Torus Quotients of Flag Varieties - a Computational Approach Based on Lattice Basis

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1 Abstract

Consider the Plücker embedding, $Gr_{3,7}$, of the Grassmannian of three dimensional subspaces of a seven dimensional space in projective space $\mathbb{P}(\bigwedge^3 \mathbb{C}^7)$. Consider the action of the maximal torus of diagonal matrices in $SL(7, \mathbb{C})$ on $Gr_{3,7}$. We give a computational proof that the GIT quotient of $Gr_{3,7}$ is projectively normal. Our computations rely on the connection between generators of rings of invariants under the action of a torus and Hilbert basis of polyhedral cones. We use Sage for our computation. We describe the geometry of the torus quotient of partial flag varieties, $SL(5, \mathbb{C})/Q$, when Q is the intersection of two maximal parabolic subgroups of $SL(5, \mathbb{C})$ containing a Borel subgroup. The proofs are computational, based again on computing generators for rings of invariants using the connection to Hilbert basis of polyhedral cone. However, unlike in $Gr_{3,7}$, these can in principle be checked by hand.

2 Introduction

Let X be a projective variety over $\mathbb{C}$. Let T be an algebraic torus over $\mathbb{C}$ acting on X and let $\mathcal{L}$ be a T-equivariant ample line bundle on T. A point x in X is said to be $\mathcal{L}$-semistable if there is a T-invariant section of a positive power of $\mathcal{L}$ that does not vanish at p. The set of $\mathcal{L}$-semistable points of X is denoted by $X_T^{ss}(\mathcal{L})$. An $\mathcal{L}$-semistable point p is said to be stable if its stabilizer in T is finite and the orbit of p in $X_T^{ss}(\mathcal{L})$ is closed. The GIT quotient of X is defined to the categorical quotient of $X_T^{ss}(\mathcal{L})$ by T, in which two points are identified if the closures of their T-orbits intersect. We denote the GIT quotient of the polarized variety by $T \backslash\backslash X_T^{ss}(\mathcal{L})$.

Let $G = SL(n)$ and T be the subgroup of diagonal matrices in G. There is a natural action of G on the n-dimensional vector space $\mathbb{C}^n$. There is a natural action of G on flags in $\mathbb{C}^n$. The motivating question of this paper is understanding the GIT quotient of these flag varieties under the action of T. To describe the problem statements and our results we first introduce some notation.

2.1 Notations

We first recall some well known results. The proofs of these statements can be found in [LR07] and [Ses07]. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. Let T be the maximal torus consisting

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of all diagonal matrices in G. Let B be the Borel subgroup of upper triangular matrices. We denote the opposite Borel subgroup of all lower triangular matrices in G determined by B and T by B^-. The Weyl group of G with respect to T is the symmetric group S_n and we have the Bruhat decomposition $G = \bigcup_w BwB$, where $w \in S_n$. Let R be the set of roots for the adjoint action of T on G. Let $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$ denote the set of simple roots in R^+. The simple reflection in W corresponding to α_i is denoted by s_i. Then (W, S) is a Coxeter group (see [Hum75, Theorem 29.4, p.180]). For a subset J of S, we denote by W_J the subgroup of W generated by $\{s_\alpha : \alpha \in J\}$. Let $W^J := \{w \in W : w(\alpha) \in R^+ \ for\ all\ \alpha \in J\}$. For each $w \in W_J$, choose a representative element $n_w \in N_G(T)$. Let $N_J := \{n_w : w \in W_J\}$. Let $P_J := BN_JB$. We denote by P^{α_r}, the maximal parabolic subgroup of G corresponding to $S \setminus \{\alpha_r\}$. Let $\{\omega_i : 1 \leq i \leq n-1\}$ be the set of fundamental dominant weights corresponding to $\{\alpha_i : 1 \leq i \leq n-1\}$. We use the notation W^{P_J} for W^J and, in particular, $W^{S \setminus \{\alpha_r\}}$ is denoted by $W^{P^{\alpha_r}}$.

Let $I_{r,n} := \{\underline{\mathbf{i}} = (i_1, i_2, \ldots, i_r) | 1 \leq i_1 < i_2 \cdots < i_r \leq n\}$ be the set of all strictly increasing sequences of legth r with entries in $[n]$. A canonical basis of $\bigwedge^r \mathbb{C}^n$ is given by $\{e_{\underline{\mathbf{i}}} = e_{i_1} \wedge \ldots \wedge e_{i_r}, \underline{\mathbf{i}} \in I_{r,n}\}$. The Grassmannian $G_{r,n}$ of r-dimensional subspaces of $\mathbb{C}^n$ can be viewed as a subvariety of $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$, given by sending an r-dimensional subspace of $\mathbb{C}^n$ with basis $v_1, v_2, \ldots, v_r$ to the class $[v_1 \wedge v_2 \wedge \cdots v_r] \in \mathbb{P}(\bigwedge^r \mathbb{C}^n)$. This identifies the Grassmannian with the homogeneous space G/P^{α_r}, since P^{α_r} is the stabiliser in G of the vector subspace spanned by $[e_1, e_2, \ldots, e_r]$. We denote this polarization of $G_{r,n}$ by $\mathcal{L}(\omega_r)$. When r is clear from the context we also use the notation G/P for the Grassmannian.

Note that the Weyl group of P^{α_r} is $W_{S \setminus \{\alpha_r\}}$. The coset representatives of $W/W_{S \setminus \{\alpha_r\}}$ of minimal length, $W^{P^{\alpha_r}}$, can be identified with $I_{r,n}$, with the coset representative w corresponding to the subspace spanned by $e_{w(1)}, \ldots, e_{w(r)}$. Note that $e_w := [e_{w(1)} \wedge e_{w(2)} \cdots \wedge e_{w(r)}]$ is a T fixed point for any $w \in W^{P^{\alpha_r}}$. The B orbit closure of e_w is called a Schubert variety in G/P and we denote it by $X(w)$. G/P is itself a Schubert variety for $w_0^{S \setminus \alpha_r}$, the minimal representative of the longest element w_0 in $W^{S \setminus \alpha_r}$.

The last definition we need to recall is that of projective normality of a projective variety. Let X be a projective variety in $\mathbb{P}^m$. We denote by $\hat{X}$ the affine cone of X. X is said to be projectively normal if $\hat{X}$ is normal. For a reference, see exercise 3.18, page. 23 of [Har77].

2.2 Background

The GIT quotient of $Gr_{r,n}$ under the action of T is well studied. Gelfand and Macpherson [GM82] studied this by considering the GIT quotient of n points spanning projective space $\mathbb{P}^{r-1}$, with respect to the group $PGL(r, \mathbb{C})$. Their results were extended by Gelfand et al in [GGMS87]. Hausmann and Knutson[HK97] study the GIT quotient of $Gr_{2,n}$ and related the resulting GIT quotient to the moduli space of polygons in $\mathbb{R}^3$. Skorobogatov [Sko93] gave combinatorial conditions determining when a point in $G_{r,n}$ is semistable with respect to the T-linearized bundle $\mathcal{L}(\omega_r)$. As a corollary he showed that when r and n are coprime, semistability is the same as stability. This was proved independently by Kannan [Kan98]. Howard et al. [HMSV05] showed that the relations among the ring of invariants for the diagonal action of $SL(2)$ on $(\mathbb{P}^1)^n$ are generated in degree four. For Q a parabolic subgroup of G, Howard [How05] considered the problem of determining which line bundle on G/Q descends to an ample line bundle of the GIT quotient of G/Q by T. Kumar [Kum08] extended these results to other algebraic groups. We use the notation $\mathcal{M}$ to denote the descent of the ample bundle to the GIT quotient. Kannan and Sardar [KS09] studied T-quotients of Schubert varieties in $G_{r,n}$ and showed that, when r and n are coprime there is a minimal Schubert variety in $Gr_{r,n}$ with semistable points. Kannan and Pattanayak extended these results to the case when G is of type B, C or D, and when P is a maximal parabolic subgroup of G. In [KPPU18], Kannan et al extended the results in [KS09] to Richardson varieties in the Grassmanian $G_{r,n}$. In [BKS20] the authors show that $(T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}_{n\omega_2}), \mathcal{M})$ is projectively normal when n is odd, using combinatorial methods. They show that the torus quotient of the minimal Schubert variety in $Gr_{3,7}$ is projectively normal for the embedding given by $\mathcal{M}$. Here $\mathcal{M}$ is the descent of $\mathcal{L}(7\omega_3)$. They leave open the question of whether $(T\backslash\backslash(G_{3,7})^{ss}_T(\mathcal{L}_{\omega_3}), \mathcal{M})$ is projectively normal.

