geometric Complexity Theory: a high level overview.

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Roadmap

Roadmap

1. *Basics in Complexity theory, Algebraic Geometry*
   - Complexity Theory
   - Representation theory
   - Algebraic geometry
2. Algebraizing the formula complexity question
   - Reduction
   - Geometry and class varieties
3. From lower bounds to obstructions
   - Geometry of class varieties is tractable
   - The first flip
4. Saturated Integer programming
   - Overcoming the Razbarov-Rudich barrier
   - Non-zeroness of LR-coeffs in poly time
   - Saturated functions
5. Implementing the flip for $\#P$ vs NC^2
   - Saturation of Kronecker coefficients

Functions and Algorithms

Functions considered in Complexity theory:

- Number Functions $f : \mathbb{Z}^n \mapsto \mathbb{Z}$
  - The determinant of a matrix.
  - Optimal value of an LP.
- Decision functions $f : \mathbb{Z}^n \mapsto \{0, 1\}$.
  - Is the determinant of a matrix zero?
  - Is the optimum of an LP greater than a certain value?
  - Is there an assignment of values to the variables of a Boolean formula so that the formula is true?

In general consider a family of functions - example det_m.

An Algorithm is a program which computes a family of functions.

Measures

Every problem for which we need to design an algorithm comes with:

- The input instance
  - The matrix whose determinant is to be computed.
  - The specific Boolean formula.
- Input size
  - for det_m, the size, m^2, of the matrix.
  - Also the number of digits needed to write an entry of the matrix

What is important?

- The resources consumed by the algorithm to solve the problem.
  - Time taken by the algorithm.
  - How many digits are needed to store each intermediate value; how many intermediate values need to be stored.

Computing with formulas

Let $p(X_1, \ldots, X_n)$ be a polynomial. A formula is a particular way of writing it using $+$ and $*$.

$$\mathit{formula} = \mathit{formula} \ * \ \mathit{formula} \| \mathit{formula} \ + \ \mathit{formula}$$

Formula size: The number of $*$ and $+$ operations.

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$$formula = formula \ * \ formula || formula \ + \ formula$$

Formula size: The number of $*$ and $+$ operations.

Example

$$\begin{array}{lcl} a^3 - b^3 & = & (a-b) * (a^2 + a * b + b^2) \\ \text{Van-der-Monde } (\lambda_1, \ldots, \lambda_n) & = & \Pi_{i \neq j}(\lambda_i - \lambda_j) \end{array}$$

The permanent function

- $M = (m_{ij})$, a square $n \times n$ matrix Then

$$Perm_n(M) = \sum_{\sigma \in S_n} \Pi_{1 \leq i \leq n} m_{i,\sigma(i)}$$

$$Det_n(M) = \sum_{\sigma \in S_n} \Pi_{1 \leq i \leq n} (-1)^{l(\sigma)} m_{i,\sigma(i)}$$

Question

Does $Perm_n$ have a formula of size polynomially bounded in n?

If NOT, what is the proof?

The complexity theory of det_m *and* $Perm_n$

- The class of functions which can be computed in polynomial time in input size - P. Intuitively, what can be considered feasibly computable.
- det_m is easy to compute - The standard Gaussian elimination for example shows feasibility. A combinatorial algorithm (Meena, Vinay) - allows us to compute it by a polynomial number of computers running in $poly(\log n)$time.
- The permanent is believed to be hard. Intuitively, among the hardest among functions f for which:
  - There is an expression for f which involves only positive coeffecients.
  - Each term of f is computable in polynomial time.
  - Number of terms is at most exponential in input size.

Roadmap

G a group. V a vector space.

- $\rho : G \mapsto GL(V)$
  - V is a representation of G.
- $W \subseteq V$ a subspace of V is a subrepresentation.
  - $\rho(g) \cdot w \in W, \forall g \in G, \forall w \in W$.
- Irreducible representation.
  - No proper non-trivial subrepresentation.

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  - $V = \oplus_\lambda m_\lambda V_\lambda(G)$
  - λ - labels of irreducible representations of G - building blocks in the representation theory of reductive groups.

