The Littlewood-Richardson Rule

Aman Barot  
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Abstract

We motivate and prove the Littlewood-Richardson rule for Schur polynomials. We describe how to explicitly compute these numbers in two different ways.

1 INTRODUCTION

The Littlewood-Richardson rule combinatorially describes the coefficients which arise when a product of two Schur polynomials is expressed as a linear combination of Schur polynomials in the ring of symmetric polynomials. These coefficients as we shall see are just the number of skew tableaux of a certain type. These coefficients arise in the decomposition of skew Schur functions in terms Schur polynomials and in the decomposition of tensor product of irreducible representations of general linear groups as well.

We begin with giving a description about Schur polynomials. We shall see how results about tableaux give us identities for Schur polynomials. In section 3, we define exactly what the Littlewood-Richardson numbers are. It tells us how to express a product of Schur polynomials as a linear combination of Schur polynomials. Section 4 gives a description of computing Littlewood-Richardson numbers by counting skew tableau of a certain type. Section 5 gives a method of recursively computing Littlewood-Richardson numbers. This method connects these numbers with counting standard tableau with some special properties. Finally, we indicate how these numbers come up in other places as well.

2 SCHUR POLYNOMIALS

We begin by defining Schur polynomials.
Let \( \lambda \) be a partition of \( n \). To every numbering \( T \) of the Young diagram corresponds a monomial \( x^T \) defined formally as

\[
x^T = \prod_{i=1}^{m} (x_i)^{\text{number of times } i \text{ occurs in } T}
\]

The Schur polynomial \( s_\lambda(x_1, x_2, \ldots, x_m) \) then is the sum

\[
s_\lambda(x_1, x_2, \ldots, x_m) = \sum_T x^T
\]  

We recall the three ways of defining a product on the set of tableaux (which agree with each other):

1. By row-insertion
2. By sliding
3. By Knuth words

Since, we have a product on tabloids, we consider the monoid of tableaux with entries in \( \{1, 2, \ldots, m\} \). With this monoid we associate the ring \( R_{[m]} \), and call it the tableau ring. This is the free \( \mathbb{Z} \)-module with basis the tableaux with entries in \( \{1, 2, \ldots, m\} \), with multiplication determined by the multiplication of tableaux. There is a canonical homomorphism from \( R_{[m]} \) onto the ring \( \mathbb{Z}[x_1, \ldots, x_m] \) of polynomials that takes a tableau \( T \) to its monomial \( x^T \), where \( x^T \) is the product of the variables \( x_i \), each occurring as many times in \( x^T \) as \( i \) occurs in \( T \).

If by \( S_\lambda \) in \( R_{[m]} \) we denote the sum of all tableau of shape \( \lambda \) with entries in \( [m] \), then its image under the canonical homomorphism above is exactly the Schur polynomial \( s_\lambda(x_1, x_2, \ldots, x_m) \).

Now we can state the Pieri formulas which motivate the main result in the next section. Historically, they came before the more general Littlewood-Richardson formulas for an arbitrary tableau in Theorem 2.1 in place of a row or a column.

**Theorem 2.1 (Pieri formulas).** If \( (p) \) and \( 1^p \) denote the Young diagrams with one row and one column of length \( p \), then the following hold in the tableau ring \( R_{[m]} \):

1. \( S_\lambda \cdot S_{(p)} = \sum_\mu S_\mu \) where the sum is over all \( \mu \)'s that are obtained from \( \lambda \) by adding \( p \) boxes, with no two in the same column.

2. \( S_\lambda \cdot S_{1^p} = \sum_\mu S_\mu \) where the sum is over all \( \mu \)'s that are obtained from \( \lambda \) by adding \( p \) boxes, with no two in the same row.

**Proof.** The product of a tableau \( T \) and a tableau \( V \) of shape of a row is a tableau of shape \( \mu \) such that the elements of \( V \) lie in different columns. Also, given a tableau \( S \) of shape \( \mu \) such that \( \lambda \subset \mu \) and if the elements of \( \mu/\lambda \) lie in different columns, then there is a unique way of writing \( S = T \cdot V \), where \( T \) is tableau of shape \( \lambda \) and \( V \) is a tableau of shape of a row. Therefore, we have the first of the Pieri formula.