2.3 Main results

We give a computational proof that the polarised variety $(T\backslash\backslash(G_{3,7})^{ss}_T(\mathcal{L}_{7\omega_3}), \mathcal{M})$ is projectively normal. We also study $T\backslash\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda)$. Here $Q = P^{\alpha_r} \cap P^{\alpha_s}$ for some integers $1 \le r < s \le 4$ and $\lambda = a\omega_r + b\omega_s$, with a and b positive integers such that $ar + bs = 5$. We give complete descriptions of the GIT quotients in these cases.

2.4 Organisation

In Section 3.1 we recall relevant background on standard monomials that we use. In Section 3.1.1 we describe the subring of T-invariants. We then recall definitions from polyhedral theory. This allows us to set up a bijection between T-invariant monomials in the coordinate ring of $Gr_{r,n}$ and unions of lattice points of pointed cones. In Section 5 we use this connection to give a computational proof that $(T\backslash\backslash(G_{3,7})^{ss}_T\mathcal{L}(7\omega_3), \mathcal{M})$ is projectively normal. In Section 6 we give complete descriptions of $T\backslash\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda)$ when Q is the intersection of two maximal parabolic subgroups of $SL(5,\mathbb{C})$, and λ is as described in the above paragraph.

3 Preliminaries

We recall some notions from standard monomial theory that we will use, see [Ses07] and [LR07]. The main theorem of standard monomial theory for $Gr_{r,n}$ allows us to describe the subring of T invariants of $Gr_{r,n}$ for the Plücker embedding and for flag varieties in general. We then recall a theorem proved independently by Skorobogatov [Sko93] and Kannan [Kan98], alluded to in the background section, which gives sufficient conditions for when semistability and stability coincide for the action of T on $Gr_{r,n}$.

3.1 Standard monomials

We define a partial order "$\le$" on $I_{r,n}$ by defining $\underline{\mathbf{i}} \le \underline{\mathbf{j}} \Leftrightarrow i_t \le j_t$ for all $t = 1, 2, \cdots, r$. The set $I_{r,n}$ with this partial order is called the Brühat poset. It will be useful to refine the partial order to a total order which we will also denote by the same symbol. Then there is a natural lexicographic order on $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ which is a monomial order.

Denote by $\{p_{\underline{\mathbf{i}}} | \underline{\mathbf{i}} \in I_{r,n}\}$ the dual basis of the canonical basis in $(\bigwedge^r \mathbb{C}^n)^*$. The $p_{\underline{\mathbf{i}}}$'s are called Plücker coordinates on $Gr_{r,n}$. The homogeneous coordinate ring of $\mathbb{C}[Gr_{r,n}]$ is the quotient of $\mathbb{C}[p_{\underline{\mathbf{i}}} | \underline{\mathbf{i}} \in I_{r,n}]$ by the ideal of polynomials $I(Gr_{r,n})$ vanishing on $Gr_{r,n}$ for its embedding in $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$.

A monomial $p_{\underline{\mathbf{i}}^1} p_{\underline{\mathbf{i}}^2} \cdots p_{\underline{\mathbf{i}}^s}$ of degree s in the Plücker coordinates is said to be standard in $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ iff $\underline{\mathbf{i}}^1 \le \underline{\mathbf{i}}^2 \cdots \le \underline{\mathbf{i}}^s$. If a product of monomials $p_{\underline{\mathbf{i}}} p_{\underline{\mathbf{j}}}$ is not standard then it is known that

$$p_{\underline{\mathbf{i}}} p_{\underline{\mathbf{j}}} = p_{\underline{\mathbf{i}} \cup \underline{\mathbf{j}}} p_{\underline{\mathbf{i}} \cap \underline{\mathbf{j}}} + \text{other quadratic terms modulo } I(G_{r,n}).$$

Here $(\underline{\mathbf{i}} \cup \underline{\mathbf{j}})_t = max(i_t, j_t), t = 1, \ldots, r$ and $(\underline{\mathbf{i}} \cap \underline{\mathbf{j}})_t = min(i_t, j_t), t = 1, \ldots, r$. Furthermore the other quadratic terms in the above expression are products of monomials which are strictly bigger than $p_{\underline{\mathbf{i}}} p_{\underline{\mathbf{j}}}$ in the monomial order on $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$. Quadratic terms in the above expression which are nonstandard can be refined again using the same rule and, since the terms are increasing in lex order, we see that any non standard monomial has an expression in terms of standard monomials. The difference between the two expressions goes by the name ”straightening law” and this is an element of the ideal $I(Gr_{r,n})$. As a result we have the following theorem, see [Ses07, Proposition 1.3.6].

Theorem 3.1.

1. *Standard monomials of degree m, $m \geq 0$, form a basis of the homogeneous coordinate ring of the Grassmannian.*
2. *The ideal $I(Gr_{r,n})$ is generated by straightening laws.*

Let $w \in W^{P^{\alpha_r}}$ corresponds to the element $\underline{\mathbf{i}} \in I_{r,n}$. The Schubert variety $X(w) \subset Gr_{r,n}$ is given by the vanishing of the $p_{\underline{\mathbf{j}}}, \underline{\mathbf{j}} \not\leq \underline{\mathbf{i}}$. It follows from [Ses07, Proposition 1.4.5] that standard monomials $p_{\underline{\mathbf{i}}^1} p_{\underline{\mathbf{i}}^2} \cdots p_{\underline{\mathbf{i}}^s}$ of degree s with $\underline{\mathbf{i}}^s \leq \underline{\mathbf{i}}$ span a basis of $H^0(X(w), s\omega_r)$.

3.1.1 T-invariants

Let $p_{\underline{i}_1}^{a_{i_1}} \cdots p_{\underline{i}_s}^{a_{i_s}}$ be a monomial in $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ of degree s. We associate with this monomial a column standard tableau S of shape $(\sum_j a_{\underline{i}_j})^r$ to this monomial by stacking $a_{\underline{i}_1}$ copies of $\underline{i}_1$ then $a_{\underline{i}_2}$ copies of $\underline{i}_2$ and so on, from left to right. Conversely for any column standard rectangular tableau S of shape $(\sum_j a_{\underline{i}_j})^r$ we get a monomial p_S of degree s in $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ by taking the product of the Plücker coordinates indexed by each column.

Denote by p_S the monomial associated to a column standard tableau S. The weight of a tableau S is defined to be the n-dimentional vector $wt(S)$, such that any $i \in [n]$ appears $wt(S)_i$ times in S. The weight of a monomial in $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ is the weight of the column standard tableau corresponding to the monomial.

It is easy to describe the action of T on $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$. For any $t = diag(t_1, \cdots, t_n) \in T$ and $\underline{i} \in I_{r,n}$ define $t \cdot p_{\underline{i}} = (t_{i_1} \cdots t_{i_r})^{-1} p_{\underline{i}}$. It follows that for any monomial $p_{_S}$, $t \cdot p_{_S} = (t_i^{-wt(S)_1} \cdots t_n^{-wt(S)_n}) p_{_S}$. We have,

Proposition 3.2. *A monomial $p_{_S} \in \mathbb{C}[\bigwedge^r \mathbb{C}^n]$ is T-invariant iff $wt(S)$ is a uniform vector i.e. if $wt(S)$ is a vector of the form $(k, k, \ldots, k)$ for some positive integer k.*

We recall the following proposition, see Skorobogatov [Sko93] and Kannan [Kan98].

Proposition 3.3. *Let $(r, n) = 1$. Then we have $(G/P)_T^{ss}(\mathcal{L}(\omega_r)) = (G/P)_T^{s}(\mathcal{L}(\omega_r))$.*

If all semistable points are stable, the GIT quotient would in fact be a geometric quotient, with each point in the quotient variety corresponding to an orbit under T. We will assume henceforth that $(r, n) = 1$. In this setting if $p_{_S} = p_{\underline{i}_1} p_{\underline{i}_2} \cdots p_{\underline{i}_s}$ is T invariant then s is multiple of n and $wt(S) = (\frac{sr}{n}, \cdots, \frac{sr}{n})$. It is easy to see that there are monomials of degree n in the Plücker coordinates which are T-invariant, see [BKS20] for example. We have,

Theorem 3.4. *Let $R(k) = H^0(G/P, \mathcal{L}(kn\omega_r))^T$. Then $T \backslash\backslash (G/P)_T^{ss} \mathcal{L}(\omega_r) = Proj(\oplus_k R(k))$.*

This gives us a geometric description of the quotient. However we would like to say more, and for that we will need to compute an explicit set of generators of the ring $\oplus_k R(k)$ and relations between them. When $r = 2$ this is known, see [HMSV05], [BKS20]. However this becomes difficult when r is bigger than or equal to three since there are too many generators and this has to be done more systematically. This is what we address in the next section.

4 Generating set of T-invariants using polyhedral theory

Since $H^0(Gr_{r,n}, \mathcal{L}(kn\omega_r))$ is spanned by standard monomials, the homogeneous coordinate ring of the T-quotient, $R = \oplus_k R(k)$, is spanned by T-invariants which are standard. We call these standard T-invariants. Now, standardness is defined by the "$\leq$" relation on the Brühat poset. A T-invariant monomial which is the product of Plücker coordinates with support in a maximal chain in the Brühat poset is naturally standard.