Examples of representations of $GL_n(\mathbb{C})$

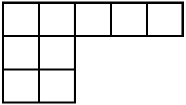
- G acting on an $\mathbb{C}^n$ by $A \cdot v = A * v$.
  - The zero vector is left fixed. A subrepresentation.
  - Ireducible representation.
- G acting on the vector space V of $n \times n$ matrices, $A \cdot X = AXA^{-1}$.
  - The vector space of scalar matrices is left invariant.
  - Not irreducible.
- $X = (x_{i,j})$ an $n \times n$ matrix of indeterminates. G acts on the vector space of functions $\mathbb{C}[x_{i,j}]$ by $A \cdot x_{i,j} = \sum_{k=1}^{k=n} a_{k,i} x_{k,j}$.

Representations of $GL_n(\mathbb{C}), S_d$

$GL(n)$ – the general linear group over $\mathbb{C}$

S_d – the symmetric group on d letters

Partition – a decreasing sequence of positive integers

Pictorially, partition $(5, 2, 2)$ is represented as 

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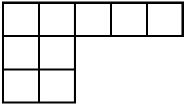
Pictorially, partition $(5, 2, 2)$ is represented as 

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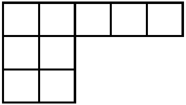
Pictorially, partition $(5, 2, 2)$ is represented as 

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- Irreducible polynomial representations of $GL(n)$ are parameterized by partitions with at most n rows - denoted $V_\lambda(GL_n)$

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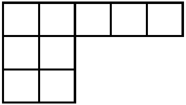
Pictorially, partition $(5,2,2)$ is represented as 

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- Irreducible polynomial representations of $GL(n)$ are parameterized by partitions with at most n rows - denoted $V_\lambda(GL_n)$
- Irreducible representations of S_d are parameterized by partitions of d - denoted $W_\lambda(S_d)$

The irreducible representations of GL_n

- Z - an $n \times n$ variable matrix.
- $\mathbb{C}[Z]$ - ring of polynomial functions in the entries of Z - a representation of GL_n.
  - $(\sigma \cdot f)(Z) = f(Z\sigma)$
- Semi-standard tableau T -
  -

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 5 \\ \cline{1-2}\end{array}$$

- C_T is the product of determinants of minors such as

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Theorem

V_λ is the subrepresentation spanned by C_T, T-semi standard.

Irreducible representations of S_d

- $\mathbb{C}[x_1, \cdots, x_d]$ - polynomials in n variables - a representation of S_d.
  - $(\sigma \cdot f) = f_(x_{\sigma(1)}, \cdots, x_{\sigma(d)})$.

  Standard tableau T -
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W_λ - the subrepresentation spanned by f_T, T-standard.

Tensor Products

- V a representation of G.
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Question

Find an explicit decomposition of the tensor product in terms of irreducible representations of G.

Littlewood-Richardson coefficients

- $G = GL_n((C))$.
- Find the multiplicity, $c^{\gamma}_{\alpha,\beta}$, of the irreducible representation of shape γ in $V_\alpha \otimes V_\beta$.
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Theorem

$c^{\gamma}_{\alpha,\beta} = \#$ LR skew-tableau of shape $\gamma \backslash \alpha$ with content β

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Checking non-zeroness of LR coeff is in P

- The precursor to Saturated linear programming.

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Conjecture

- $k^{\gamma}_{\alpha,\beta}$ is in $\#P$.
- non-zeroness of $k^{\gamma}_{\alpha,\beta}$ is in P.

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Affine varieties

- $V = \mathbb{C}^n$, $X = (x_1, x_2, \ldots, x_n)$ be the coordinates of V.
- An affine algebraic set, $Z \subseteq V$, is the zero set of a collection of polynomials in $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$.
- Irreducible - if it is not the union of two proper affine algebraic sets - Affine variety
- Coordinate ring $\mathbb{C}[Z]$ - $\mathbb{C}[X]/I(Z)$ where $I(Z)$ is the set of all polynomial functions vanishing on Z.
- Elements of $\mathbb{C}[Z]$ - polynomial functions on Z.
- Examples
  - X axis union Y axis is the zero set of $xy = 0$ in $\mathbb{C}[x, y]$.
  - The parabola is the zero set of $y^2 = x$ in $\mathbb{C}[x, y]$.

