Orbit closures of quiver representations

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1. Orbit closures
2. Geometric technique
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4. Results
- Quiver $Q = (Q_0, Q_1)$ is a directed graph with set of vertices $Q_0$ and set of arrows $Q_1$. (Notation: $ta \rightarrow ha$)
- Dynkin quiver - underlying unoriented graph is a Dynkin diagram.
$A_n$

$D_n$

$E_6$

$E_7$

$E_8$
- **Quiver** $Q = (Q_0, Q_1)$ is a directed graph with set of vertices $Q_0$ and set of arrows $Q_1$. (Notation: $ta \xrightarrow{a} ha$)
- **Dynkin quiver** - underlying unoriented graph is a Dynkin diagram.
- A **representation** of $Q$ is a pair $V = ((V_i)_{i \in Q_0}, (V(a))_{a \in Q_1})$, where $V_i$ are finite-dimensional vector spaces over algebraically closed field $k$ and $V_{ta} \xrightarrow{V(a)} V_{ha}$.
- **Dimension vector** $d = (\dim V_i)_{i \in Q_0}$.
- The **representation space** of dimension type $d = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$

$$
\text{Rep}(Q, d) = \{ V = ((V_i)_{i \in Q_0}, (V(a))_{a \in Q_1}) | \dim(V_i) = d_i \} \\
= \prod_{a \in Q_1} \text{Hom}(V_{ta}, V_{ha}) \cong \mathbb{A}^N.
$$
\begin{itemize}
  \item \( \text{GL}(d) = \prod_{i \in Q_0} \text{GL}(d_i) \)
  \item \( \text{GL}(d) \) acts on \( \text{Rep}(Q, d) \) by simultaneous change of basis at each vertex
    \[
    ((g_i)_{i \in Q_0}, (V(a))_{a \in Q_1}) = (g_{ha} V(a) g_{ia}^{-1})_{a \in Q_1}
    \]
  \item \( \{ \text{GL}(d) - \text{orbits} \} \leftrightarrow \{ \text{isomorphism classes of representations of } Q \} \)
    \[
    O_V \leftrightarrow [V]
    \]
\end{itemize}

Orbit closure \( \overline{O}_V \) is a subvariety of the affine space \( \text{Rep}(Q, d) \).
Example

\[ Q = (Q_0, Q_1); Q_0 = \{1, 2, 3\}, Q_1 = \{a, b\} \]

\[ \text{Rep}(Q, d) = \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_3) \]
\[ d = (\text{dim } V_1, \text{dim } V_2, \text{dim } V_3) \]

\[ \text{Rep}(Q, (3, 4, 3)) = \text{Hom}(k^3, k^4) \times \text{Hom}(k^4, k^3) \cong \mathbb{A}^{24} \]

\[ \text{GL}(d) = \text{GL}(3) \times \text{GL}(4) \times \text{GL}(3) \]
\[ (g_1, g_2, g_3) \cdot (V_a, V_b) := (g_2 V_a g_1^{-1}, g_3 V_b g_2^{-1}) \]

\[ \overline{O}_V \text{ is an affine variety in } \mathbb{A}^{24}. \]
Example

- Rep($Q, d$) = $\mathbb{M}_{d \times d}(k)$
- Group action: conjugation
- Orbits: conjugacy classes of matrices in $\mathbb{M}(d, k)$
- For nilpotent $V(a)$,
  - if char $k > 0$ then $\overline{O}_{V(a)}$ is a Frobenius split variety.
  - if char $k = 0$ then $\overline{O}_{V(a)}$ is Gorenstein; defining ideal generated by minors of various sizes.
Example

\[ V_1 \xrightarrow{V(a)} V_2 \]

\[ d = (d_1, d_2) \]

- \( \text{Rep}(Q, d) = \mathbb{M}_{d_2 \times d_1}(k) \)
- Group action: \((g_1, g_2) \ast V(a) = g_1 V(a) g_2^{-1} \)
- Orbits: \( O_r = \text{matrices of rank } r, \quad 0 \leq r \leq m = \min(d_1, d_2) \)
- \( \overline{O}_r = \bigcup_{j \leq r} O_j \) (determinantal varieties)
  \[ = \left\{ W \in \text{Rep}(Q, d) \mid \text{rank } W(a) \leq r \right\} \]
- Geometry: normal, Cohen-Macaulay varieties with rational singularities;
  Gorenstein if \( r = 0, r = m \) or \( d_1 = d_2 \);
  regular if \( r = 0 \) or \( r = m \);
- Resolution of defining ideal (Lascoux);
  defining ideal generated by \((r + 1) \times (r + 1)\) minors.
Why study orbit closures?

