The Expressive Power of Linear-time Temporal Logic

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Linear-time Temporal Logic

**LTL** — convenient specification language
- Atomic propositions, boolean connectives, temporal modalities.
**Linear-time Temporal Logic**

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- Atomic propositions, boolean connectives, temporal modalities.
- Models are words.
Linear-time Temporal Logic

**LTL** — convenient specification language

- Atomic propositions, boolean connectives, temporal modalities.
- Models are words.

Formulas are interpreted at positions of a word.

\[ w = w_1 w_2 w_3 \ldots \quad \text{with } w_i \in \Sigma \]

\[ w, i \models \varphi ? \]
Syntax and Semantics

Atomic propositions: elements of $\Sigma$.

\[ w, i \models a \iff w_i = a \]

\[ a \ b \ b \ c \ b \ a \ c \ b \ b \]

\[ a, \neg b, \neg c \]
Syntax and Semantics

Atomic propositions: elements of $\Sigma$.

$$w, i \models a \iff w_i = a$$

$$a, b, b, c, b, a, c, b, b$$

The Next state operator:

$$w, i \models X\varphi \iff w, i + 1 \models \varphi$$
Syntax and Semantics

The *Until* operator:

\[ w, i \models \varphi U \psi \iff \exists j \geq i. \ w, j \models \psi \text{ and } \forall i \leq k < j. \ w, k \models \varphi \]
Syntax and Semantics

The Until operator:

\[ w, i \models \varphi U \psi \iff \exists j \geq i. \ w, j \models \psi \text{ and } \forall i \leq k < j. \ w, k \models \varphi \]

Boolean Connectives:

\[ \varphi \land \psi, \ \neg \varphi, \ \ldots \]

with the usual interpretation.
Other Modalities

The **Future** modality

\[ w, i \models F\varphi \iff \exists j \geq i. w, j \models \varphi \]

\[ F\varphi \]

\[ \varphi \quad \varphi \quad \cdots \quad \varphi \quad \psi \]
Other Modalities

The **Future** modality

\[ F\varphi \quad = \quad T U \varphi \]

\[
\begin{array}{ccc}
F\varphi \\
\varphi & \varphi & \cdots & \varphi & \psi \\
\end{array}
\]
Other Modalities

The **Future** modality

\[ F\varphi = TU\varphi \]

\[ \begin{array}{ccccc}
  & o & o & o & o & \cdots \\
  \varphi & \varphi & \varphi & \varphi & \varphi & \cdots
\end{array} \]

Henceforth modality:

\[ w, i \models G\varphi \iff \forall j \geq i. \ w, j \models \varphi \]

\[ \begin{array}{ccccccc}
  & o & o & o & o & \cdots & o \\
  \varphi & \varphi & \varphi & \varphi & \varphi & \cdots
\end{array} \]
Other Modalities

The **Future** modality

\[ F\varphi = TU\varphi \]

![Diagram of Future modality]

**Henceforth** modality:

\[ G\varphi = \neg F\neg \varphi \]

![Diagram of Henceforth modality]
The Universal Modality

The **Next-Until** modality:

\[ w, i \models \varphi XU \psi \equiv \exists j > i. \ w, j \models \psi \text{ and } \forall i < k \leq j. \ w, k \models \varphi \]

\[ \varphi XU \psi \]

0 → 0 → 0 → [ ] → 0 → · · · → 0 → · · ·

\[ \varphi \quad \varphi \quad \psi \]
The Next-Until modality:

\[ \varphi X \Upsilon \psi \]

\[ \varphi \quad \cdots \quad \varphi \quad \psi \]

\[ \varphi X \Upsilon \psi = X(\varphi \Upsilon \psi) \]
The Universal Modality

The Next-Until modality:

\[ \varphi X U \psi \]

Next-Until can express everything else:

\[ X \varphi = \bot X U \varphi \]
\[ \varphi U \psi = \psi \lor (\varphi \land \varphi X U \psi) \]
A word satisfies $\varphi$ if the initial position satisfies $\varphi$

$w \models \varphi \iff w,1 \models \varphi$
A word \textit{satisfies} $\varphi$ if the initial position satisfies $\varphi$.

\[ w \models \varphi \iff w, 1 \models \varphi \]

Formulas define languages. For example,

\[ G(a \implies Fb) \]

describes words in which there is a \textit{b} somewhere to the right of every \textit{a}.

\[ b^*(aa^* bb^*)^* \]
LTL formulas are interpreted over both finite and infinite words.
Finite/Infinite Words

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- Satisfiability of a formula may depend on the class of models.

$$GX^\top$$

is satisfied only over infinite words.

$$F\neg X^\top$$

is satisfied only by finite words.