Projective Varieties

- $P^{n-1} \stackrel{def}{=} P(V)$, the projective space of lines in V through the origin.
- The homogeneous coordinate ring of $P(V)$ is defined to be $\mathbb{C}[X]$.
- A projective algebraic set $Y \subseteq V$ is the set of zeros of *homogeneous* functions on V.
- The *affine cone* $\hat{Y} \subseteq V$ is the union of lines in V corresponding to points in Y.
- The *homogeneous* coordinate ring $R(Y) = \mathbb{C}[X]/I(Y)$.
- Elements of $R[Y]$ - homogeneous functions on the cone over Y.
- Degree d component of $R[Y]$, is the space of homogeneous polynomials of degree d on Y.

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- $\mathbb{C}[X]/I(Y)$ - a representation of G.
- For $p \in Y$, $\sigma(p)$ is in Y - G moves points of Y around.

Orbit closures as special G-varieties

- $v \in P(V)$, a point and Gv the orbit of v.
- $$Gv = \{gv | g \in G\}$$
- $H = G_v \stackrel{def}{=} \{g \in G | gv = v\}$
- $\Delta_V[v] = \overline{Gv} \subseteq P(V)$, the orbit closure of v.

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Example: The Grassmanian

Let V_λ the irreducible GL_n module given by shape λ. Take the point v_λ. Then the orbit of v_λ is closed, and is isomorphic to G/P. (Very well studied in algebraic geometry)

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Valiant's reduction

Theorem

If $p(X_1, \ldots, X_n)$ has a formula of size $m/2$ then there is a matrix A of size $2m \times 2m$, with entries being constants (from the underlying field), or variables with $p(X_1, \ldots, X_n) = det_m(A)$.

Lets homogenize this construction:

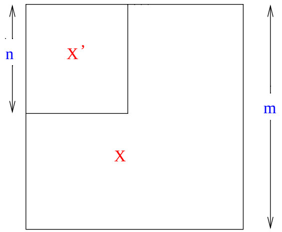
- Add an extra variable X_0.
- Let $p^m(X_0, X_1, \ldots, X_n)$ be the degree m homogenization of p.
- Homogenize A_{ij} using X_0.

Then $p^m(X_0, \ldots, X_n) = det_m(A)$.

The $\preceq_{hom}$ reduction

Let $X = (X_1, \ldots, X_r)$. For two forms $f, g \in Sym^d(X)$ we say we say $f \preceq_{hom} g$ if $f(X) = g(AX)$, where $A \in gl(X)$.
If $perm_n$ has a formula of size $m/2$ -
Consider the space of all $m \times m$ matrices. For $m > n$ we construct a new function, $perm_n^m \in Sym^m(X)$.

- Let X' be the principal $n \times n$ sub matrix of X.
- $perm_n^m(X) = x_{mm}^{m-n} perm_n(X')$.



Thus $perm_n$ has been inserted into $Sym^m(X)$, of which $det_m(X)$ is a special element. $perm_n^m \preceq_{hom} det_m$

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Group actions

Let $V = Sym^m(X)$.
Recall $GL(X)$ action on V:
$(\sigma \cdot f)(X) = f(\sigma^{-1}X)$.
Two notions:

- Orbit: $O(g) = \{\sigma \cdot g | \sigma \in GL(X)\}$
- The projective orbit closure: $\Delta_V(g) = \overline{O(g)}$

If $f \preceq_{hom} g$ then $f = g(\mu \cdot X)$, so:

- If μ is full rank, then f is in the $GL(X)$ orbit of g.
- If not, then μ is approximated by elements of $GL(X)$.

In either case,
$f \preceq_{hom} g \implies f \in \Delta(g)$

A faithful algebraization

- If $perm_n$ has a formula of size $m/2$ then $perm_n^m \in \Delta(det_m)$.
- On the other hand if $perm_n^m(X) \in \Delta(det_m)$ then for all $\epsilon > 0$, there is a a $\sigma \in GL(X)$ such that $\sigma \cdot det_m$ approximates $perm_n^m$ to ϵ.

  Gives a poly time approximation algorithm for permanent
- Recall $perm_n$ is $\#P$-complete.

Approach

To show $perm_n$ has no formula of size $n^c/2$ it suffices to show that $perm_n^{n^c} \notin \Delta(det_{n^c})$

Roadmap

Roadmap

Group theoretic varieties, and orbit closure membership

To determine if a form f belongs to $\Delta(g)$ (assuming both live in the same space), is in generalhopeless!