This gives us a natural recipe to find generators for the ring of invariants. In the first step, for every maximal chain $\mathcal{C}$ in the Brühat poset find a generating set of T-invariant monomials which can be expressed as a product of Plücker coordinates with support in $\mathcal{C}$. Then take the union of these generating sets of T-invariant monomials over all maximal chains in the Brühat poset. This follows, since every standard monomial $p_S$which is T-invariant has Plücker coordinates which are pairwise comparable, and so there is a maximal chain in the Brühat poset which contains the support of p_S. This is the approach we take. In the final step we prune the generating set using the straightening relations. To carry out the first step, we use polyhedral theory. We describe that in the next section. It will be clear that with a little work this approach can be extended to partial flag varieties also. While this approach is general, it is computationally prohibitive. We are able to carry this program out for $Gr_{3,7}$ and flag varieties of of the form $SL(5,\mathbb{C})/Q$, where Q is the intersection of two maximal parabolic subgroups of $SL(5,\mathbb{C})$ containing B.

4.1 T invariants with support in a maximal chain

We formulate the question of obtaining a generating set of T-invariant monomials with support in a fixed maximal chain $\mathcal{C}$, of the Bruhat poset using notions from polyhedral theory. We recall the relevant definitions. A standard and wonderful reference for all this is Schrijver's book [Sch98].

Definition 4.1 (Cone)**.** *A subset $C \subseteq \mathbb{R}^n$ is a cone if for any $x, y \in C$ and non-negative real numbers λ_1, λ_2, we have $\lambda_1 x + \lambda_2 y \in P$. A cone C is pointed if $C \cap -C = \{0\}$. A cone C is polyhedral if $C = \{x \in \mathbb{R}^n | Mx \geq 0\}$ for some $m \times n$ real matrix M. A set $\{g_1, ...g_k\} \subset C$ is called a (conical) generating set of C if for all $x \in C$ there exist non-negative real numbers $\lambda_1, ..., \lambda_k$ such that $x = \sum_{i=1}^{k} \lambda_i g_i$.*

Definition 4.2 (Hilbert basis)**.** *Let $C \in \mathbb{R}^n$ be a polyhedral cone with rational generators. We call a finite set $H = \{h_1, ..., h_t\} \subset \mathbb{Z}^n \cap C$ a Hilbert basis of C, if for every integral vector $v \in C$ there exist non-negative integers $\lambda_1, ..., \lambda_t$ such that $v = \sum_{i=1}^{t} \lambda_i h_i$.*

The following is well known, see [Sch98, Theorem ?].

Proposition 4.3. *Every rational polyhedral pointed cone has a unique inclusion minimal Hilbert basis.*

We will show that T-invariant monomials in a fixed maximal chain $\mathcal{C}$ in the Brühat order are in bijection with integer points in a polyhedral cone $C_{\mathcal{C}}$. It will then follow that monomials corresponding to the Hilbert basis of integer points in $C_{\mathcal{C}}$ will be a generating set for invariants whose support is in $\mathcal{C}$, and these will be standard by the discussion in the previous section.

Let $\mathcal{C} = \underline{i}_1 \leq \cdots \leq \underline{i}_l$ be a chain in the Bruhat poset $I_{r,n}$. For every standard monomial $p_{\underline{i}_1}^{a_{\underline{i}_1}} p_{\underline{i}_2}^{a_{\underline{i}_2}} ... p_{\underline{i}_l}^{a_{\underline{i}_l}}$ of degree k, we have a semistandard tableau S of rectangular shape (k^r) such that column $\underline{i}_j$ appears $a_{\underline{i}_j}$ times in S, for each $1 \leq j \leq l$. Note that some of $a_{\underline{i}_j}$ may be 0. Associate vectors $v'_S = (a_{\underline{i}_1}, ..., a_{\underline{i}_l})$ and $v_S = (v'_S, \frac{k}{n})$ with this standard monomial. If X, Y are tableaux with support in $\mathcal{C}$ and XY is the tableau associated with monomial $p_X p_Y$ then clearly $v'_{XY} = v'_X + v'_Y$ and $v_{XY} = v_X + v_Y$.

We associate with $\mathcal{C}$ an $n \times l$ matrix $A'_{\mathcal{C}}$ with rows indexed by $[n]$ and columns indexed by Plücker coordinates with support in the chain $\mathcal{C}$ as follows: for $i \in [n], \underline{j} \in \mathcal{C}$,

$$(A'_{\mathcal{C}})_{i,\underline{j}} = \begin{cases} 1 & i \in \underline{j} \\ 0 & i \notin \underline{j} \end{cases}$$

That is the $\underline{j}^{th}$ column of $A'_{\mathcal{C}}$ is weight vector of column tableau $\underline{j}$. Let $v = (-r, \cdots, -r)$ be a n-dimentional vector. Let $A_{\mathcal{C}} := [A'_{\mathcal{C}}|v]$ be the matrix $A'_{\mathcal{C}}$ augmented with column v. Let $C_{\mathcal{C}} = \mathbb{R}^{l+1}_{\geq 0} \cap ker(A_{\mathcal{C}})$. We have following observations.

Observation 4.4. *Let S be a tableau whose columns are in $\mathcal{C}$. For $i \in [n]$ we have*

$$(A'_{\mathcal{C}} v'_S)_i = \textit{number of columns in } S \textit{ which contain } i.$$

Hence, $wt(S) = A'_{\mathcal{C}} v'_S$. If S is T-invariant then v_S is a nonzero integral vector in $C_{\mathcal{C}}$.

Proof.

$$(A'_{\mathcal{C}} v'_S)_i = \sum_{\underline{j} \in \mathcal{C}} (A'_{\mathcal{C}})_{i,\underline{j}} (v'_S)_{\underline{j}} = \sum_{i \in \underline{j}, \underline{j} \in \mathcal{C}} (v'_S)_{\underline{j}}.$$

This proves $wt(S) = A'_{\mathcal{C}} v'_{\mathcal{C}}$.

To prove that v_S is integral vector for a T-invariant tableau S, we need to show that $\frac{k}{n}$ is integer. Observe that since $wt(S) = (\frac{kr}{n}, \cdots, \frac{kr}{n})$ is an integer vector we have $n|k$ since $(r, n) = 1$. $v_S \in ker(A_{\mathcal{C}})$ follows from definition of $A_{\mathcal{C}}$. The non-negativity of v_S follows from definition of v_S. $\square$

Observation 4.5. *Let $x = (x', d) \in \mathbb{Z}^{l+1}_{\geq 0} \cap C_{\mathcal{C}}$ be a nonzero integer vector and suppose $d = \frac{k}{n}$ for some k. Then there exists a T-invariant semistandard tableau S of degree k such that $x = v_S$.*

Proof. Define the tableau S with $\underline{j} \in \mathcal{C}$ appearing in S, $x'_{\underline{j}}$ times. Then we have $x' = v'_S$. Since the columns of S have support in $\mathcal{C}$, these columns can be sorted so that S is semistandard. It remains to show that S is T-invariant and of degree k. That is $wt(S) = (\frac{kr}{n}, \cdots, \frac{kr}{n})$. This follows from calculation below.

$$A_{\mathcal{C}} x = 0 \implies [A'_{\mathcal{C}}|w](x', \frac{k}{n}) = 0 \implies A'_{\mathcal{C}} x' = -\frac{k}{n} w \implies wt(S) = (\frac{kr}{n}, \cdots, \frac{kr}{n}).$$

$\square$

Corollary 4.6. *We have a bijection between T invariant monomials with support in $\mathcal{C}$ and nonzero integral points in $C_{\mathcal{C}}$ given by tableau $S \mapsto v_S$.*

Proof. This follows from the above observations. $\square$

Corollary 4.7. $C_{\mathcal{C}} = ker(A_{\mathcal{C}}) \cap \mathbb{R}^{l+1}_{\geq 0}$ *is a pointed cone. Hence $C_{\mathcal{C}}$ has a unique inclusion minimal Hilbert basis.*

Proof. This follows from the observation that any subspace intersected with a pointed cone (the non-negative orthant in our case) results in a pointed cone. $\square$

4.2 Generating set of T-invariants

For a maximal chain $\mathcal{C}$, let $H_{\mathcal{C}}$ denote the unique Hilbert basis of the pointed polyhedral cone $C_{\mathcal{C}}$.

Definition 4.8. *We say a T invariant semistandard tableau S splits directly if there are two T invariant semistandard tableaux X, Y such that $S = XY$ up to rearrangement of columns.*

The following lemma is now immediate.

Lemma 4.9. *Let S be any T invariant semistandard tableau. S splits directly iff $v_{_S} \notin H_{\mathcal{C}}$ for any maximal chain $\mathcal{C}$ in $I_{r,n}$.*

Proof. Let $\mathcal{C}$ be any maximal chain such that all columns of S are in $\mathcal{C}$.