- types of singularities
- degenerations
- desingularization
- normality, C-M, unibranchness
- tangent spaces
- defining equations
- etc.
Some results

(1981) Abeasis, Del Fra, Kraft
Equi-oriented quiver of type $A_n$: orbit closures are normal, Cohen-Macaulay with rational singularities.

(1998) Laxmibai, Magyar
Extended the above result to arbitrary characteristic.

Proved the above result for $A_n$ with arbitrary orientation and for $D_n$.

- Defining ideals?
- What about quivers of type $E$?
Goal

To study orbit closures by calculating resolutions

Strategy

Use geometric technique
- $X$: affine space
- $Y$: subvariety
- $\mathcal{V}$: a projective variety
Let $Z = \text{tot}(\eta)$ and $X \times \mathcal{V} = \text{tot}(\mathcal{E})$.

Exact sequence of vector bundles over $\mathcal{V}$:

$$0 \to \eta \to \mathcal{E} \to \tau \to 0$$

Define $\xi = \tau^*$

$$K(\xi) : 0 \to \bigwedge^t \xi \to \cdots \to \bigwedge^2(p^*\xi) \to p^*\xi \to \mathcal{O}_{X \times \mathcal{V}}$$

resolves $\mathcal{O}_Z$ as $\mathcal{O}_{X \times \mathcal{V}}$-module.
Let $Z = \text{tot}(\eta)$ and $X \times \mathcal{V} = \text{tot}(\mathcal{E})$.

**Exact sequence of vector bundles over $\mathcal{V}$:**

$$0 \to \eta \to \mathcal{E} \to \tau \to 0$$

Define $\xi = \tau^*$

**$K(\xi)$:**

$$0 \to \bigwedge^t \xi \to \cdots \to \bigwedge^2(p^*\xi) \to p^*\xi \to \mathcal{O}_{X \times \mathcal{V}}$$

$$\downarrow q_*$$

$\mathbf{F}$. 
Main theorem [Weyman]

- \( F_i = \bigoplus_{j \geq 0} H^j(\mathcal{V}, \bigwedge^{i+j} \xi) \otimes_K A(-i-j) \) where \( A = K[X] \).
- \( F_i = 0 \) for \( i < 0 \) \( \Rightarrow \) \( F_\bullet \) is a finite free resolution of the normalization of \( K[Y] \).
- If \( F_i = 0 \) for \( i < 0 \) and \( F_0 = A \) \( \Rightarrow \) \( Y \) is normal, Cohen-Macaulay and has rational singularities.
(Reineke desingularization)

\[
Z \subset \text{Rep}(Q, d) \times \prod_{x \in Q_0} \text{Flag}(d_s(x), ..., K^d(x))
\]

\[
q' \quad q \\
\downarrow \quad \downarrow \\
\overline{O}_V \subset \text{Rep}(Q, d) \\
\prod_{x \in Q_0} \text{Flag}(d_s(x), ..., K^d(x))
\]

\[
F_i = \bigoplus_{j \geq 0} H^j(V, \wedge^{i+j} \xi) \otimes_K A(-i-j) \quad \text{where} \quad A = K[\text{Rep}(Q, d)].
\]

If \( F_i = 0 \) for \( i < 0 \) \( \Rightarrow \mathbf{F} \) is a finite free resolution of the normalization of \( K[\overline{O}_V] \).

If \( F_i = 0 \) for \( i < 0 \) and \( F_0 = A \) \( \Rightarrow \overline{O}_V \) is normal and has rational singularities.
Steps in the construction

- Construct the Auslander-Reiten quiver of $Q$.
- Find a directed partition of the AR quiver (with $s$ parts).
- Define a subset of $Z_{I^*,V}$ of $\text{Rep}(Q,d) \times \prod_{x \in Q_0} \text{Flag}(\beta, V_x)$

$$Z_{I^*,V} = \{(V, (R_s(x) \subset \cdots \subset R_2(x) \subset V_x)_{x \in Q_0})$$
$$| \forall a \in Q_1, \forall t, \ V(a)(R_t(ta)) \subset R_t(ha)\}$$

$$
\begin{array}{ccc}
Z_{I^*,V} & \longrightarrow & \text{Rep}(Q,d) \times \prod_{x \in Q_0} \text{Flag}(\beta, V_x) \\
\downarrow q & & \downarrow q \\
\overline{O}_V & \longrightarrow & \text{Rep}(Q,d)
\end{array}
$$

Theorem [Reineke]

- $q(Z_{I^*,V}) = \overline{O}_V$.
- $q$ is a proper birational isomorphism of $Z_{I^*,V}$ and $\overline{O}_V$. 
Example - determinantal variety