- $\Delta(det_{n^c})$ - the *class variety* associated to NC^2 is group theoretic.
- $\Delta(perm_n^{n^c})$ - the extended $\#P$ variety is also group theoretic.

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- $\Delta(det_{n^c})$ - the *class variety* associated to NC^2 is group theoretic.
- $\Delta(perm_n^{n^c})$ - the extended $\#P$ variety is also group theoretic.
- This is why we expect this problem to be tractable!

Rich stabilizers

The stabilizer G_{det_m} of the determinant in $G = GL(m^2)$

-
  - $X \mapsto AXB$, $A, B \in GL(m)$;
  - $X \mapsto X^T$
- $det_m \in Sym^m(m^2)$ determined by its stabilizer.

The stabilizer of the permanent in $GL(n^2)$.

-
  - $X' \mapsto PX'Q$, P, Q permutation matrices in $GL(n)$.
  - $X' \mapsto D_1X'D_2$, D_1, D_2 diagonal matrices in $GL(n)$.
  - $X' \mapsto X'^T$
- $perm_n \in Sym^n(n^2)$ is determined by its stabilizer.

The embedding of G_{det} is (almost) the natural embedding

$$GL(\mathbb{C}^m) \times GL(\mathbb{C}^m) \mapsto GL(\mathbb{C}^m \otimes \mathbb{C}^m)$$

Consequences of permanent having poly-sized formulas

Facts

- If $perm_n^m \in \Delta(det_m)$ then $\Delta(perm_n^m) \subseteq \Delta(det)$.

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- If $perm_n^m \in \Delta(det_m)$ then $\Delta(perm_n^m) \subseteq \Delta(det)$.
- Let $R(n, m^2)$ denote the coordinate ring of $\Delta(perm_n^m)$ and S the coordinate ring of $\Delta(det)$. Then $R_d \hookrightarrow S_d$ for all d.

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- Recall: Both R and S are $G = GL(m^2)$ representations.
- So every irreducible representation $V_\lambda \stackrel{def}{=} V_\lambda(G)$ that occurs in R_d as a subrepresentation, also occurs in S_d as a subrepresentation.

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Obstructions as witnesses

Definition

We say V_λ is an obstruction for the pair n, m^2 and the pair $(perm, det)$ in degree d if it occurs in R_d and not in S_d.

Conjecture

An obstruction for n, m^2 and the pair $(perm, det)$ exists if $m = 2^{n^a}$, for a small constant $a > 0$ as $n \to \infty$. There exists such an obstruction of a small degree m^b, $b > 0$, a large constant.

The specification of an obstruction is given in the form of its label λ.

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  - $\#$P by $G_{perm_n} \hookrightarrow GL(n^2) \hookrightarrow GL(V)$
  - NC^2 by $G_{det_m} \hookrightarrow GL(m^2) \hookrightarrow GL(W)$
- A *witness* for the non-existence of the embedding ought to be present in the representation-theoretic datum, assuming $\#P \neq NC^2$.

Representation theory of homogeneous spaces, $GL(V)/H$

- Suppose H_1, H_2 are reductive subgroups of a reductive group $G = GL(\mathbb{C}^l)$.
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  - $\mathbb{C}[G] = \oplus_\lambda V_\lambda \otimes V_\lambda$.
  - $\mathbb{C}[G/H] = \oplus_\lambda V_\lambda \otimes V_\lambda^H$

The work of Larsen and Pink

$V_\lambda(G)$ is an obstruction for the pair $(G/H_1, G/H_2)$ is it contains an H_1 invariant and does not contain an H_2 invariant.

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Conjecture, GCT II

Let $V = Sym^m(x_{11}, x_{12}, \ldots, x_{mm})$. Let Π be the set of $G = GL(m^2)$ submodules of $\mathbb{C}[V]$ whose duals do not contain a G_{det} invariant. Let $X(\Pi) \subseteq P(V)$ be the zero set of forms in Π. Then $X(\Pi) = \Delta(det_m)$.

What is known

Theorem, GCTII

There is a dense open neighbourhood $U \subseteq P(V)$ of the orbit of the determinant such that $\Delta(det_m) \cap U = X(\Pi) \cap U$.