For the second one, a similar argument follows. In this case, instead of a row we have a column. □
Applying the canonical homomorphism from $R[m]$ onto the ring $\mathbb{Z}[x_1, \ldots, x_m]$, we get the following corollary:

**Corollary 2.2.** If $(p)$ and $1^p$ denote the Young diagrams with one row and one column of length $p$, then the following hold in $\mathbb{Z}[x_1, \ldots, x_m]$:

1. $s_\lambda(x_1, \ldots, x_m) \cdot h_p(x_1, \ldots, x_m) = \sum_\mu s_\mu(x_1, \ldots, x_m)$ where the sum is over all $\mu$'s that are obtained from $\lambda$ by adding $p$ boxes, with no two in the same column.

2. $s_\lambda(x_1, \ldots, x_m) \cdot e_p(x_1, \ldots, x_m) = \sum_\mu s_\mu(x_1, \ldots, x_m)$ where the sum is over all $\mu$'s that are obtained from $\lambda$ by adding $p$ boxes, with no two in the same row.

**Example 2.1.** Let $\lambda = (2, 1)$. We wish to calculate

- $S_{(2,1)} \cdot S_{(2)}$
- $S_{(2,1)} \cdot S_{(1,1)}$

By the Pieri rule, for the first one, the sum is over partitions $\mu$ that are obtained by adding two boxes, with no two in the same column. They are the following (coloured boxes are the added ones):

```
+ + + |
+ + + |
+ + + |
+ + + |
```

So, we get the following expansion of the product in $R[m]$ and $\mathbb{Z}[x_1, x_2, x_3]$ respectively:

$S_{(2,1)} \cdot S_{(2)} = S_{(3,2)} + S_{(2,2,1)} + S_{(3,1,1)} + S_{(4,1)}$

$s_{(2,1)}(x_1, x_2, x_3) \cdot s_{(2)}(x_1, x_2, x_3) = s_{(3,2)}(x_1, x_2, x_3) + s_{(2,2,1)}(x_1, x_2, x_3) + s_{(3,1,1)}(x_1, x_2, x_3) + s_{(4,1)}(x_1, x_2, x_3)$

For the second one, we list $\mu$'s that are obtained by adding two boxes, with no two in the same row:

```
+ + + |
+ + + |
+ + + |
+ + + |
```

The two equations for this one are:

$S_{(2,1)} \cdot S_{(1,1)} = S_{(3,2)} + S_{(2,2,1)} + S_{(3,1,1)} + S_{(2,1,1,1)}$

$s_{(2,1)}(x_1, x_2, x_3) \cdot s_{(1,1)}(x_1, x_2, x_3) = s_{(3,2)}(x_1, x_2, x_3) + s_{(2,2,1)}(x_1, x_2, x_3) + s_{(3,1,1)}(x_1, x_2, x_3) + s_{(2,1,1,1)}(x_1, x_2, x_3)$
3 Correspondences between Skew Tableau

We shall be constructing correspondence between tableaux, which will give us results about symmetric polynomials. Our aim is to investigate in what ways a tableau can be expressed as a product of two tableaux of certain shapes. The crucial result needed towards that is as follows:

**Theorem 3.1** (Chapter 5, Proposition 3.1 [Ful97]). Suppose \( \omega = (u_1 \ u_2 \ \ldots \ u_r \ v_1 \ v_2 \ \ldots \ v_r) \) is an array in lexicographic order, corresponding to the pair \((P, Q)\) by the R-S-K correspondence. Let \( T \) be any tableau, and if we perform the row-insertions \( \ldots (T \leftarrow v_1) \leftarrow v_2 \ldots) \) and place \( u_1, u_2, \ldots, u_m \) successively in the new boxes. Then the entries \( u_1, u_2, \ldots, u_m \) form a skew tableau \( S \) whose rectification is \( Q \).

