The Expressive Power of Linear-time Temporal Logic

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Linear-time Temporal Logic

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- Atomic propositions, boolean connectives, temporal modalities.

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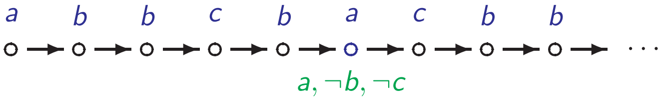
  Formulas are interpreted at positions of a word.

$$w = w_1 w_2 w_3 \ldots \qquad \text{with } w_i \in \Sigma$$

$$w, i \models \varphi \; ?$$

Atomic propositions: elements of Σ.

$$w, i \models a \iff w_i = a$$



Syntax and Semantics

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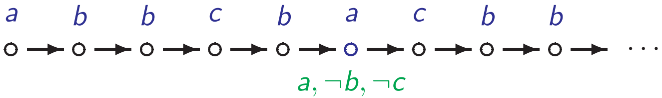
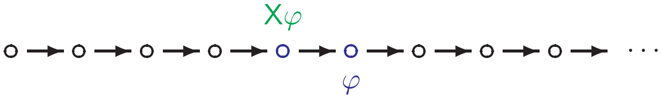


The Next state operator:

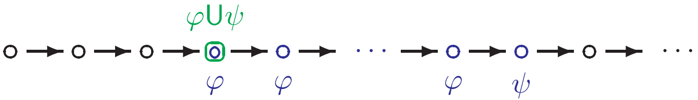
$$w, i \models \mathrm{X}\varphi \iff w, i+1 \models \varphi$$



Syntax and Semantics

The Until operator:

$$w, i \models \varphi \mathsf{U} \psi \iff \exists j \geq i.\ w, j \models \psi \text{ and } \forall i \leq k < j.\ w, k \models \varphi$$



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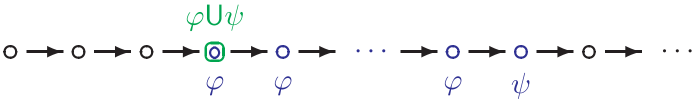


Boolean Connectives:

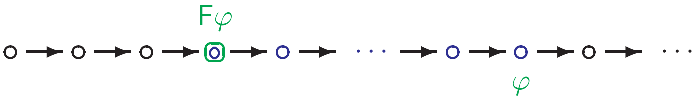
$$\varphi \wedge \psi, \quad \neg\varphi, \quad \ldots$$

with the usual interpretation.

Other Modalities

The Future modality

$$w, i \models F\varphi \iff \exists j \geq i.\ w, j \models \varphi$$



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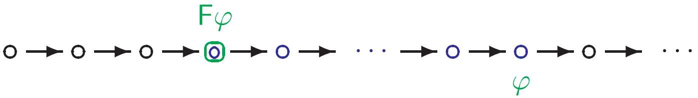

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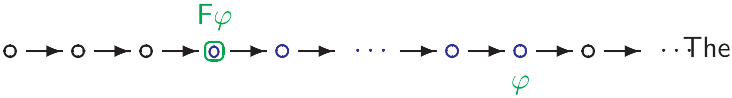
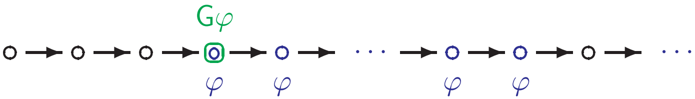


Henceforth modality:

$$w, i \models G\varphi \quad \Longleftrightarrow \quad \forall j \geq i.\ w, j \models \varphi$$



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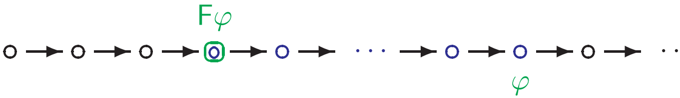

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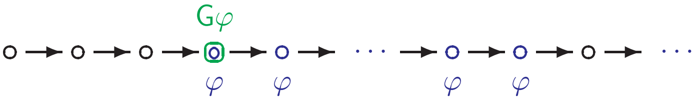
Henceforth modality:

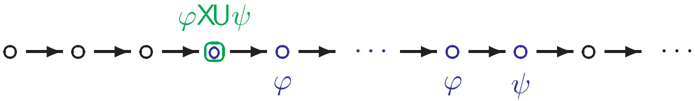
$$G\varphi \quad = \quad \neg F \neg \varphi$$



The Universal Modality

The Next-Until modality:

$$w, i \models \varphi \mathsf{XU} \psi \quad \equiv \quad \exists j > i.\ w, j \models \psi \text{ and } \forall i < k \leq j.\ w, k \models \varphi$$



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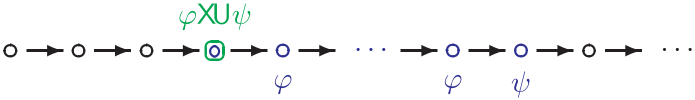

$$\varphi \mathsf{XU} \psi \quad = \quad \mathsf{X}(\varphi \mathsf{U} \psi)$$

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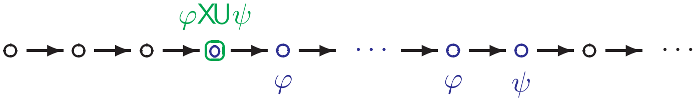

$$\varphi XU \psi \quad = \quad X(\varphi U \psi)$$

Next-Until can express everthing else

$$\begin{aligned} X\varphi &= \bot XU \varphi \\ \varphi U \psi &= \psi \vee (\varphi \wedge \varphi XU \psi) \end{aligned}$$

LTL definable languages

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Formulas define languages. For example,

$$\mathsf{G}(\mathsf{a} \implies \mathsf{F}\mathsf{b})$$

describes words in which there is a b somewhere to the right of every a.

$$b^*(aa^*bb^*)^*$$

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We restrict ourselves to finite word models (for now!).