($\Longrightarrow$) Let S splits directly then there are T-invariant semistandard tableaux X, Y such that $S = XY$ up to rearranging columns. This implies $v_{_S} = v_{_X} + v_{_Y}$ in the chain $\mathcal{C}$ and thus $v_{_S}$ is decomposible hence $v_{_S} \notin H_{\mathcal{C}}$.

($\Longleftarrow$) If $v_{_S} \notin H_{\mathcal{C}}$ then there are non-negative integers $\lambda_1, ..., \lambda_t$ such that $v_{_S} = \sum_{i=1}^{t} \lambda_i h_i$ and $\sum_{i=1}^{t} \lambda_i \geq 2$ (otherwise $v_{_S} \in H$). This implies there is $u \in \mathbb{Z}^{l+1} \cap C_{\mathcal{C}}$ and $h, h' \in H$ such that $v_{_S} = h + h' + u$. Hence $S = XYZ$ up to rearranging columns where X, Y, Z are T-invariant tableaux such that $v_{_X} = h, v_{_Y} = h', v_{_Z} = u$. So S splits, completing the proof. $\square$

We have,

Theorem 4.10. *The coordinate ring of $T \backslash\backslash (G_{r,n})^{ss}_{T} \mathcal{L}(n\omega_r)$ is generated by the union of standard monomials corresponding to elements in the Hilbert basis $H_{\mathcal{C}}$, the union taken over all maximal chains $\mathcal{C}$ in $I_{r,n}$.*

Proof. The proof follows from the definition of a Hilbert basis of a polyhedral cone, the definition of a split tableau and Corollary 4.6, Corollary 4.7 and Lemma 4.9.

$\square$

In fact the above proof works for any Schubert variety in the Grassmannian. We state this for completeness.

Corollary 4.11. *Let $w \in I_{r,n}$ and $\mathcal{P} \subset I_{r,n}$ be the sub-poset consisting of elements $v \in I_{r,n}$ such that $v \leq w$. The coordinate ring of $T \backslash\backslash X(w)^{ss}_{T}(\mathcal{L}(n\omega_r))$ is generated by the union of standard monomials corresponding to elements in the Hilbert basis $H_{\mathcal{C}}$, where the union is taken over all maximal chains $\mathcal{C}$ in the poset $\mathcal{P}$.*

Proof. Clear. $\square$

5 The case of $Gr_{3,7}$

In this section we give a computational proof that $(T \backslash\backslash (G_{3,7})^{ss}_{T} \mathcal{L}(7\omega_3), \mathcal{M})$ is projectively normal. Before proceeding we illustrate the notation from the previous section.

5.0.1 Example

For $G_{3,7}$, the Brühat poset is on the set $I(3,7)$ which has cardinality 35. There are 462 maximal chains in the Brühat poset with lowest element $[1, 2, 3]$ and top element $[5, 6, 7]$. The length of every maximal chain is 13. Let $\mathcal{C}_0$ be the following maximal chain in $I_{3,7}$.

$$\mathcal{C}_0 = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 5\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 6\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline 6\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 6\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 3\\ \hline 5\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 4\\ \hline 5\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 4\\ \hline 6\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 5\\ \hline 6\\ \hline 7\\ \hline\end{array}$$

The matrix $A'_{\mathcal{C}}$ is of order 7×13 and $v = (-3, \cdots, -3)$. Hence we get the following matrix $A_{\mathcal{C}} = [A'_{\mathcal{C}}|v]$ of order 7×14.

$$A_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \end{pmatrix}$$

The monomial indexed by the following tableau is an example of a T-invariant polynomial with support in $\mathcal{C}$.

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&4&4&4&4&4\\ \hline 2&2&2&3&3&3&3&3&3&5&5&5&5&5\\ \hline 4&5&6&6&6&6&6&6&7&7&7&7&7&7\\ \hline\end{array}\,, \quad \boldsymbol{v}_S = [0,1,1,1,3,2,1,0,0,0,5,0,0,2].$$

Observe that S is degree 14 monomial hence $\frac{k}{n} = 2$ which is the last entry in v_S.

5.1 Projective normality

Theorem 4.10 and Corollary 4.11 gives us effective ways of computing a generating set of the ring of torus invariants for the Grassmannian and Schubert varieties in the Grassmannian. We were able to run computational experiments in the case of $Gr_{3,7}$.

Theorem 5.1. *Let $w = [3,6,7]$. The line bundle $\mathcal{L}(7\omega_3)$ descends to a line bundle on the GIT quotient $T\backslash\backslash X(w)^{ss}_T(\mathcal{L}(7\omega_3))$. The polarised variety $(T\backslash\backslash X(w)^{ss}_T(\mathcal{L}(7\omega_3)), \mathcal{M})$ is projectively normal.*

Proof. The proof is computational. We continue to use the notation from Corollary 4.11. Let $\mathcal{P} \subset I_{3,7}$ be the sub-poset consisting of $v \in I_{r,n}$, $v \leq w$. We verified that there are 252 maximal chains in $\mathcal{P}$. For each maximal chain $\mathcal{C}$ we compute $H_{\mathcal{C}}$, the Hilbert basis of the the pointed cone $C_{\mathcal{C}}$. It follows from Corollary 4.11 that the coordinate ring of $T\backslash\backslash X(\omega)^{ss}_T(\mathcal{L}(7\omega_3))$ is generated by the union of standard monomials corresponding to elements in $H_{\mathcal{C}}$, the union taken over maximal chains $\mathcal{C}$ in $\mathcal{P}$.

We observe that the union has 31 standard monomials in degree 7 and 8 standard monomials in degree 14. Using straightening laws, we showed that monomials of degree 14 are in algebra generated by monomials in degree 7. In Appendix A we give details of the straightening relations used to straighten the 8 degree 14 monomials.This completes the proof. □

To compute the Hilbert basis in the example above we used the algorithm given in [Hem02]. This algorithm is implemented in package 4ti2.

We also compute the Hilbert basis of T-invariants in $\oplus_m H^0(G_{3,7}, \mathcal{L}(7m\omega_3))^T$ with support in each maximal chain $\mathcal{C}$ in $I_{3,7}$. Data of that computation is available on github. We have following observations.

Observation 5.2. *In $G_{3,7}$ the following holds:*

1. *There are 462 maximal chains of which 131 have no T-invariants.*

2. *The cardinality of union of standard monomials corresponding to the Hilbert basis of chains $\mathcal{C}$ in $I_{r,n}$ is 390. There are 225 monomials in degree 7, 157 in degree 14 and 8 in degree 21.*

We have following theorem.

Theorem 5.3. *The polarised variety $(T\backslash\!\backslash(G_{3,7})^{ss}_T\mathcal{L}(7\omega_3), \mathcal{M})$ is projectively normal, where $\mathcal{M}$ is the descent of $\mathcal{L}(7\omega_3)$ to $T\backslash\!\backslash(G_{3,7})^{ss}_T\mathcal{L}(7\omega_3)$.*

Proof. Using straightening laws, we observe that each monomial of degree 14 and 21 in the generating set of T-invariants given by Observation 5.2 is in the polynomial span of T-invariant monomials of degree 7. So the ring of T invariants is generated in degree 7. This completes the proof. □

Note that Theorem 5.1 follows from the above theorem. This follows because we have a surjective map from $H^0(G/P, \mathcal{L}(kn\omega_r))$ to $H^0(X(\omega), \mathcal{L}(kn\omega_r))$ for every k, which remains surjective after taking T invariants since T is reductive. Nevertheless, we have given a separate proof since we can show explicitly in this case the straightening laws used to prove that degree 14 invariants in the Hilbert basis are in the polynomial span of degree 7 invariants in the Hilbert basis.

6 T-quotients of partial flag varities

We use ideas similar to what we used in the previous section to study GIT quotients $T\backslash\!\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda)$, for certain parabolic subgroups Q.