**Proof.** Let \( T_0 \) be a tableau of same shape as \( T \) with entries from the set of integers less than all the entries in \( T \). By R-S-K correspondence the pair \((T, T_0)\) corresponds to some lexicographic array \((s_1 \ s_2 \ \ldots \ s_n \ t_1 \ t_2 \ \ldots \ t_n)\). The lexicographic array \((s_1 \ s_2 \ \ldots \ s_n \ u_1 \ u_2 \ \ldots \ u_m \ t_1 \ t_2 \ \ldots \ t_n \ v_1 \ v_2 \ \ldots \ v_m)\) corresponds to a pair \((T \cdot P, V)\) where \( T \cdot P \) is the tableau obtained by successively row-inserting \( v_1, v_2, \ldots, v_m \) into \( T \), and \( V \) is the tableau whose entries \( s_1, s_2, \ldots, s_n \) make up \( T_0 \), and whose entries \( u_1, u_2, \ldots, u_m \) make up \( S \).

Now, invert the array and put it in lexicographic order. By Schutzenberger’s symmetry theorem, this corresponds to \((V, T \cdot P)\) and if we remove \( \left(\begin{array}{c}s_j \\ t_j\end{array}\right)\)’s, the array will correspond to \((Q, P)\). The word on the bottom of this array is Knuth equivalent to \( w_{row}(V) \), and when the \( s_j \)'s are removed from this word, we get a word Knuth equivalent to \( w_{row}(Q) \). However, removing the \( m \) smallest letters from \( w_{row}(V) \) leaves the word \( w_{row}(S) \). Since, removing the \( m \) smallest letters of Knuth equivalent words gives Knuth equivalent words, we conclude \( w_{row}(S) \) is Knuth equivalent to \( w_{row}(Q) \), which means precisely that the rectification of \( S \) is \( Q \).

Now, for any tableau \( U_0 \) of shape \( \mu \), define

\[ \mathcal{S}(\nu/\lambda, U_0) = \{ \text{skew tableaux } S \text{ on } \nu/\lambda : \text{Rect}(S) = U_0 \} \]

and for any tableau \( V_0 \) of shape \( \nu \), define

\[ \mathcal{T}(\lambda, \mu, V_0) = \{ [T, U] : T \text{ is a tableau on } \lambda, U \text{ is a tableau on } \mu, \text{ and } T \cdot U = V_0 \} \]

Then, the following result gives a natural bijection between these two sets:

**Theorem 3.2** (Chapter 5, Proposition 3.2 [Ful97]). For any tableau \( U_o \) on \( \mu \) and \( V \) on \( \nu \), there is a canonical one-one correspondence

\[ \mathcal{T}(\lambda, \mu, V_0) \leftrightarrow \mathcal{S}(\nu/\lambda, U_0) \]

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Proof. Given \([T, U]\) in \(\mathcal{F}(\lambda, \mu, V_0)\) consider the lexicographic array corresponds to the pair \((U, U_0)\):

\[
(U, U_0) \leftrightarrow \begin{pmatrix} u_1 & u_2 & \ldots & u_m \\ v_1 & v_2 & \ldots & v_m \end{pmatrix}
\]

Successively row-insert \(v_1, v_2, \ldots, v_m\) into \(T\), and let \(S\) be the skew tableau obtained by successively placing \(u_1, u_2, \ldots, u_m\) into the new boxes. Since, \(T \cdot U = T \leftarrow v_1 \leftarrow \ldots \leftarrow v_m = V_0\) has shape \(v\), by Theorem 3.1 \(S \in \mathcal{F}(v/\lambda, U_0)\).

Conversely, given \(S \in \mathcal{F}(v/\lambda, U_0)\), let \(T_0\) be an arbitrary tableau such that all the entries in \(T_0\) are less than the entries in \(S\). Let \(T_0 S\) be the tableau on \(v\) that is simply \(T_0\) on \(\lambda\) and \(S\) on \(v/\lambda\). Under R-S-K correspondence,

\[
(V_0, T_0 S) \leftrightarrow \begin{pmatrix} t_1 & t_2 & \ldots & t_n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}
\] (3.1)

We now claim that

\[
(T, T_0) \leftrightarrow \begin{pmatrix} t_1 & t_2 & \ldots & t_n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}
\] (3.2)

and

\[
(U, U_0) \leftrightarrow \begin{pmatrix} u_1 & u_2 & \ldots & u_m \\ v_1 & v_2 & \ldots & v_m \end{pmatrix}
\] (3.3)

for some tableau \(T\) and \(U\) of shapes \(\lambda\) and \(\mu\) with \(T \cdot U = V_0\). By construction, entries of \(T_0\) are less than the entries of \(S\), so the second tableau in (3.2) is \(T_0\) by the product construction of tableau via row-insertion. In (3.3), it can be seen that the second tableau is \(U_0\) by using Theorem 3.1. This gives us the pair \([T, U]\) in \(\mathcal{F}(\lambda, \mu, V_0)\), and clearly both the constructions are inverses to each other. So, the claim follows.