- The empty word is not a model.
Consider the First-Order formula

\[ \varphi = \forall x. (a(x) \implies \exists y. ((y > x) \land b(x))) \]

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- The formula \(a(x)\) asserts that the letter at position \(x\) is \(a\).
First-Order Logic of Words

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- The formula $a(x)$ asserts that the letter at position $x$ is $a$.
- The quantifiers have the usual meaning.
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- The variables $x$, $y$ etc. refer to positions in the word.
- The formula $a(x)$ asserts that the letter at position $x$ is $a$.
- The quantifiers have the usual meaning.
- The formula $y > x$ is true if the position $y$ appears somewhere to the right of the position $x$.

A word $w$ satisfies $\varphi$ only if for any position ($x$) with the letter $a$, there is some position to its right ($y$) with the letter $b$.

$$L(\varphi) = b^* (aa^* bb^*)^*$$
First-Order Logic over words

The formula

\[ \forall x. \forall y. (a(x) \land a(y)) \implies x = y \]

is true of all words that have at most one \( a \).
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The formula

\[ \text{First}(x) \equiv \forall y.(x = y) \lor (x < y) \]

evaluates to true at a position \( x \) if and only if it is the first position in the word.
The formula

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is true of all words that have at most one $a$. The formula

$$\text{First}(x) \triangleq \forall y. (x = y) \lor (x < y)$$

evaluates to true at a position $x$ if and only if it is the first position in the word. Thus

$$\forall x. (\text{First}(x) \implies a(x))$$

identifies all the words that begin with an $a$. 

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- Translated to FO formulas with a single free variable.
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$$\begin{align*}
T(a) &= a(x) \\
T(X\alpha) &= \exists y. (y = x + 1) \land T(\alpha)[y/x] \\
T(\varphi U \psi) &= \exists y. (y \geq x) \land T(\psi)[y/x] \land \\
&\quad \forall z. (x \leq z < y) \implies T(\varphi)[z/x]
\end{align*}$$
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\]

- $w, i \models T(\varphi) \iff w, i \models \varphi$. 
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T(a) &= a(x) \\
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&\quad \forall z. (x \leq z < y) \implies T(\varphi)[z/x]
\end{align*}
\]

- $w, i \models T(\varphi) \iff w, i \models \varphi$.
- $T(\varphi)$ uses at the most 3 variables. So, LTL is expressible in FO(3).
Complexity of LTL and FO

Satisfiability: Given a formula $\varphi$ determine whether there is some word $w$ such that $w \models \varphi$. 
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Not very different from the best known for propositional formulas.
Complexity of LTL and FO

**Satifiability:** Given a formula \( \varphi \) determine whether there is some word \( w \) such that \( w \models \varphi \).

**Theorem:** (Clarke-Sistla) Satisfiability problem for LTL formulas is PSPACE complete.

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What about FO?
Satisfiability: Given a formula $\varphi$ determine whether there is some word $w$ such that $w \models \varphi$.

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In particular, there is a satisfiability checking algorithm that runs in time $2^{|\varphi|}$.

Not very different from the best known for propositional formulas.

Theorem: (Albert Meyer) Satisfiability checking for FO over words is non-elementary.
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In particular, there is a satisfiability checking algorithm that runs in time \( 2^{|\varphi|} \).

Not very different from the best known for propositional formulas.

Theorem: (Albert Meyer) Satisfiability checking for FO over words is non-elementary.

Conclusion: FO seems to be a stronger logic than LTL.
Theorem: (Kamp) LTL is as expressive as FO over words.
Expressive Completeness of LTL

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- Kamp’s logic uses “future” and “past” modalities.
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Our presentation shall follow a variation of Wilke’s proof due to Volker Diekert and Paul Gastin.
Expressive Completeness of LTL

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Our presentation shall follow a variation of Wilke’s proof due to Volker Diekert and Paul Gastin.

The rest of this talk and the next would be devoted to proving this result.
An Outline of the proof

1. Characterize the languages defined by FO.
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   3. Every FO formula defines a regular language recognized by an aperiodic monoid.
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Ehrenfeucht-Fraisse Games
An Outline of the proof

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   *Ehrenfeucht-Fraisse Games*

2. Transform aperiodic monoids into equivalent LTL formulas.
   *Wilke’s technique.*
Formulas with free variables

Let $\varphi$ be a FO formula with free variables $x_1, \ldots, x_k$. 
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Example: $\phi = (x < y) \land a(x) \land b(y)$.