Assuming the conjecture and the belief that the permanent cannot be approximated infinitely closely by circuits of poly-logarithmic depth,

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Obstructions do exist. GCT II.

We need to show obstructions exist, Unconditionally.

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Obstructions do exist. GCT II.

We need to show obstructions exist, Unconditionally.

- Important to note that existence of obstructions depends upon the special nature of det_m and so, also of the variety $\Delta(det_m)$.
- The conjecture will not be true for varieties arising out of many other NC^2-complete forms - the class variety may not be group theoretic.

The Kronecker problem and NC^2 vs $\#P$

- We need to understand which $GL(m^2)$ modules contain the stabilizer of the determinant form.

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- Now irreducible representations of $GL(m) \times GL(m)$ are of the form $V_\alpha \otimes V_\beta$, α, β, shapes, with at most m rows.
- Given an $GL(m^2)$ module of shape γ, we need to understand the multiplicity $k^\gamma_{\alpha,\beta}$ - this is exactly the Kronecker problem using Schur Weyl duality.

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Flip

- Non existence of algorithms reduced to existence of representation theoretic obstructions

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- This will overcome the naturalizability barrier!

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Roadmap

The GCT VI hypothesis

Hypothesis, GCT VI

The following problems belong to P.

- Verification There is a $poly(m^2, n, < d >, < \lambda >)$ algorithm for deciding, given m, n, d, λ, if V_λ is an obstruction of degree d for (n, m^2) and the pair $(perm_n, det_m)$.
- Explicit construction of obstructions Suppose $m = 2^{n^a}$ for a small constant $a > 0$. Then, for every $n \to \infty$, a label $\lambda(n)$ of an obstruction for (n, m^2) and the pair $(perm_n, det_m)$ can be constructed in time $poly(m)$, thereby proving the existence of an obstruction for every such n, m^2.
- Discovery of obstructionsThere exists a $poly(n, m)$ algorithm for deciding, if there exists an obstruction for (n, m^2) and the pair $(perm_n, det_m)$, and for constructing the label of one, if it exists.

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- By exhibiting an obstruction. SAME IDEA, take $\Delta(h_n^m)$ and show this is not embeddable in $\Delta(det_m)$.
- Of course, such obstructions exist in plenty! thanks to the special nature of determinant - , however

The P-barrier thesis

- It is unlikely that the hypothesis holds for a general h.

The P-barrier thesis

- It is unlikely that the hypothesis holds for a general h.
- Using currently available techniques, Gröbner basis, etc, we only get an algorithm which is double exponential in m, or triple exponential in n to verify, given λ, if V_λ is an obstruction for the pair (h, det_m)!
- Unlikely we will be able to do better, given general lower bounds for construction of Gröbner basis

The P-barrier thesis

For any approach towards $P \neq NP$ to be viable and non-naturalizeable, at least the problem of verifying an obstruction should be in P.

Crossing the P-barrier

Theorem

Hypothesis (a) is true assuming certain mathematical positivity hypothesis.

Crossing the P-barrier

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Theorem, GCT III

Checking non-zeroness of *LR* coefficients can be done in polynomial time - that is, given partitions α, β, γ,and n, checking if $c^{\gamma}_{\alpha,\beta}$ is non-zero can be done in time polynomial in $n, < \alpha >, < \beta >, < \gamma >$.

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Consequences of the Knutson-Tao Saturation conjecture

Theorem, GCT III

The polytope P described by the linear inequalities above has an integer point iff it is non-empty

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- However number of integral points in qP is precisely $c_{q\alpha,q\beta}^{q\gamma}$
- $c_{q\alpha,q\beta}^{q\gamma}$ is non-zero!
- Theorem, Knutson, Tao: $c_{q\alpha,q\beta}^{q\gamma} \neq 0 \implies c_{\alpha,\beta}^{\gamma} \neq 0$.

$\square$

Roadmap

Generalized Littlewood Richardson coefficients

- Given weights α, β, γ of a semi-simple Lie algebra $\mathcal{G}$ the generalized LR coefficient is the multiplicity of V_γ in $V_\alpha \otimes V_\beta$, where V_μ denotes the irreducible representation of $\mathcal{G}$ with highest weight μ.