Since, the correspondence holds for all \(U_0\) and \(V_0\) of the specified shapes, the cardinalities the two sets depends only on the choice of \(\lambda, \mu, \) and \(v\). Thus, we have:

**Corollary 3.3.** The cardinalities of the sets \(\mathcal{F}(v/\lambda, U_0)\) and \(\mathcal{F}(\lambda, \mu, V_0)\) are independent of the choice of \(U_0\) or \(V_0\), and depend only on the shapes \(\lambda, \mu, \) and \(v\).

The number(cardinality of the sets) in the above corollary will be denoted by \(c_{\lambda\mu}^v\). These are called the Littlewood-Richardson numbers.

Since, each tableau \(V\) of shape \(v\) can be written in exactly \(c_{\lambda\mu}^v\) ways as a product of a tableau of shape \(\lambda\) times a tableau of shape, we have the corollary:

**Corollary 3.4.** The following identities hold in the tableau ring \(R_{(m)}\):

1. \(S_{\lambda} \cdot S_{\mu} = \sum_v c_{\lambda\mu}^v S_v\)
2. \(S_{v/\lambda} = \sum_{\mu} c_{\lambda\mu}^v S_{\mu}\)
The second result above follows from the fact that each tableau of shape $\mu$ occurs precisely $c_{\lambda\mu}^\nu$ times as the rectification of a skew tableau of shape $\nu/\lambda$.

We naturally have corresponding identities for Schur polynomials. Consider the canonical homomorphism from $R_m$ to $\mathbb{Z}[x_1, \ldots, x_m]$. Define $s_{\nu/\lambda}(x_1, \ldots, x_m)$ to be the image of $S_{\nu/\lambda}$ under this homomorphism. Using the previous corollary and applying the homomorphism we have:

**Corollary 3.5 (Littlewood-Richardson rule).** The following identities hold in the ring $\mathbb{Z}[x_1, \ldots, x_m]$:

1. $s_\lambda(x_1, \ldots, x_m) \cdot s_\mu(x_1, \ldots, x_m) = \sum \nu c_{\lambda\mu}^\nu s_\nu(x_1, \ldots, x_m)$

2. $s_{\nu/\lambda}(x_1, \ldots, x_m) = \sum \mu c_{\lambda\mu}^\nu s_\mu(x_1, \ldots, x_m)$

### 4 Reverse lattice words

**Definition 4.1.** A **lattice word** is a sequence of positive integers $\pi = i_1 i_2 \ldots i_n$ such that, for any prefix $\pi_k = i_1 i_2 \ldots i_k$ and any positive integer $l$, the number of $l$'s in $\pi_k$ is at least as large as the number of $(l + 1)$'s in that prefix. A **reverse lattice word** is a sequence $\pi$ such that $\pi^{rev}$ is a lattice word.

We call a skew tableau $T$ a **Littlewood-Richardson skew tableau** if its word $w(T)$ is a reverse lattice word.

Before, we prove the main result, we need a couple of lemmas:

**Lemma 4.1** (Chapter 5, Lemma 1 [Ful97]). If $w$ and $w'$ are two Knuth equivalent words, then $w$ is reverse lattice word if and only if $w'$ is a reverse lattice word.

**Proof.** Consider a Knuth transformation:

$$w = uxzyv \rightarrow uzxyv = w' \text{ with } x \leq y < z.$$  

We need only check for possible changes in the numbers of consecutive integers $k$ and $k + 1$. If $x < y < z$ there is no change, and the only case to check is when $x = y = k$ and $z = k + 1$. For either of them to be reverse lattice words, the number of $k$'s appearing in $v$ must be at least as large as the number of $(k + 1)$'s appearing in $v$. In this case, both $xzyv$ and $zxyv$ are reverse lattice words.