Theorem 6.1. *We have the following:*

1. *Let $Q = P^{\alpha_3} \cap P^{\alpha_2}$ and $\lambda = \omega_3 + \omega_2$. Let $X = T\backslash\!\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda)$ and $\mathcal{M}$ be the descent of $\mathcal{L}(\lambda)$ to X. The homogeneous coordinate ring of $(X, \mathcal{M})$ is an integral extension of a polynomial ring in five variables.*

2. *Let $Q = P^{\alpha_4} \cap P^{\alpha_1}$ and $\lambda = \omega_4 + \omega_1$. Let $X = T\backslash\!\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda)$ and $\mathcal{M}$ be the descent of $\mathcal{L}(\lambda)$ to X. The polarized variety $(X, \mathcal{M})$ is projectively normal.*

3. *Let $Q = P^{\alpha_3} \cap P^{\alpha_1}$ and $\lambda = \omega_3 + 2\omega_1$. Let $X = T\backslash\!\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda)$ and $\mathcal{M}$ be the descent of $\mathcal{L}(\lambda)$ to X. The polarized variety $(X, \mathcal{M})$ is a projectively normal.*

4. *Let $Q = P^{\alpha_2} \cap P^{\alpha_1}$ and $\lambda_1 = 2\omega_2 + \omega_1$ and $\lambda_2 = \omega_2 + 3\omega_1$. For $i = 1, 2$ let $X_i = T\backslash\!\backslash(SL(5,\mathbb{C})/Q)^{ss}_T\mathcal{L}(\lambda_i)$ and $\mathcal{M}_i$ be the descent of $\mathcal{L}(\lambda_i)$ to X_i. The polarized variety $(X_i, \mathcal{M}_i)$ is projectively normal.*

Proof. In all cases we follow the procedure outlined in Section 4. For each Q we set up an appropriate poset $\mathcal{P}$. Given λ, for each maximal chain $\mathcal{C}$ in $\mathcal{P}$, we construct a pointed cone $C_{\mathcal{C},\lambda}$. We compute a Hilbert basis of $C_{\mathcal{C},\lambda}$. The standard monomials corresponding to the Hilbert basis elements have support in $\mathcal{C}$, and the shape of the tableau is a non-negative integral multiple of the shape of λ. The union of standard monomials corresponding to these Hilbert basis gives us a generating set for the ring of T-invariants of $SL(5,\mathbb{C})/Q$, with respect to the polarization λ.

For each Q and λ as in the statement of the theorem, details of the construction of $\mathcal{P}$ and details of the construction of $C_{\mathcal{C},\lambda}$ for each maximal chain $\mathcal{C}$ are given in Appendix B. [1]

[1] To keep the notation simple, we omit the additional subscript λ when referring to $C_{\mathcal{C},\lambda}$ in the appendix.

For each Q, λ, in Appendix C we write down explicitly the generators of the ring of T-invariants obtained. The theorem follows from the calculations given there. Here, we only present details of part 1 of the theorem, and illustrate via an example the construction of $C_{\mathcal{C},\lambda}$. Again, for simplicity of notation we omit the subscript λ in $C_{\mathcal{C},\lambda}$.

Let $Q = P^{\alpha_3} \cap P^{\alpha_2}$ and $\lambda = \omega_3 + \omega_2$. The poset in this case is on set $I_Q := I_{3,5} \cup I_{2,5}$. We extend the Brühat order on $I_{2,5}$ and $I_{3,5}$ to I_Q as follows:

For $\underline{i} = [i_1, i_2, i_3]$ and $\underline{j} = [j_1, j_2]$ define $\underline{i} \leq \underline{j}$ if $i_1 \leq j_1$ and $i_2 \leq j_2$.

The poset I_Q contains 20 elements and 42 maximal chains. We fix the following maximal chain of length $l = 10$.

$$\mathcal{C} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 5\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 4\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 5\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 5\\ \hline\end{array} \leq \begin{array}{|c|}\hline 3\\ \hline 5\\ \hline\end{array} \leq \begin{array}{|c|}\hline 4\\ \hline 5\\ \hline\end{array}$$

Following the construction given in Appendix B, the matrices $A_{\mathcal{C}}$ and $B_{\mathcal{C}}$ are as follows.

$$A_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

$$B_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

The matrix $A_{\mathcal{C}}$ is of dimension 5×11 and the matrix $B_{\mathcal{C}}$ has dimension 2×11. Set $P_{\mathcal{C}} = ker(A_{\mathcal{C}}) \cap ker(B_{\mathcal{C}}) \cap \mathbb{R}^{l+1}_{\geq 0}$. We get

$$H_{\mathcal{C}} = \left\{ \begin{array}{c} (0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1) \\ (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1) \end{array} \right\}$$

and the two tableaux corresponding to above vectors are:

$$t_0 = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 5\\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad t_1 = \begin{array}{|c|c|}\hline 1 & 4\\ \hline 2 & 5\\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Similarly, we compute the Hilbert basis for all maximal chains in I_Q and we get following generating set for ring of T-invariants. (Observe that tableaux t_0 and t_1 below have support in the chain $\mathcal{C}$ above.)

$$t_0 = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 5\\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad t_1 = \begin{array}{|c|c|}\hline 1 & 4\\ \hline 2 & 5\\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad t_2 = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 4\\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad t_3 = \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 & 4\\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad t_4 = \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 & 5\\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$x_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4\\ \hline 2 & 3 & 3 & 5\\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \qquad x_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3\\ \hline 2 & 4 & 4 & 5\\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

From Theorem 4.10, the ring of invariants is generated by $t_0, ..., t_4, x_1, x_2$, hence it is contained in $\mathbb{C}[t_0, \cdots, t_4, x_1, x_2]$. To prove part 1 of the claim we further study the ring of invariants. Observe that only t_2t_3 and x_1x_2 are the two nonstandard products amongst $t_0, \cdots, t_4, x_1, x_2$. It can be easily seen that

$$x_1x_2 = f_1(t_0, t_1, t_2, t_3, t_4),$$

$$t_2t_4 = x_1 + x_2 + f_2(t_0, t_1, t_2, t_3, t_4).$$

Here f_1 and f_2 are in the ring $\mathbb{C}[t_0, ..., t_4]$. Note that the ring $\mathbb{C}[t_0, ..., t_4]$ is a polynomial ring in five variables. It also follows from the above calculations that $x_1 - x_2$ satisfies a monic polynomial of degree 2 with coefficients in the ring $\mathbb{C}[t_0, ..., t_4]$. So x_1, x_2 are integral over $\mathbb{C}[t_0, \cdots, t_4]$ and hence $\mathbb{C}[t_0, ..., t_4, x_1, x_2]$ is an integral extension of $\mathbb{C}[t_0, ..., t_4]$. This completes the proof of the theorem. We give details of the polynomials f_1 and f_2 in Appendix C.1. Proofs for the other parts are similar and are presented later in Appendix C.

□

7 Flag varieties whose T quotients are Grassmannians

7.1 Shape $(r, 1^s)$

Let $G = SL(n, \mathbb{C})$. We look at flag varieties of the form G/Q, where Q is the intersection of two maximal parabolic subgroups, as in Section 6. We show that for certain Q and a chosen polarization that the T-quotients are Grassmannians.

Let r be a positive integer such that $1 \leq r \leq n-1$ and set $s = n - r$. Let $Q = P^{\alpha_r} \cap P^{\alpha_1}$. Let $\lambda = \omega_r + s\omega_1$. Let

$$I'_{r,n} = \{\underline{i} \in I_{r,n} : i_1 = 1\}.$$

$$I'_{r-1,n-1} = \{(i_2, \cdots, i_r) : 2 \leq i_2 < \cdots < i_r \leq n\}.$$

The indexing set for Plücker coordinates on $G_{r-1,n-1}$ is $[1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n-1]$, and this set can be seen to be in bijection with the set $I'_{r-1,n-1}$ by the map sending $[i_1, i_2, \ldots, i_{r-1}]$ to $[i_1 + 1, i_2 + 1, \ldots, i_{r-1} + 1]$. Let ϕ' be natural bijection between $I'_{r,n}$ and $I'_{r-1,n-1}$, sending $[1, i_2, \ldots, i_n]$ to $[i_2, \ldots, i_n]$. Composing this bijection with the inverse of the earlier bijection, we have the bijection

$$\phi'[1, i_2, i_3, \ldots, i_r] \mapsto [i_2 - 1, i_3 - 1, \ldots, i_r - 1].$$

Note that $\oplus H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})$ is generated by standard monomials of shape $(r^k, 1^{sk})^t$ for k positive integer. Here $(,)^t$ denotes taking the transpose shape. In the rest of this section we will identify standard monomials with the semistandard tableaux associated to them.

Recall from Proposition 3.2 that a standard monomial is T invariant only when its weight is uniform. We have following observation.

Observation 7.1. *$p_X = p_{\underline{i}_1} p_{\underline{i}_2} \cdots p_{\underline{i}_k} p_{i'_1} p_{i'_2} \cdots p_{i'_{sk}} \in H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})^T$ be a standard monomial, where $\underline{i}_1, \cdots, \underline{i}_k \in I_{r,n}$ and $i'_1, \cdots, i'_{sk} \in I_{1,n} = [n]$.*

1. *$wt(p_X) = (k, \cdots, k)$.*

2. *$X_{1,1} = X_{1,2} = \cdots = X_{1,k} = 1$, i.e. first entry in columns of length r is 1.*

Proof. Since the shape of X is $(r^k, 1^{ks})^t$, the number of boxes in X is $kr + ks = kn$. Since p_X is T-invariant each integer in $[n]$ appears an equal number of times and $wt(X) = (k, \cdots, k)$. Since p_X is a standard monomial the first k entries of first row in X must be 1. □

With the help of above observation, the bijection from $\phi' : I'_{r,n} \to I_{r-1,n-1}$ can be extended to a $\mathbb{C}$-linear map $\phi : H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})^T \to H^0(G'/P', \mathcal{L}(\omega_{r-1})^{\otimes k})$, where G'/P' is Grassmannian $G_{r-1,n-1}$, as follows:

$$\phi(p_{\underline{i}_1} p_{\underline{i}_2} \cdots p_{\underline{i}_k} p_{i'_1} p_{i'_2} \cdots p_{i'_{sk}}) = p_{\phi'(\underline{i}_1)} p_{\phi'(\underline{i}_2)} \cdots p_{\phi'(\underline{i}_k)}$$

Noe that the map ϕ is well defined. We show that ϕ is ring map.