The $bacabc$ with $x$ assigned position 2 and $y$ assigned position 5 satisfies $\phi$. 
Formulas with free variables

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Example: \( \phi = (x < y) \land a(x) \land b(y) \).

The \textit{bacabc} with \( x \) assigned position 2 and \( y \) assigned position 5 satisfies \( \phi \).

Model as a word decorated with the variables \( x \) and \( y \).

\[
\begin{array}{ccccccc}
& b & a & c & a & b & c \\
\text{x} & & & & & & \\
\end{array}
\]
Formulas with free variables

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Example: $\phi = (x < y) \land a(x) \land b(y)$.

Another decorated word:

```
    b  a  c  a  b  c
    x
    y
```

$\phi$ is not satisfied by this word.
Formulas with free variables

Let $\varphi$ be a FO formula with free variables $x_1, \ldots, x_k$. A model of $\varphi$: A word $w$ along with an assignment of positions to $x_1, x_2 \ldots x_k$.

Example: $\phi = (x < y) \land a(x) \land b(y)$.

Any formula defines a language of decorated words
A decorated word is a word over the alphabet $\Sigma \times 2^V$, where $V$ is a set of free variables.

Words corresponding to the decorated words:

$$b \quad a \quad c \quad a \quad b \quad c$$

$$x \quad y$$

is $(b, \emptyset)(a, \{x\})(c, \emptyset)(a, \emptyset)(b, \{y\})(c, \emptyset)$. 

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A decorated word is a word over the alphabet $\Sigma \times 2^V$, where $V$ is a set of free variables.

Words corresponding to the decorated words:

$$b \ a \ c \ a \ b \ c$$

$$x$$

$$y$$

is $(b, \emptyset)(a, \emptyset)(c, \emptyset)(a, \emptyset)(b, \{x, y\})(c, \emptyset)$.

A $V$-word is a word $(a_1, U_1)(a_2, U_2)\ldots(a_k, U_k)$ with

1. $U_i \cap U_j = \emptyset$ for $i \neq j$.
2. $\bigcup_{1 \leq i \leq k} U_i = V$.

$L(\varphi)$ is a language of $V$-words for any $V$ with $\text{free}(\varphi) \subseteq V$. 
A natural measure of the complexity of a FO formula is its quantifier-depth.

\[
\begin{align*}
qd(\varphi) &= 0 & \text{if } \varphi \text{ is an atomic formula} \\
qd(\varphi \land \psi) &= \text{Maximum}(qd(\varphi), qd(\psi)) \\
qd(\neg \varphi) &= qd(\varphi) \\
qd(\exists x. \varphi) &= 1 + qd(\varphi)
\end{align*}
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\]

**Theorem:** For any \( i \) there are only finitely many formulas of quantifier depth \( i \) or less (upto logical equivalence).
Stratifying FO formulas

Why are we doing all this?
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This allows us to establish properties of FO via induction.

For example, we could show, by induction on quantifier-depth, that any language definable in FO is a regular language.
Stratifying FO formulas

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For example, we could show, by induction on quantifier-depth, that any language definable in FO is a regular language.

To do this we need an alternative characterization of quantifier-depth.
Question: When is $L$ definable in FO(k)? 

or equivalently

Question: When is $L$ not definable in FO(k)?
**FO(k) definability**

**Question:** When is $L$ definable in FO(k)?  
or equivalently  
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Find a pair of words $w, w'$ such that

1. $w \in L, w' \notin L$.  
2. $\forall \phi \in FO(k). (w \models \phi) \iff (w' \models \phi)$.  

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1. $w \in L$, $w' \notin L$.
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Question: When are two words distinguishable by FO(k)?
FO(k) definability

**Question:** When is $L$ definable in FO(k)?

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Find a pair of words $w$, $w'$ such that

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**Question:** When are two words distinguishable by FO(k)?

**EF-Games:** Set up $k$-round two player game (between say player 0 and player 1) based on $w$ and $w'$. Relate winning strategies to distinguishability.
The Game

Let \( w, w' \) be two words \( V \)-words and let \( k \) be an integer. The \( k \) round EF-game consists of the two players making \( k \) moves. In round \( i \):
Let $w, w'$ be two words $V$-words and let $k$ be an integer. The $k$ round EF-game consists of the two players making $k$ moves. In round $i$:

- Player 0 (who is trying to show that the two words are distinguishable) picks one of the two words and a position $p$ in that word and labels it with $x_i$. 