For the other Knuth transformation:

$$w = uyxzv \rightarrow uyxzv = w' \text{ with } x < y \leq z.$$  

again the only non-trivial case is when $x = k$ and $y = z = k + 1$. In this case none of them will be a reverse lattice word unless the number of $k$'s appearing in $v$ is strictly larger than the number of $(k + 1)$'s, in which case both of the words $yxzv$ and $yzxv$ have at least as many $k$'s as $(k + 1)$'s.

**Lemma 4.2** (Chapter 5, Lemma 2 [Ful97]). A skew tableau $S$ is a Littlewood-Richardson skew tableau of content $\mu$ if and only if its rectification is the tableau $U(\mu)$. 

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Proof. Suppose the given skew-tableau is actually a tableau. In this case, we have to show that the only tableau on a given Young diagram whose word is a reverse lattice word is the one whose first row consists of 1’s, second row of 2’s and so on. But if the word is a reverse lattice word, the last entry of the first row is a 1 and by property of a tableau, all the entries in the first row are 1’s. The last entry in the second row must be a 2 since it must be strictly larger than 1 since it is a tableau and it must be a 2 since it is a reverse lattice word. This implies all entries in second row are 2’s and so on row by row.

To conclude the proof, it is sufficient to show that skew-tableau is a Littlewood-Richardson skew-tableau if and only if its rectification is a Littlewood-Richardson tableau. Since, the rectification process preserves the Knuth equivalence class of the word, we are done by Lemma 4.1.

The following result gives another way of seeing how Littlewood-Richardson numbers come up:

**Theorem 4.3** (Chapter 5, Proposition 3 [Ful97]). The number $c_{\lambda \mu}^\nu$ is the number of Littlewood-Richardson skew tableau on the shape $\nu \setminus \lambda$ of content $\mu$.

**Proof.** By Lemma 4.2, a skew tableau $S$ is Littlewood-Richardson tableau if and only if its rectification is $U(\mu)$. But by Theorem 3.2, the number of skew tableau $S$ of shape $\lambda \setminus \mu$ with rectification $U(\mu)$ (of shape $\mu$) is exactly the Littlewood-Richardson number $c_{\lambda \mu}^\nu$. Hence, the claim follows.

**Example 4.1.** Let $\lambda = \mu = (2, 1)$. We want to calculate the product $S_{(2,1)} \cdot S_{(2,1)}$.

We list out the Littlewood-Richardson skew-tableaux of shape $\nu/(2,1)$ and content $(2,1)$, where $\nu$ is varying. The following are all such tableaux:

```
   • • 1 1
   • 2
   • • 1 1
   • 2
   • • 1 1
   • 1
   • • 1 1
   • 1
```

By the above theorem, count of the tableaux for each shape $\nu$ gives us the number $c_{\lambda \mu}^\nu$. This helps us in expanding the product as:

$$S_{(2,1)} \cdot S_{(2,1)} = S_{(2,1)} + S_{(4,1,1)} + S_{(3,3)} + 2S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2,2,2)} + S_{(2,2,1,1)}$$

**Example 4.2.** Let $\nu = (4,3,2)$ and $\lambda = (2,1)$. We want to expand $S_{(4,3,2)/(2,1)}$ as a linear sum of Schur polynomials.

For this one, we list out all the Littlewood-Richardson skew-tableaux of shape $(4,3,2)/(2,1)$:
The count of the tableaux for each shape \( \mu \) gives us the number \( c^\nu_{\lambda \mu} \). This gives us the expansion as:

\[
S_{(4,3,2)/(2,1)} = S_{(2,2,2)} + 2S_{(3,2,1)} + S_{(3,3)} + S_{(4,2)} + S_{(4,1,1)}
\]

5 A Recursive Method for Computation

There is a canonical one-to-one correspondence between reverse lattice words and standard tableaux. We can see this as given a reverse lattice word \( w = x_r \ldots x_1 \), put the number \( p \) in the \( x_p \)th row of the standard tableau, for \( p = 1, \ldots, r \). Denote this tableau by \( U(w) \). And given a standard tableau \( S \), we can get a reverse lattice word \( w(S) = y_r \ldots y_1 \) where \( y_i \) is the row number in which \( i \) is in \( S \). Clearly, this construction is inverse to the previous one.