Lemma 7.2. *Let ϕ be map as above and X, Y be tableaux such that $p_X \in H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})^T$ and $p_Y \in H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k'})^T$. We have*

1. *ϕ is vector space isomorphism.*
2. *$\phi(p_X p_Y) = \phi(p_X)\phi(p_Y)$.*

Hence ϕ is a $\mathbb{C}$-algebra isomorphism.

Proof. Observe that the tableau corresponding to the standard monomial $\phi(p_X)$ is obtained by removing the first row of the tableau X and decreasing every entry of the resulting rectangular tableau by 1. We claim this is a bijection. For every standard monomial p_Z in $H^0(G'/P', \mathcal{L}(\omega_{r-1})^{\otimes k})$, we claim that there is unique semistandard monomial p_X in $H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})^T$ such that $\phi(p_X) = p_Z$. Start with the tableau Z corresponding to p_Z. Increase each entry of Z by 1, to get a semistandard rectangular tableau Z' with entries in $[2, \ldots, n]$. Let the weight of Z' be $wt(Z')$. We construct the tableau X as follows: construct the semistandard tableau $X_{1,*}$ of weight $v = (k, k, ..., k) - wt(Z')$, having exactly one row. Note that this has at least k columns and the entries in the first k columns are all 1's. Tableau X is constructed by appending the tableau Z' below $X_{1,*}$. Clearly X is a semistandard tableau. It follows from the construction that p_X is T-invariant standard monomial. Clearly, $\phi(p_X) = p_Z$.

2. Let X_1, X_2 be the semistandard tableau corresponding to T-invariant monomials p_{X_1}, p_{X_2}. If the product $p_{X_1} p_{X_2}$ is standard, the columns of X_1, X_2 can be rearranged so that the tableau X corresponding to the product monomial is semistandard. Note that X has $2k$ columns of length r and $2sk$ columns of length 1. The image of p_X is the monomial corresponding to the tableau (of size $r-1 \times 2k$)obtained from X by taking the subtableau of X of size $r \times 2k$, removing the first row and then decreasing every entry of the resulting tableau. But this is precisely the tableau corresponding to the product of the monomials $\phi(p_{X_1})$, $\phi(p_{X_2})$.

If $p_{X_1} p_{X_2}$ is non standard, we can express it as the sum of products of standard monomials. Note that the straightening relations needed to straighten the product only involves columns of length r in X_1, X_2. Now, the first rows of both X_1 and X_2 have 1's. So this row remains untouched during straightening. Recall ϕ is a bijection. If a straightening relation is applied on the left hand side to straighten two columns, we can apply the same straightening relation to the images of these columns under ϕ on the right hand side. This completes the proof. □

We have,

Theorem 7.3. *Let $(r, n) = 1$, and let $s = n - r$. Set $Q = P^{\alpha_r} \cap P^{\alpha_1}$ and let $\lambda = \omega_r + s\omega_1$. Consider the embedding of G/Q given by the line bundle $\mathcal{L}(\lambda)$. Let $X = T\backslash\backslash(SL(n, \mathbb{C})/Q)^{ss}_T \mathcal{L}(\lambda)$ be the GIT quotient. The line bundle $\mathcal{L}(\lambda)$ descends to an ample line bundle $\mathcal{M}$ on X, and the polarized variety $(X, \mathcal{M})$ is isomorphic to $Gr_{r-1,n-1}$.*

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A The Schubert variety [367] in $G_{3,7}$

We list of degree 2 tableaux which cannot be split directly.

$$t_0 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&5&5&5&6\\\hline 3&5&5&6&6&6&6&6&7&7&7&7&7&7\\\hline\end{array} \qquad t_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&5&5&5&5\\\hline 3&5&6&6&6&6&6&6&7&7&7&7&7&7\\\hline\end{array}$$

$$t_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&6&6&6&6\\\hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\\hline\end{array} \qquad t_3 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&5&6&6&6\\\hline 3&5&5&5&5&6&6&6&7&7&7&7&7&7\\\hline\end{array}$$

$$t_4 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&5&5&6&6\\\hline 3&5&5&5&6&6&6&6&7&7&7&7&7&7\\\hline\end{array} \qquad t_5 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&6&6&6&6&6\\\hline 3&5&5&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

$$t_6 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&5&5&6&6&6&6&6\\\hline 3&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array} \qquad t_7 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 4&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

A.1 Factoring t_0, t_3, t_4

We will use the following Plücker relation:

$$R_1: \quad 0 = \begin{array}{|c|c|}\hline 2&3\\\hline 5&4\\\hline 7&7\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline 7&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline 7&7\\\hline\end{array}$$

On the Schubert variety $[3, 6, 7]$ the last monomial in the above relation is 0.

Applying R_1 on t_0, t_3, t_4 we get

$$t_0 \;=\; \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&5&4&5&5&5&6\\\hline 3&5&5&6&6&6&6&6&7&7&7&7&7&7\\\hline\end{array}$$

$$t_3 \;=\; \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&5&4&5&6&6&6\\\hline 3&5&5&5&5&6&6&6&7&7&7&7&7&7\\\hline\end{array}$$

$$t_4 \;=\; \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&5&4&5&5&6&6\\\hline 3&5&5&5&6&6&6&6&7&7&7&7&7&7\\\hline\end{array}$$

Note that the following T-invariant tableau is a factor of all of the above tableaux

$$f_{034} \;=\; \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&2&4&4&4&5&6\\\hline 5&5&6&6&7&7&7\\\hline\end{array}$$

A.2 Factoring t_2

We will use the following Plücker relation.

$$R_2 : \quad 0 = \begin{array}{|c|c|}\hline 2&3\\\hline 6&4\\\hline 7&7\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 4&6\\\hline 7&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&4\\\hline 3&6\\\hline 7&7\\\hline\end{array}$$

On $[3, 6, 7]$ the last monomial is 0.

Applying R_2 on t_2 we get

$$t_2 \;=\; \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&6&5&4&6&6&6\\\hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\\hline\end{array}$$

Note that the following T-invariant tableau is a factor of above tableau:

$$f_{034} \;=\; \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&2&4&4&4&6&6\\\hline 5&5&5&6&7&7&7\\\hline\end{array}$$

A.3 Factoring t_5

We use the following Plücker relation:

$$R_3 : \quad 0 = \begin{array}{|c|c|}\hline 2&3\\\hline 5&4\\\hline 6&7\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline 6&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline 6&7\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&4\\\hline 3&6\\\hline 5&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&3\\\hline 4&6\\\hline 5&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&5\\\hline 3&6\\\hline 4&7\\\hline\end{array}$$

Observe that the $3^{rd}, 4^{th}$ and 6^{th} monomials are 0 on [3,6,7]. Hence after multiplying following tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&&2&2&&3&3&3&3\\\hline 2&2&2&4&4&4&&4&4&&6&6&6&6\\\hline 3&5&5&5&5&5&&6&7&&7&7&7&7\\\hline\end{array}$$

to the above relation (i.e. substituting the relation in the empty columns of the tableau above) we get

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&5&4&4&4&6&6&6&6\\\hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\\hline\end{array}$$
$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&6&6&6&6\\\hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\\hline\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&6&6&6&6&6\\\hline 3&5&5&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array} = t_2 - t_5.$$

Observe that following is a factor of the left hand side tableau

$$f_5 \;=\; \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&2&4&4&4&6&6\\\hline 5&5&5&6&7&7&7\\\hline\end{array}$$

Hence t_5 factored.