Let \( w, w' \) be two words \( V \)-words and let \( k \) be an integer. The \( k \) round EF-game consists of the two players making \( k \) moves. In round \( i \):

1. Player 0 (who is trying to show that the two words are distinguishable) picks one of the two words and a position \( p \) in that word and labels it with \( x_i \).

2. Player 1 must then pick the other word (i.e. the one not picked by player 0 in round \( i \)), pick some position \( p' \) and label it with \( x_i \).
The Game

Let \( w, w' \) be two words \( V \)-words and let \( k \) be an integer. The \( k \) round EF-game consists of the two players making \( k \) moves. In round \( i \):

1. **Player 0** (who is trying to show that the two words are distinguishable) picks one of the two words and a position \( p \) in that word and labels it with \( x_i \).

2. **Player 1** must then pick the other word (i.e. the one not picked by player 0 in round \( i \)), pick some position \( p' \) and label it with \( x_i \).

Let \( W \) and \( W' \) be the two \( V \cup \{x_1, x_2 \ldots, x_k\} \) words resulting from the \( k \)-round game.
The Game

Let $w, w'$ be two words $V$-words and let $k$ be an integer. The $k$-round EF-game consists of the two players making $k$ moves. In round $i$:

1. Player 0 (who is trying to show that the two words are distinguishable) picks one of the two words and a position $p$ in that word and labels it with $x_i$.

2. Player 1 must then pick the other word (i.e. the one not picked by player 0 in round $i$), pick some position $p'$ and label it with $x_i$.

Let $W$ and $W'$ be the two $V \cup \{x_1, x_2 \ldots, x_k\}$ words resulting from the $k$-round game.

If $W$ and $W'$ are distinguishable by atomic formulas then Player 0 is the winner.
The Game

Let \( w, w' \) be two words \( V \)-words and let \( k \) be an integer. The \( k \) round EF-game consists of the two players making \( k \) moves. In round \( i \):

1. Player 0 (who is trying to show that the two words are distinguishable) picks one of the two words and a position \( p \) in that word and labels it with \( x_i \).

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Let \( W \) and \( W' \) be the two \( V \cup \{x_1, x_2 \ldots, x_k\} \) words resulting from the \( k \)-round game.

1. If \( W \) and \( W' \) are distinguishable by atomic formulas then Player 0 is the winner.

2. Otherwise Player 1 is the winner.
An Example

Consider the words \textit{abba} and \textit{ababa}. Here is a winning strategy for Player 0.
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- Pick the first word and position 3.
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- No matter how Player 1 responded, pick the first word and position 2.
An Example

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- Pick the first word and position 3.
- No matter how Player 1 responded, pick the first word and position 2.
- If the positions picked by player 1 are not 2 and 4, Player 0 has already won.
Consider the words \textit{abba} and \textit{ababa}. Here is a winning strategy for Player 0.

- Pick the first word and position 3.
- No matter how Player 1 responded, pick the first word and position 2.
- If the positions picked by player 1 are not 2 and 4, Player 0 has already won.
- Otherwise, pick the second word and position 3.
Theorem: (Ehrenfeucht-Fraisse) Player 0 has a winning strategy in the $k$ round game on $w, w'$ if and only if there is a FO($k$) formula that distinguishes $w$ and $w'$. 
Theorem: (Ehrenfeucht,Fraisse) Player 0 has a winning strategy in the $k$ round game on $w$, $w'$ if and only if there is a FO($k$) formula that distinguishes $w$ and $w'$.

The proof is an easy inductive argument.

Note that any distinguishing formula dictates a winning strategy for player 0.
Theorem:  (Ehrenfeucht,Fraisse) Player 0 has a winning strategy in the $k$ round game on $w$, $w'$ if and only if there is a FO($k$) formula that distinguishes $w$ and $w'$.

Example:  Consider the words

$$\begin{align*}
a & b & b & a & b & b & a & b \\
a & b & a & b & b & a & b & b
\end{align*}$$

Here is a distinguishing formula:

$$\exists x_1. ( b(x_1) \land \exists x_2. (x_1 < x_2) \land \forall x_2 > x_1. b(x_2) )$$
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Conversely, winning strategies for Player 0 can be turned into distinguishing formulas.
Two words $w$ and $w'$ are said to be $k$-equivalent if they are indistinguishable by formulas with quantifier depth $k$ or less.

$$w \equiv_k w'$$

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- $\equiv_k$ is of finite index.
- Let $\varphi$ be a FO($k$) formula. Then $L(\varphi)$ is a (disjoint) union of some of the equivalence classes of $\equiv_k$. 
**Theorem:** (Myhill-Nerode) A language $L$ is regular if and only if it is the union of some of the equivalence classes of a right-invariant equivalence relation of finite index.