Given a skew shape \( \nu/\lambda \), number the boxes from left to right in each row, working from the top to the bottom; call it the reverse numbering of the shape.

**Example 5.1.** Let \( \nu = (4, 3, 2) \) and \( \lambda = (2, 1) \). Then the reverse numbering for \( \nu/\lambda \) is given as:

\[
\begin{array}{ccc}
\bullet & \bullet & 1 \\
\bullet & 2 & 2 \\
& 3 & 3 \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & 1 \\
\bullet & 1 & 2 \\
& 2 & 3 \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & 1 \\
\bullet & 1 & 2 \\
& 1 & 3 \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & 1 \\
\bullet & 2 & 2 \\
& 1 & 3 \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & 1 \\
\bullet & 1 & 2 \\
& 1 & 3 \\
\end{array}
\begin{array}{ccc}
\bullet & \bullet & 1 \\
\bullet & 2 & 2 \\
& 1 & 3 \\
\end{array}
\]

Then we have the following result:

**Theorem 5.1** (Chapter 5, Proposition 4 [Ful97]). The value of the coefficient \( c^\nu_{\lambda \mu} \) is equal to the number of standard tableaux \( T \) on the shape \( \mu \) such that:

1. If \( k - 1 \) and \( k \) appear in the same row of the reverse numbering of \( \nu/\lambda \), then \( k \) occurs weakly above and strictly right of \( k - 1 \) in \( T \).
2. If \( k \) appears in the box directly below \( j \) in the reverse numbering of \( \nu/\lambda \), then \( k \) occurs strictly below and weakly left of \( j \) in \( T \).

**Proof.** Corresponding to each Littlewood-Richardson skew-tableau \( S \) on \( \nu/\lambda \), we have a reverse lattice word \( w(S) = x_r \ldots x_1 \) and so a standard tableau \( U(w(S)) \). The fact that \( S \) is a skew tableau translates precisely to the conditions (1) and (2) for \( U(w(S)) \). This because if \( k - 1 \) and \( k \) are in the same row of the reverse numbering, then \( x_k \leq x_{k-1} \) as \( S \) is a skew-tableau which means that \( k \) is entered weakly above \( k - 1 \) in \( U \). Since, \( k \) occurs weakly above \( k - 1 \), it has to be strictly right of \( k - 1 \) since it is a tableau. Similarly, if \( j \) is directly above \( k \) in the reverse numbering, the tableau condition gives that \( x_j < x_k \), which translates to the fact that \( k \) goes in a lower row of \( j \) in \( U \).
Example 5.2. Let $\nu = (4, 3, 2)$ and $\lambda = (2, 1)$. We compute the numbers $c_{\lambda \mu}^{\nu}$.

Using the previous theorem, starting with a tableau containing 1, we construct all the standard tableaux satisfying the two conditions inductively. We use the two conditions to see how to enlarge the tableau. In this case, there is only one place to insert a 2 by condition 1 and similarly for 3 by condition 2.

```
1 2
3
```

There are two places to insert a 4 after this:

```
1 2 3 4
1 2 3
```

We continue the process and the tableaux we get during this are:

```
1 2 3 4
1 2 3 5
1 2 3 4 5
1 2 3 4 6
```

Thus, comparing the shapes $\mu$ of these standard tableaux, we infer that $c_{(2,1)\mu}^{(4,3,2)}$ equals

- 2 when $\mu = (3, 2, 1)$
- 1 when $\mu = (2, 2, 2), (3, 3), (4, 2), (4, 1, 1)$
- 0 otherwise.

We notice that this answer is the same as we had calculated in Example 4.2.

6 Conclusion

We saw how to define a product on the set of tableau and how formal sums of tableau relate to symmetric polynomials. This helped us in getting results about Schur polynomials from results about tableaux. We thus proved the Littlewood-Richardson rule and saw how to compute the Littlewood-Richardson numbers.
Apart from coming up in the product of Schur polynomials, Littlewood-Richardson coefficients come up in many other places as well. In particular, it helps in describing the induced representation of the tensor product of Specht modules, or in the decomposition of tensor product of irreducible representations of the general linear group or in Schubert varieties. The interested reader is encouraged to read more about them.

REFERENCES