A.4 Factoring t_7

We will use the following Plücker relation:

$$R_4: \quad 0 = \begin{array}{|c|c|}\hline 1&3\\\hline 3&5\\\hline 7&6\\\hline\end{array} - \begin{array}{|c|c|}\hline 1&3\\\hline 3&5\\\hline 6&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 3&6\\\hline 5&7\\\hline\end{array}$$

We multiply the following tableau to the above relation (in the empty columns)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1& &1&1&2&2&2& &3&3&3&3\\\hline 2&2&2& &4&4&4&5&5& &6&6&6&6\\\hline 4&4&4& &5&5&5&6&7& &7&7&7&7\\\hline\end{array}$$

then we get following

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 4&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$
$$= - \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&5&6&6&6&6\\\hline 4&4&4&7&5&5&5&6&7&6&7&7&7&7\\\hline\end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&5&6&6&6&6\\\hline 4&4&4&6&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

Observe that the tableau on left hand side is t_7 and also observe that the following two tableaux are factors of two tableaux on right side in above relation

$$f_{71} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 2&3&4&4&5&6&6\\\hline 4&7&5&5&6&7&7\\\hline\end{array} \qquad f_{72} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 2&3&4&4&5&6&6\\\hline 4&6&5&5&7&7&7\\\hline\end{array}$$

A.5 Factoring t_6

We use the following Plücker relation:

$$R_5: \quad 0 = \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 5&4\\\hline\end{array} - \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4&5\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline 3&5\\\hline\end{array}$$

We multiply the following tableau to the above relation (in the empty columns)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline &1&1& &1&1&2&2&2&3&3&3&3&3\\\hline &2&2& &4&4&4&5&5&6&6&6&6&6\\\hline &4&4& &5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

We get

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 5&4&4&4&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$
$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 4&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&5&5&6&6&6&6&6\\\hline 3&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array} = t_7 - t_6.$$

Observe that following tableau ia factor of left hand side tableau in above relation

$$f_6 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&2&4&4&6&6&6\\\hline 4&5&5&5&7&7&7\\\hline\end{array}$$

B Hilbert basis of flag varieties

We define a few notations and make some observations.

Let $Q = P^{\alpha_r} \cap P^{\alpha_s}$ be a parabolic subgroup of G (we assume $r > s$). Let $\lambda = a\omega_r + b\omega_s$ be a dominant weight. Denote by γ_k the Young diagram such that $\gamma_k^t = (ka, kb)$ for $k \geq 1$. Let S be a semistandard tableau of shape γ_k. Let $\underline{i} = (i_1, ..., i_d)$ denote a column in S and $p_{\underline{i}}$ be the Plücker coordinate corresponding to $\underline{i}$ where $d \in \{r, s\}$ and $1 \leq i_1 < \cdots < i_d \leq n$. Let $p_S = \prod_{\underline{i} \in S} p_{\underline{i}}$ be the monomial associated to semistandard tableau S. Then $p_S \in H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})$. It is known that the set of all standard monomials of shape γ_k forms a basis of $H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})$ (see [LB15] Theorem 8.2.4).

Assume $ar + bs = n$. It is clear that a monomial $p_S \in H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})$ is T invariant if and only if each integer $i \in [n]$ appears exactly k times in tableau S. Define the weight of tableau S to be the n-dimentional vector $wt(S)$ such that each $i \in [n]$ appears $wt(S)_i$ times in tableau S.

Let $I_Q = I_{r,n} \cup I_{s,n}$. We extend the Brühat order on $I_{r,n}$ and $I_{s,n}$ to I_Q as follows: For $\underline{i} \in I_{r,n}, \underline{j} \in I_{s,n}$ define

$$\underline{i} \leq \underline{j} \text{ if } \underline{i}_l \leq \underline{j}_l \text{ for } 1 \leq l \leq s.$$

Let $\mathcal{C} = \underline{i}_1 \leq \cdots \leq \underline{i}_l$ be a chain in the poset I_Q and $p_{\underline{i}_1}^{a_{\underline{i}_1}} p_{\underline{i}_2}^{a_{\underline{i}_2}} ... p_{\underline{i}_l}^{a_{\underline{i}_l}} \in H^0(G/Q, \mathcal{L}(\lambda)^{\otimes k})$ be a standard monomial and let the corresponding semistandard tableau be S. Associate vectors $v'_S = (a_{\underline{i}_1}, ..., a_{\underline{i}_l})$ and $v_S = (v'_S, k)$ with this monomial.

We associate the $n \times l$ matrix $A'_{\mathcal{C}}$ indexed by $[n] \times \mathcal{C}$ with chain $\mathcal{C}$ in the same way as we did in the Grassmannian case: For $\underline{j} \in \mathcal{C}$, the $\underline{j}^{th}$ column of $A'_{\mathcal{C}}$ is the weight vector of column tableau $\underline{j}$, Let $v = (-1, \cdots, -1)$ be a n-dimentional vector denote $A_{\mathcal{C}} := [A'_{\mathcal{C}}|v]$, the $A'_{\mathcal{C}}$ augmented with w. We have following observation:

Observation B.1. *Let S be a tableau whose columns are in $\mathcal{C}$. For $i \in [n]$ we have*

$$(A'_{\mathcal{C}} v'_S)_i = \textit{number of columns in } S \textit{ which contain } i.$$

Hence, $wt(S) = A'_{\mathcal{C}} v'_S$. Also if S is T-invariant then v_S is nonzero integral vector in $ker(A_{\mathcal{C}})$.

Proof. The proof is similar to the proof of Observation 4.4 and is omitted. □

To make sure that the shape of the tableau is that of γ_k for some k, we need to associate another matrix $B'_{\mathcal{C}}$ with chain $\mathcal{C}$ whose entries are indexed by the set $\{r, s\} \times \mathcal{C}$. $B'_{\mathcal{C}}$ is defined as follows: For $i \in \{r, s\}, \underline{j} \in \mathcal{C}$ define

$$(B'_{\mathcal{C}})_{i,\underline{j}} = \begin{cases} 1 & |\underline{j}| = i \\ 0 & |\underline{j}| \neq i \end{cases}$$

Here $|\underline{i}|$ is the length of column $\underline{i}$. Let $u = (-a, -b)$ be a 2-dimentional vector. Denote $B_{\mathcal{C}} := [B'_{\mathcal{C}}|u]$, $B'_{\mathcal{C}}$ augmented with vector u. We have the following observations:

Observation B.2. *Let S be a tableau whose columns are in $\mathcal{C}$. For $i \in \{r, s\}$ we have,*

$$(B'_{\mathcal{C}}v'_{S})_i = \textit{number of columns in } S \textit{ of length } i.$$

Hence the shape of tableau S is $((B'_{\mathcal{C}}v'_{S})_r, (B'_{\mathcal{C}}v'_{S})_s)^t$.

Proof.

$$(B'_{\mathcal{C}}v'_{S})_i = \sum_{\underline{j}\in\mathcal{C}}(B'_{\mathcal{C}})_{i,\underline{j}}(v'_{S})_{\underline{j}} = \sum_{|\underline{j}|=i}(v'_{S})_{\underline{j}}.$$

$\square$

Let $C_{\mathcal{C}} = \mathbb{R}^{l+1}_{\geq 0} \cap ker(A_{\mathcal{C}}) \cap ker(B_{\mathcal{C}})$. From the above two observations we see that if S is a T-invariant tableau whose columns are in $\mathcal{C}$ then $v_S \in C_{\mathcal{C}}$. Conversely we have:

Observation B.3. *Let $x = (x', k)$ be a nonzero vector in $\mathbb{Z}^{l+1}_{\geq 0} \cap C_{\mathcal{C}}$. There exists a T-invariant semistandard tableau S such that $x = v_S$.*

Proof. Define tableau S such that, $\underline{j} \in \mathcal{C}$ appears $x'_{\underline{j}}$ times. Hence we have $x' = v'_S$. Note that since the columns of S are from a chain, S is semistandard. We are done if we show that $wt(S) = (k, \cdots, k)$ and the shape of S is $(ka, kb)^t$. This follows from calculations below.

$$A_{\mathcal{C}}x = 0 \implies [A'_{\mathcal{C}}|v](x', k) = 0 \implies A'_{\mathcal{C}}x' = -kv \implies wt(S) = -kv = (k, \cdots, k).$$

$$\begin{aligned} B_{\mathcal{C}}x = 0 \implies [B'_{\mathcal{C}}|u](x', k) = 0 &\implies B'_{\mathcal{C}}x' = -ku = (ka, kb) \\ &\implies ((B'_{\mathcal{C}}v'_{S})_r, (B'_{\mathcal{C}}v'_{S})_s) = (ka, kb). \end{aligned}$$

$\square$

Corollary B.4. *We have a bijection between T invariant monomials with support in $\mathcal{C}$ and nonzero integral points in $C_{\mathcal{C}}$ given by tableau $S \mapsto v_S$.*

Corollary B.5. *$C_{\mathcal{C}}$ is a pointed cone. Hence $C_{\mathcal{C}}$ has a unique inclusion minimal Hilbert basis.*

Let $H_{\mathcal{C}}$ denote Hilbert basis of $C_{\mathcal{C}}$.

Lemma B.6. *Let S be any T invariant semistandard tableau and assume support of S is in chain $\mathcal{C}$. Then S splits directly iff $v_S \notin H_{\mathcal{C}}$.*

Proof. Proof is the same as that of Lemma 4.9. $\square$

Theorem B.7. *The ring $\oplus_{k\geq 0}H^0(SL(n,\mathbb{C})/Q, \mathcal{L}(k\lambda))^T$ is generated by the union of standard monomials corresponding to the Hilbert basis of the cone $C_{\mathcal{C},\lambda}$. Here the union is taken over all maximal chains in the poset I_Q.*

Proof. Follows from Observation B.1, Corollary B.4 and Lemma B.6.