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- $C^\gamma_{\alpha,\beta}$ is exactly the number of integral points in this polytope.

No saturation type theorem known for these coefficients.

A mathematical positivity hypothesis

- Let $C^{n\gamma}_{n\alpha,n\beta}$ be the stretching function associated with $C^{\gamma}_{\alpha,\beta}$. Then the stretching function is a quasi-polynomial with period at most two i.e.there exists two polynomials C_1, C_2 such that

$$C^{n\gamma}_{n\alpha,n\beta} = \begin{cases} C_1(n), n \text{ odd} \\ C_2(n), n \text{ even} \end{cases}$$

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Positivity hypothesis, De Loera, McAllister

The quasi-polynomial is positive - i.e. the coefficients of C_1, C_2 are positive.

Polynomiality under the hypothesis

Theorem GCT V

Assume $\mathcal{G}$ is simple of type B,C,D. Under the positivity hypothesis the following are equivalent.

1. $C^{\gamma}_{\alpha,\beta} \geq 1$
2. There exists an odd integer n such that $C^{n\gamma}_{n\alpha,n\beta} \geq 1$.
3. The BZ polytope contains a rational point whose denominators are all odd.
4. The affine span of the BZ polytope contains a rational point whose denominators are all odd.

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Proof.

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Proof.

(2) $\implies$ (1): If for some n odd, nP has an integral point then $C_1()$ should be a non-zero polynomial. Its coefficients are all positive. So $C_1(1)$ is also non-zero! □

Saturated quasi-polynomials

- A function $f(n)$ is called a quasi-polynomial if there exist l polynomials $f_j(n)$, $1 \leq j \leq l$, such that $f(n)=f_j(n)$ if $n = j$ mod l. Here l is supposed to be the smallest such integer, and is called the period of $f(n)$.
- The smallest j, $1 \leq j \leq l$, such that f_j is not identically zero, is called the index of the quasi-polynomial.
- We say that the quasi-polynomial $f(n)$ which is not identically zero is saturated , if $f(index(f)) \neq 0$.
- Positivity $\implies$ Saturation. If each f_j is guaranteed to have positive coefficients, f is saturated.

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- Integer Programming: Does the polytope $\{x|Ax \leq b\}$ contain an integer point?

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- Saturated integer programming has a polynomial time algorithm.

Questions associated with quasi-polynomials

Given a quasi-polynomial $g(n)$ we may ask:

- Is g convex? Is there a polytope P whose Ehrhart polynomial $f_P(n)$ coincides with $g(n)$.
- Is g positive?
- Is g saturated?
- Does $G(t) \stackrel{def}{=} \sum_n g(n)t^n$, have a reduced, positive form?

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Kirillov's conjecture

- Associate with the Kronecker constant $k^{\gamma}_{\alpha,\beta}$ the following stretching function.

$$\tilde{k}^{\gamma}_{\alpha,\beta}(n) = k^{n\gamma}_{n\alpha,\beta}$$

Conjecture, Kirillov

The generating function, $K^{\gamma}_{\alpha,\beta}(t) = \sum_{n\geq 0} k^{n\gamma}_{n\alpha,\beta} t^n$ is a rational function.

Kirillov's conjecture verified, GCT VI

Theorem

(a) Rationality The generating function $K^{\gamma}_{\alpha,\beta}(t)$ is rational.

(b) Quasi-polynomiality The stretching function $\tilde{k}^{\gamma}_{\alpha,\beta}(n)$ is a quasi-polynomial function of n.

(c) There exist graded, normal $\mathbb{C}$-algebras $S = S(k^{\gamma}_{\alpha,\beta}) = \oplus_n S_n$, and $T = T(k^{\gamma}_{\alpha,\beta}) = \oplus_n T_n$ such that:

1. *The schemes $spec(S)$ and $spec(T)$ are normal and have rational singularities.*
2. *$T = S^{GL_n(\mathbb{C})}$, the subring of $GL_n(\mathbb{C})$-invariants in S.*
3. *The quasi-polynomial $\tilde{k}^{\gamma}_{\alpha,\beta}(n)$ is the Hilbert function of T. In other words, it is the Hilbert function of the homogeneous coordinate ring of the projective scheme $Proj(T)$.*

(d) Positivity The rational function $K^{\gamma}_{\alpha,\beta}(t)$ can be expressed in a positive form:

Kirillov .....

$$K^{\gamma}_{\alpha,\beta}(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{\prod_j (1 - t^{a(j)})^{d(j)}},$$

where $a(j)$'s and $d(j)$'s are positive integers, $\sum_j d(j) = d + 1$, where d is the degree of the quasi-polynomial $\tilde{k}^{\gamma}_{\alpha,\beta}(n)$, $h_0 = 1$, and h_i's are nonnegative integers.