$V = a(0K) \oplus b(KK) \oplus c(K0)$

$\beta_i = \dim R_i$

$Z \in \text{Rep}(Q,d) \times \prod_{i=1}^{2} \text{Flag}(\beta, V_i) \times \text{Gr}(\beta, V_i)$
Example - determinantal variety

\[ V_1 \xrightarrow{\phi} V_2 \]
\[ \dim V_1 = 2 \]
\[ \dim V_2 = 3 \]
\[ r = \text{rank } \phi = 2 \]

\[ X = \text{Hom}(V_1, V_2) = V_1^* \otimes V_2 \]
\[ \text{Orbit closure } \overline{O}_V = \{ \phi : V_1 \to V_2 \mid \text{rk } \phi \leq 2 \} \]
\[ \mathcal{V} = \text{Gr}(2, V_2) \]
\[ Z = \text{tot}(V_1^* \otimes \mathcal{R}) = \{ (\phi, R) \mid \text{im}(\phi) \subseteq R \} \]
\[ A = K[X] = \text{Sym}(V_1 \otimes V_2^*) \]

Lascoux resolution

\[ 0 \longrightarrow S_{21} V_1 \otimes \wedge^3 V_2^* \otimes A(-3) \xrightarrow{d_2} \wedge^2 V_1 \otimes \wedge^2 V_2^* \otimes A(-2) \xrightarrow{d_1} A \longrightarrow 0 \]

\[ d_1 = (x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{21}x_{32} - x_{22}x_{31}) \]

\[ d_2 = \begin{pmatrix} x_{31} & -x_{32} \\ -x_{21} & -x_{22} \\ x_{11} & x_{12} \end{pmatrix} \]
Non-equioriented $A_3$

$$V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha$$

$$\beta_i = \dim R_i$$

$$Z \subset \text{Rep}(Q, d) \times \prod_{i=1}^{3} \text{Eng}(\beta_i, V_i)$$
Non-equioriented $A_3$

$$V = \bigoplus_{\alpha \in R^+} X_\alpha$$

$$d = (3, 4, 3)$$

$$\beta = (2, 1, 2)$$

$$\xi = \mathcal{R}_1 \otimes Q_2^* \oplus \mathcal{R}_3 \otimes Q_2^*$$
A_4 (source-sink)

\[ V = \bigoplus_{\alpha \in R^+} m_{\alpha} X_{\alpha} \]

\[ Z \subset \text{Rep}(Q, d) \times \prod_{i=1}^{4} \text{Flag}(\beta_i^2, \beta_i^1, V_i) \]
\[ V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha \]

\[ \xi = (\mathcal{R}_2^1 \otimes Q_1^{1*} + \mathcal{R}_2^2 \otimes Q_1^{2*}) \oplus (\mathcal{R}_2^1 \otimes Q_3^{1*} + \mathcal{R}_2^2 \otimes Q_3^{2*}) \oplus (\mathcal{R}_4^1 \otimes Q_3^{1*} + \mathcal{R}_4^2 \otimes Q_3^{2*}) \]

\[ Z: \quad V_1 \leftarrow V_2 \rightarrow V_3 \leftarrow V_4 \]
\[ U \quad U \quad U \quad U \quad U \]
\[ R_1 \leftarrow R_2 \rightarrow R_3 \leftarrow R_4 \]
\[ U \quad U \quad U \quad U \]
\[ S_1 \leftarrow S_2 \rightarrow S_3 \leftarrow S_4 \]
\[ \xi = \bigoplus_{a \in Q_1} \left( \bigoplus_{t=1}^s \mathcal{R}_t^{ta} \otimes Q_t^{ha*} \right) \]

\[ \mathcal{R}_t^{ta} \otimes Q_t^{ha*} \]

\[ \mathcal{R}_t^{ta} \otimes Q_t^{ha*} \]

\[ \bigwedge^t \xi = \bigoplus_{\sum a \in Q_1 |\lambda(a)| = t} S_{\lambda(a)} \mathcal{R}_t^{ta} \otimes S_{\lambda(a)'} Q_t^{ha*} \]

Assign weights to each summand.
Apply Bott’s theorem to these weights to calculate

\[ F_i = \bigoplus_{j \geq 0} H^j(\mathcal{V}, \bigwedge \xi) \otimes_K A(-i - j), \quad j = \# \text{exchanges} \]

\[ D(\lambda) := \sum_{a \in Q_1} |\lambda(a)| - j = (i + j) - j = i. \]

Rewrite

\[ F_{D(\lambda)} = \bigoplus_{j \geq 0} H^j(\mathcal{V}, \bigwedge \xi) \otimes_K A(-i - j) \]
Non-equioriented $A_3$

$$V_1 \longrightarrow V_3 \leftarrow V_2$$

$$F_{D(\lambda)} = \bigoplus_{|\lambda|+|\mu|=t} c_{\lambda,\mu}^{\nu} \left( S_{\lambda} V_1 \otimes S_{\mu} V_2 \otimes S_{\nu} V_3^{*} \right)$$