$\square$

C Generators of T quotient of partial Flag varieties.

In this section we will present the calculations leading to Theorem 6.1 for $\lambda = a\omega_r + b\omega_s$ such that $ar + bs = 5$. We have following possible values of λ: 1. $\omega_3 + \omega_2$, 2. $\omega_4 + \omega_1$, 3. $\omega_3 + 2\omega_1$, 4. $2\omega_2 + \omega_1$ and 5. $\omega_2 + 3\omega_1$ We have dealt the first case in Section 6 where we showed calculations in one chain in poset $I_{\lambda,5}$ for $\lambda = \omega_3 + \omega_2$. The only details missing in that proof were about the polynomials f_1, f_2. We describe those polynomials first in Section C.1. The other cases are actually easier.

C.1 $\lambda = \omega_3 + \omega_2$

The polynomial f_1 is obtained by straightening the monomial x_1x_2

$$x_1x_2 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\ \hline 2&2&3&4&3&4&5&5\\ \hline 3&4&5&5\\ \cline{1-4}\end{array}$$

Observe that the monomial is nonstandard and non-comparable pair is $[1,4,5],[2,3]$. We will use following Plúcker relation.

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 4&3\\ \hline 5\\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline 5\\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&3\\ \hline 2&4\\ \hline 5\\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&2\\ \hline 3&5\\ \hline 4\\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&3\\ \hline 2&5\\ \hline 4\\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&4\\ \hline 2&5\\ \hline 3\\ \cline{1-1}\end{array} = 0$$

We get

$$\begin{aligned} x_1x_2 &= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\ \hline 2&2&3&4&3&4&5&5\\ \hline 3&4&5&5\\ \cline{1-4}\end{array} \\ &= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\ \hline 2&2&3&3&4&4&5&5\\ \hline 3&4&5&5\\ \cline{1-4}\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&3&4\\ \hline 2&2&2&3&4&4&5&5\\ \hline 3&4&5&5\\ \cline{1-4}\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\ \hline 2&2&3&3&4&5&5&5\\ \hline 3&4&4&5\\ \cline{1-4}\end{array} \\ &\qquad + \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&3&4\\ \hline 2&2&2&3&4&5&5&5\\ \hline 3&4&4&5\\ \cline{1-4}\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&4&4\\ \hline 2&2&2&3&4&5&5&5\\ \hline 3&3&4&5\\ \cline{1-4}\end{array} \\ &= t_1t_0t_3^2 - t_1t_0t_2t_3 - t_1t_0t_4t_3 + t_1t_0^2t_3 - t_1^2t_0t_3. \end{aligned}$$

Hence we define,

$$\begin{aligned} f_1(t_0,t_1,t_2,t_3,t_4) &:= x_1x_2 \\ &= t_1t_0t_3^2 - t_1t_0t_2t_3 - t_1t_0t_4t_3 + t_1t_0^2t_3 - t_1^2t_0t_3. \end{aligned}$$

C.1.1 Polynomial f_2

To get f_2 we will straigntain following monomial

$$t_2t_4 = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3&5&4\\ \hline 5&4\\ \cline{1-2}\end{array}$$

We will use following Plúcker relations

$$\begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline 5&4\\ \hline\end{array} + \begin{array}{|c|c|}\hline 1&1\\ \hline 2&4\\ \hline 3&5\\ \hline\end{array} - \begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline 4&5\\ \hline\end{array} = 0 \qquad \cdots(\text{R1})$$

$$\begin{array}{|c|c|}\hline 2&3\\ \hline 5&4\\ \hline\end{array} - \begin{array}{|c|c|}\hline 2&3\\ \hline 4&5\\ \hline\end{array} + \begin{array}{|c|c|}\hline 2&4\\ \hline 3&5\\ \hline\end{array} = 0 \qquad \cdots(\text{R2})$$

$$\ytableaushort{12,43,5} - \ytableaushort{12,34,5} + \ytableaushort{13,24,5} + \ytableaushort{12,35,4} - \ytableaushort{13,25,4} + \ytableaushort{14,25,3} = 0 \qquad \cdots\text{(R3)}$$

We get

$$\begin{aligned}
t_2t_4 &= \ytableaushort{1123,2354,54} \\
&= -\ytableaushort{1123,2454,35} + \ytableaushort{1123,2354,45} && \cdots\text{(After applying R1)} \\
&= \ytableaushort{1123,2354,45} - \ytableaushort{1123,2445,35} + \ytableaushort{1124,2435,35} && \cdots\text{(Term 1, Apply R2)} \\
&= -\ytableaushort{1123,2445,35} + \ytableaushort{1124,2435,35} + \ytableaushort{1123,2345,45} - \ytableaushort{1124,2335,45} && \cdots\text{(Term 1, Apply R2)} \\
&= -\ytableaushort{1123,2445,35} + \ytableaushort{1123,2345,45} - \ytableaushort{1124,2335,45} + \ytableaushort{1124,2345,35} \\
&\qquad - \ytableaushort{1134,2245,35} - \ytableaushort{1124,2355,34} + \ytableaushort{1134,2255,34} - \ytableaushort{1144,2255,33} \\
& && \cdots\text{(Term 2, Apply R3)} \\
&= -x_2 + t_0t_3 - x_1 + t_1t_3 - t_1t_2 - t_1t_4 + t_1t_0 - t_1^2.
\end{aligned}$$

Define f_2 as follows:

$$\begin{aligned}
f_2(t_0, t_1, t_2, t_3, t_4) &:= x_1 + x_2 \\
&= -t_2t_4 + t_0t_3 + t_1t_3 - t_1t_2 - t_1t_4 + t_1t_0 - t_1^2.
\end{aligned}$$

C.2 $\lambda = \omega_4 + \omega_1$

The tableaux corresponding to the generators of the ring of T-invariants obtained from the union of Hilbert basis $H_{\mathfrak{C}}$ are,

$$\ytableaushort{15,2,3,4}\ , \quad \ytableaushort{14,2,3,5}\ , \quad \ytableaushort{13,2,4,5}\ , \quad \ytableaushort{12,3,4,5}$$

Hence ring of invariants is generated by first graded component.

C.3 $\lambda = \omega_3 + 2\omega_1$

The tableaux corresponding to the generators of the ring of T-invariants obtained from the union of Hilbert basis $H_{\mathcal{C}}$ are

$$\begin{ytableau} 1 & 4 & 5 \\ 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 & 5 \\ 2 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 & 4 \\ 2 \\ 5 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 5 \\ 3 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 4 \\ 3 \\ 5 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 \\ 4 \\ 5 \end{ytableau}$$

The ring of invariants is generated by the first graded component.

C.4 $\lambda = 2\omega_2 + \omega_1$

In this case, the tableaux corresponding to the generators of the ring of T-invariants obtained from the union of Hilbert basis $H_{\mathcal{C}}$ are

$$t_0 = \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} \quad t_1 = \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau} \quad t_2 = \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau} \quad t_3 = \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau} \quad t_4 = \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau}$$

$$x_1 = \begin{ytableau} 1 & 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 5 \end{ytableau} \qquad x_2 = \begin{ytableau} 1 & 1 & 2 & 4 & 4 & 5 \\ 2 & 3 & 3 & 5 \end{ytableau}$$

We have the following lemma.

Lemma C.1. *$x_1, x_2 \in \mathbb{C}[t_0, \cdots, t_4]$ using straightening laws. Hence the Krull dimension of $\mathbb{C}[t_0, \cdots, t_4, x_1, x_2]$ is 5*

Proof. We have the following relations:

$$\begin{aligned} t_0 t_3 &= \begin{ytableau} 1 & 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 5 & 4 \end{ytableau} \\ &= \begin{ytableau} 1 & 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 3 & 5 & 5 \\ 2 & 3 & 4 & 4 \end{ytableau} + \begin{ytableau} 1 & 1 & 3 & 3 & 4 & 5 \\ 2 & 2 & 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 \end{ytableau} \\ &= x_1 - t_2 t_4 + t_3 t_4 + t_2 t_3 - t_3^2 \\ t_1 t_3 &= \begin{ytableau} 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 \end{ytableau} \\ &= \begin{ytableau} 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 2 & 4 & 4 & 5 \\ 2 & 3 & 3 & 5 \end{ytableau} \\ &= t_2 t_4 - x_2 \end{aligned}$$

$\square$

Hence the ring of invariants is generated in degree 1.

C.5 $\lambda = \omega_2 + 3\omega_1$

In this case, the tableaux corresponding to the generators of the ring of T-invariants obtained from the union of Hilbert basis $H_{\mathcal{C}}$ are

$$\begin{ytableau} 1 & 3 & 4 & 5 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 & 5 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 \end{ytableau}$$

Hence the ring of invariants is generated by the first graded component.