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\mathcal{T}(a) &= a(x) \\
\mathcal{T}(X\alpha) &= \exists y.\ (y = x + 1) \wedge \mathcal{T}(\alpha)[y/x] \\
\mathcal{T}(\varphi \mathsf{U} \psi) &= \exists y.\ (y \geq x) \wedge \mathcal{T}(\psi)[y/x] \wedge \\
&\qquad \forall z.(x \leq z < y) \implies \mathcal{T}(\varphi)[z/x]
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- $w, i \models \mathcal{T}(\varphi) \quad \iff \quad w, i \models \varphi$.
- $\mathcal{T}(\varphi)$ uses at the most 3 variables (x, y and z). So, LTL is expressible in FO(3).

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What about FO?

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Conclusion: FO seems to be a stronger logic than LTL.

Model Checking

Given a FA A and a formula φ check if every word accepted by the automaton A satisfies the formula φ.

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Theorem:(Vardi/Wolper) The model-checking problem for LTL is solvable in time $O(|A|.2^{O(|\varphi|)})$.

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The rest of this talk and the next would be devoted to proving this result.

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How do we put together LTL formulas φ_1 and φ_2 to describe the language $L(\varphi_1).L(\varphi_2)$?

Easy if the decomposition is unambiguous. (eg.) $L_1.c.L_2$ where either L_1 or L_2 is c-free.

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The proof proceeds via a double induction: On the size of the monoid recognizing L and the size of the alphabet.

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  - L is $\{a^i \mid i \geq N\}$. Easy.

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show that L is expressible in LTL_Σ.

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Observation 1: If φ is a LTL_A formula describing the language L and $A \subseteq \Sigma$ then

$$\varphi \wedge \bigwedge_{a \in \Sigma \setminus A} \mathrm{G}\neg \mathrm{a}$$

is a LTL_Σ formula that describes L.

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It suffices to show that each of these three languages is LTL expressible.

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- L_0 is language over a smaller alphabet A, recognized by M via h.
- So, L_0 is defined by an LTL_A formula φ_0 over A.
- By Observation 1, it is expressible in LTL_Σ.

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$$L_1 = \bigcup_{\alpha.h(c).\beta \in X} (h^{-1}(\alpha) \cap A^*).c.(h^{-1}(\beta) \cap A^*)$$

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- If xcy is in the RHS then $h(xcy) = \alpha.h(c).\beta \in X$. Thus $xcy \in L$.

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Why?

- If xcy is in the RHS then $h(xcy) = \alpha.h(c).\beta \in X$. Thus $xcy \in L$.
- Let $w \in L_1$. Therefore, $w = xcy$. Take $\alpha = h(x)$ and $\beta = h(y)$.

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L_1 is a union of languages of the form $L_\alpha.c.L_\beta$ where $L_\alpha, L_\beta \subseteq A^*$ are recognized by M and hence LTL_A (and therefore LTL_Σ) expressible.
Well, almost! $L_\alpha \cap A^+$ and $L_\beta \cap A^+$ are LTL expressible. We have to deal with ϵ separately

Dealing with Unambiguous Concatenations

We may rewrite $L_\alpha.c.L_\beta$ as

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This case was easy because our modalities walk only to the right and so cannot "stray" to the left. Dealing with $L_\alpha.c.\Sigma^*$ will need a little more work.

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We cannot use φ_α to describe $L_\alpha.c.\Sigma^*$ since the modalities may walk to the right and cross the c boundary.

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\begin{array}{lcl}
a' & = & a \wedge \mathrm{XF}c \\
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$\varphi_2 = \varphi'_\alpha$ describes $(L_\alpha \cap A^+).c.\Sigma^*$. If $\epsilon \notin L_\alpha$ then φ_2 also describes $L_\alpha.c.\Sigma^*$. Otherwise, use $\varphi_2 \vee c$.

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The first and third components are LTL definable. What about the middle component?

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We use m to denote elements of M when treated as letters and m when they are treated as elements of the monoid M.

The map σ and Language K

The map σ is the obvious one:

$$\sigma(ct_1ct_2\ldots t_{k-2}ct_{k-1}c) \quad = \quad \mathrm{h}(\mathrm{t}_1)\mathrm{h}(\mathrm{t}_2)\ldots\mathrm{h}(\mathrm{t}_{\mathrm{k}-1})$$

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Localizing a Monoid at an element

The following construction is due to Diekert and Gastin.

The Monoid $\mathrm{Loc}_m(M)$: Let M be a monoid and $m \in M$. Then

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- $xm \circ xm \circ \ldots xm \; = \; x^N m$. Thus, $\mathrm{Loc}_m(M)$ is aperiodic whenever M is aperiodic.
- $1 \notin \mathrm{Loc}_m(M)$ if $m \neq 1$. This follows from the fact that $1 \neq m'm$ for any $m, m' \neq 1$.

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K is recognized by a smaller monoid and hence there is an LTL_M formula that describes K

Lifting the formula for K

We show that for any formula φ in LTL_M, there is a formula $\varphi^{\#}$ in LTL_Σ such that

$$w \models \varphi^{\#} \iff \begin{array}{l} w = ct_1ct_2c\ldots t_{k-1}ct_k, \text{ with } t_i \in A^* \\ \text{and } \sigma(ct_1ct_2\ldots t_{k-1}c) \models \varphi \end{array}$$

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Combining the Three parts

The formula describing $(L_\alpha \cap A^*).(L_\beta \cap \Delta).(L_\gamma \cap A^*)$ is the conjunction of the formulas describing the following languages.

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