**Theorem (-)**

- $D(\lambda) \geq E_Q$
- $F_0 = A$ and $F_i = 0$ for $i < 0$. Thus $\overline{O}_V$ is normal and has rational singularities.
Theorem(-)

- Minimal generators of the defining ideal:

Let $p = \text{rank}(\phi)$, $q = \text{rank}(\psi)$, $r = \text{rank}(\phi + \psi)$, the minimal generators of $\overline{O_V}$ are

- $(p + 1) \times (p + 1)$ minors of $\phi$,
- $(q + 1) \times (q + 1)$ minors of $\psi$,
- $(r + 1) \times (r + 1)$ minors of $\phi + \psi$, taken by choosing $b + k + 1$ columns of $\phi$ and $c + l + 1$ columns of $\psi$ such that $k + l = d - 1$. ($b, c, d$ are multiplicities of some indecomposable representations)

- Characterization of Gorenstein orbits.
\[ V = \bigoplus_{\alpha \in R^+} X_\alpha \]

\[ \underline{d} = (3, 4, 3) \]

\[ \xi = \mathcal{R}_1 \otimes Q_2^* \oplus \mathcal{R}_3 \otimes Q_2^* \]

\[ A = \text{Sym}(V_1 \otimes V_3^*) \otimes \text{Sym}(V_2 \otimes V_3^*) \]
\[A\]

\[\uparrow\]

\[(\wedge^3 V_1 \otimes \wedge^3 V_3^* \otimes A(-3)) \oplus (\wedge^3 V_2 \otimes \wedge^3 V_3^* \otimes A(-3)) \oplus (\wedge^2 V_1 \otimes \wedge^2 V_2 \otimes \wedge^4 V_3^* \otimes A(-4))\]

\[\uparrow\]

\[(\wedge^3 V_1 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus (\wedge^3 V_2 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus (\wedge^3 V_1 \otimes \wedge^2 V_2^* \otimes S_{2111} V_3^* \otimes A(-5)) \oplus (\wedge^2 V_1 \otimes \wedge^3 V_2 \otimes S_{2111} V_3^* \otimes A(-5)) \oplus (\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{222} V_3^* \otimes A(-6))\]

\[\uparrow\]

\[(\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{2221} V_3^* \otimes A(-7)) \oplus (\wedge^3 V_1 \otimes S_{2111} V_2 \otimes S_{2221} V_3^* \otimes A(-7)) \oplus (\wedge^2 V_1 \otimes S_{222} V_2^* \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (S_{222} V_1 \otimes \wedge^2 V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{3111} V_3^* \otimes A(-6))\]

\[\uparrow\]

\[(S_{211} V_1 \otimes S_{211} V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (S_{222} V_1 \otimes \wedge^3 V_2 \otimes S_{3222} V_3^* \otimes A(-9)) \oplus (\wedge^3 V_1 \otimes S_{222} V_2 \otimes S_{3222} V_3^* \otimes A(-9))\]

\[\uparrow\]

\[(S_{222} V_1 \otimes S_{222} V_2 \otimes S_{3333} V_3^* \otimes A(-12))\]
Source-sink quivers

Theorem(-)

$Q$ be a source-sink quiver and $V$ be a representation of $Q$ such that the orbit closure $\overline{O}_V$ admits a 1-step desingularization $Z$. Then

$$D(\lambda) \geq E_Q$$

Corollary

If $Q$ is Dynkin then $\overline{O}_V$ is normal and has rational singularities.

Orbit closures of type $E_6$, $E_7$ and $E_8$ admitting a 1-step desingularization are normal!

Corollary

If $Q$ is extended Dynkin then $F_\bullet$ is a minimal free resolution of the normalization of $\overline{O}_V$. 
Equioriented $A_n$

**Theorem(-)**

Let $Q$ be an equioriented quiver of type $A_n$ and $\beta, \gamma$ be two dimension vectors. Let $\alpha = \beta + \gamma$ and $V \in \text{Rep}(Q, \alpha)$. Then

- $D(\lambda) \geq E_Q$
- $F(\beta, \gamma)_i = 0$ for $i < 0$.
- $F(\beta, \gamma)_0 = A$.
- The summands of $F(\beta, \gamma)_1$ are of the form $\bigwedge^{\gamma_i + \beta_j + 1} V_i \otimes \bigwedge^{\gamma_i + \beta_j + 1} V_j^*$
  where $1 \leq i < j \leq n$. 
Thank you!