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The Expressive Power of Linear-time Temporal Logic
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Simulate strategy on $x$ and $y$, duplicate moves on $z$. 
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**Theorem:** Every First order definable language of words is regular.
Claim: The words $a^m$ and $a^{m+1}$ are $k$-equivalent whenever $m > 2^k$. 
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- Clearly $a \equiv_0 aa$.
- Player 0 will pick one of the two words and pick a position in that word and label it with $x$ to give

$$a^s.(a,x).a^t$$

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A Non-FO definable Language

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- Suppose $s \leq t$. Player 1 breaks up the other word as

  $$a^s.(a, x).a^{t'}$$

  with $s + t' = m$ or $s + t' + 1 = m$. 
A Non-FO definable Language

\[ a^s. (a, x). a^t \quad \text{and} \quad a^s. (a, x). a'^t \]

From now on:
A Non-FO definable Language

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Theorem: For all \( m > 2^k \) and \( w \in \Sigma^+ \), \( w^m \) and \( w^{m+1} \) are \( k \)-equivalent.

The latter asserts that FO definable languages are aperiodic.
Let \((M, ., 1)\) be a finite monoid. Let \(h : \Sigma^* \to M\) be a morphism.

**Theorem:** For any \(X \subseteq M\), \(h^{-1}(X)\) is a regular language.
Monoids as Language recognizers

Let \((M, ., 1)\) be a finite monoid. Let \(h : \Sigma^* \rightarrow M\) be a morphism.

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Let \(A_M = (M, \Sigma, \delta, 1, X)\) with \(\delta(m, a) = m.h(a)\). Then,

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- Let \(x \equiv_L y\) iff \(\forall u, v. \ uxv \in L \iff uvy \in L\).
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**The Syntactic Monoid of a Regular Language:**

- Let $x \equiv_L y$ iff $\forall u, v. \; uxv \in L \iff uvy \in L$.
- $\equiv_L$ is a congruence on $\Sigma^*$.
Monoids as Language recognizers

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The Syntactic Monoid of a Regular Language:

- Let \(x \equiv_L y\) iff \(\forall u, v. \ u x v \in L \iff u y v \in L\).
- \(\equiv_L\) is a congruence on \(\Sigma^*\).
- \(\text{SYN}(L) = (\Sigma^*/\equiv_L, ., [\epsilon]_{\equiv_L})\) is a finite monoid.
Monoids recognize Regular languages

Let $\eta_L : \Sigma^* \rightarrow \text{SYN}(L)$ be the morphism

$$\eta_L(x) = [x]_{\equiv_L}$$

Then,

$$L = \bigcup_{x \in L} \eta_L^{-1}([x]_{\equiv_L})$$

**Theorem:** A language is regular if and only if it is recognized by a finite monoid.
A Monoid $M$ is said to be aperiodic iff there is an integer $N$ such that
\[ a^k = a^{k+1} \text{ for all } k \geq N \text{ and } a \in M \]

A language $L$ is aperiodic iff it is recognized by an aperiodic monoid.
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**Theorem:** $\Sigma^*/\equiv_k$ is an aperiodic monoid. Thus, every FO definable language is aperiodic.
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**Theorem:** $\Sigma^*/\equiv_k$ is an aperiodic monoid. Thus, every FO definable language is aperiodic.

This follows from the fact that $w^m \equiv_k w^{m+1}$ for all $m > 2^k$. 

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The Expressive Power of Linear-time Temporal Logic
An useful result

If $M$ is an aperiodic monoid and $x, y \in M$ and $x \neq y$ then, $x.y \neq 1$. 
If $M$ is an aperiodic monoid and $x, y \in M$ and $x \neq y$ then, $x.y \neq 1$.

Suppose $x.y = 1$. Then, $x = x.x^N.y^N = x^N.y^N = 1$.

Similarly, $y = 1$, contradicting $x \neq y$. 
LTL is expressible in FO.
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FO definable languages are regular. (Via EF Games)
Summary

- LTL is expressible in FO.
- FO definable languages are regular. (Via EF Games)
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Summary

- LTL is expressible in FO.
- FO definable languages are regular. (Via EF Games)
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**Schützenberger’s Theorem:** A regular language $L$ is aperiodic if and only if it expressible as a star-free regular expression.