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- The rational function we get from the theorem is not unique. Depends upon a homogeneous system of parameters we choose. We have

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For the lex least system of parameters, $K^{\gamma}_{\alpha,\beta}(t)$ has a reduced positive form, with $max(a_i)$ bounded by a polynomial in the height of the shapes α, β and γ.

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- If so, then we will be able to conclude that $\tilde{k}^{\gamma}_{\alpha,\beta}$ is saturated.

Proof of quasipolynomiality of $K^{\gamma}_{\alpha,\beta}$

.

Theorem

Let $R = \oplus_d R_d$ *be a graded ring with an action of a connected reductive group* H. *Let* V_π *be a fixed irreducible* H *module with label* π. *Let* s^π_d *denote the multiplicity of the* V_π *in* R_d.*Take its stretching function* $\tilde{s}^\pi_d(n)$ *to be the multiplicity of* $V_{n\pi}$ *in* R_{nd}. *Take the generating function* $S^\pi_d(t) = \sum_{n \geq 0} \tilde{s}^\pi_d(n) t^n$. *Then* S *satisfies the claims of the theorem.*

Proof

- Consider the action of the cyclic group of order d with generator ζ; it acts on R sending $x \in R_k$ to $\zeta^k x$. The invariant ring $B = \sum_n R_{nd}$, for this action is normal and has rational singularities.

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- Let $V_{\pi^*} = V_\pi^*$. Consider the homogeneous coordinate ring $C_{\pi^*} = \sum_n V_{n\pi^*}$, of the orbit of v_{π^*}. This too is normal with rational singularities.(It is a G/P)

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- Our desired generating function is the Hilbert series of this ring T.

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Implications of the theorem - positivity hypothesis

Hypothesis; **PH1**

The Kronecker stretching function is convex.
For every (α, β, γ) there exists a polytope $P = P^{\gamma}_{\alpha,\beta} \subseteq \mathbb{R}^m$ such that:
(1) The Ehrhart quasi-polynomial of P coincides with the stretching quasi-polynomial $\tilde{k}^{\gamma}_{\alpha,\beta}(n)$.
(2) The dimension m of the ambient space, and hence the dimension of P as well, are polynomial in the bit-lengths of α, β and γ.
(3) Whether a point $x \in \mathbb{R}^m$ lies in P can be decided in $poly(<\alpha>, <\beta>, <\gamma>)$ time. That is, the membership problem belongs to the complexity class P. If x does not lie in P, then this algorithm outputs a hyper plane separating x from P.

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Polynomiality of $k^{\gamma}_{\alpha,\beta}$ under PH1,PH3

Theorem

Under *PH*1, *PH*3, determining if $k^{\gamma}_{\alpha,\beta}$ is non-zero can be done in polynomial time.

- Experimental evidence for *PH*3, and positivity of $\tilde{k}^{\gamma}_{\alpha,\beta}$.

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Theorem

Under *PH*1, *PH*3, determining if $k^{\gamma}_{\alpha,\beta}$ is non-zero can be done in polynomial time.

- Experimental evidence for *PH*3, and positivity of $\tilde{k}^{\gamma}_{\alpha,\beta}$.
- Current proofs of *PH*1 for type *B*, *C*, *D* are based on quantum groups
- The homogeneous coordinate rings of the canonical models associated with the Littlewood-Richardson coefficients have quantizations endowed with canonical bases Kashiwara and Lusztig.
- Positivity of the basis is also based on the Riemann hypothesis over finite fields, and work of Beilinson, Bernstein, Deligne.

A plan to prove $PH1, PH2, PH3$

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3. Prove PH1 and SH (and, possibly, the stronger PH2, and 3 as well) by a detailed analysis and study of these positive bases.