WEYL AND ZARISKI CHAMBERS ON PROJECTIVE SURFACES

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Abstract. Let X be a nonsingular complex projective surface. The Weyl and Zariski chambers give two interesting decompositions of the big cone of X. Following the ideas of [2] and [10], we study these two decompositions and determine when a Weyl chamber is contained in the interior of a Zariski chamber and vice versa. We also determine when a Weyl chamber can intersect non-trivially with a Zariski chamber.

1. Introduction

Let X be a nonsingular projective surface. A divisor D on X is called *big* if the global sections of mD determine a birational morphism for $m \gg 0$. The classes of all big divisors in the Néron-Severi space $N^1(X)_{\mathbb{R}}$ form a cone called the *big cone* of X. The big cone is open and contains the ample cone of X. Studying the big cone of X is useful in understanding the geometry of X. The intersection of the big and nef cones is particularly interesting.

A useful way to study the big cone is to decompose it into chambers and study individual chambers. One instance of such a decomposition is given by *Zariski chambers*. These have been defined in [4] as the set of big divisors for which the negative part of the Zariski decomposition is constant. Further, the Zariski chambers are rational locally polyhedral subcones and the big cone admits a locally finite decomposition into Zariski chambers. This decomposition turns out to have significant geometric implications. On each of the Zariski chambers, the volume function of the divisors is given by a single quadratic polynomial. Moreover, the stable base loci are constant in the interior of each Zariski chamber. See [4] for more details.

It is thus an interesting problem to study the Zariski chamber decomposition of the big cone of a given surface. For example, one can ask for the number of chambers in the decomposition. This question has been answered for Del Pezzo surfaces and some other surfaces with higher Picard number in [3] and [5]. The number of Zariski chamber is an interesting geometric invariant of X.

Another interesting decomposition of the big cone is given by *Weyl chambers*. Traditionally the (simple) Weyl chambers are defined on X if the only irreducible and reduced curves of negative self-intersection are (-2) curves. In this situation, [2] describes the the Weyl chamber decomposition for K3 surfaces. But in [10], the authors extend the definition of Weyl chambers to arbitrary surfaces.

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Since the Zariski and Weyl chambers are defined for any nonsingular surface, it is interesting to study how they are related to each other. The Zariski chambers are, in general, neither open nor closed, but the Weyl chambers are always open. So it is natural to ask when the interior of a Zariski chamber coincides with a Weyl chamber. This problem was studied for K3 surfaces in [2] and the authors establish a necessary and sufficient condition for the interior of every Zariski chamber to coincide with a Weyl chamber; see [2, Theorem 2.2]. This result is generalized to an arbitrary nonsingular surface in [10, Theorem 3].

In this note, we continue the comparison of Zariski and Weyl chambers started by [2] and [10]. In Section 2, we prove that there is a bijection between the set of Zariski chambers and the set of Weyl chambers on any nonsingular projective surface (Theorem 2.5). This generalizes [2, Theorem 1.3], which proved the bijection for K3 surfaces.

Our first main results (Theorems 3.1 and 3.2) give necessary and sufficient conditions for a *specific* Weyl chamber to be contained in a Zariski chamber and for the interior of a *specific* Zariski chamber to be contained in a Weyl chamber. We note that this result has been obtained in [2] for K3 surfaces and our proofs are similar to the proofs of [2]. We also show that our main results imply the main theorem of [10]. In another result (Theorem 3.6), we give a necessary and sufficient condition for a Zariski chamber and a Weyl chamber to have non-empty intersection. At the end, we give some examples which illustrate our results.

In Section 2, we recall the definitions of Weyl and Zariski chambers and their basic properties before proving some new results. In Section 3, we prove our main results about comparing the Weyl and Zariski chambers and give some examples.

Throughout this note, we work over the field $\mathbb{C}$ of complex numbers. If X is a nonsingular projective surface, a *negative curve* on X is an irreducible and reduced curve C satisfying $C^2 < 0$.

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2. Preliminaries

We start by recalling the main objects that we study.

2.1. **Big cone.** A line bundle L on an irreducible projective variety X is *big* if the mapping $\phi_m : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$, defined by $L^{\otimes m}$ is birational onto its image for $m \gg 0$. A Cartier divisor D on X is big if $\mathcal{O}_X(D)$ is big. Big divisors are precisely the divisors whose volume is positive. Another characterization of big divisors is given by the following: a divisor D is big if and only of mD is numerically equivalent to $A + E$ for an ample divisor A and an effective divisor E and some positive integer m. In particular, bigness is a property of the numerical class of D.

The notion of bigness can be extended to $\mathbb{R}$-divisors. An $\mathbb{R}$-divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is big if it can be written in the form $D = \sum a_i D_i$, where each D_i is a big integral divisor and a_i is a positive real number. Since bigness is preserved under numerical equivalence, we can talk about big classes in the real Néron Severi group $N^1(X)_{\mathbb{R}}$. The convex cone of all big $\mathbb{R}$-divisor classes in $N^1(X)_{\mathbb{R}}$ is called the *big cone* and it is denoted by $\mathrm{Big}(X) \subset N^1(X)_{\mathbb{R}}$. For further details on big divisors and the big cone, see [9].

2.2. **Zariski Decomposition.** Zariski [11] showed that any effective divisor on a surface can be written uniquely as a sum of a nef divisor and some negative curves. This has been generalized to a larger class of divisors than effective divisors and is known as the *Zariski decomposition*; see [7, 1].

The closure of the big cone is called the *pseudoeffective cone* and divisors in that cone are called pseudoeffective. Let D be a pseudoeffective $\mathbb{R}$-divisor on a nonsingular complex projective surface X. Then there exist $\mathbb{R}$-divisors P_D and N_D such that

$$D = P_D + N_D,$$

and the following conditions hold:

(1) P_D is nef,
(2) either $N_D = 0$ or $N_D = \sum_{i=1}^{r} a_i E_i$, where $a_i > 0$ and the intersection matrix $(E_i \cdot E_j)_{1 \leq i,j \leq r}$ is negative definite, and
(3) P_D is orthogonal to each of the components of N, i.e., $P_D \cdot E_i = 0$, for $i = 1, \cdots, r$.

P_D and N_D are called the *positive part* and the *negative part* of D respectively. They are are uniquely determined by D.

It follows from the property (2) of the definition that the negative part of the big divisor is either trivial or is supported by negative curves.

2.3. **Zariski chambers.** The variation of the Zariski decomposition over the big cone $\mathrm{Big}(X)$ leads to a partition of the big cone, given by subcones for which the negative part of the Zariski decomposition is constant. Each such subcone is called a *Zariski chamber*. A formal definition of a Zariski chamber is given below.

Let D be a big divisor and let $D = P_D + N_D$ be the Zariski decomposition of D. We first define two sets of curves associated to D as follows:

$$\mathrm{Null}(D) = \{C \subset X \mid C \text{ is an irreducible curve with } D \cdot C = 0\}, \text{ and}$$

$$\mathrm{Neg}(D) = \{C \subset X \mid C \text{ an irreducible component of } N_D\}.$$

Clearly, $\mathrm{Neg}(D) \subseteq \mathrm{Null}(P_D)$. It may happen that $\mathrm{Neg}(D) \neq \mathrm{Null}(P_D)$.

Suppose now that P is a big and nef divisor. The *Zariski chamber* $\sum_P$ associated to P is defined as

$$\textstyle\sum_P = \Big\{D \in \mathrm{Big}(X) \mid \mathrm{Neg}(D) = \mathrm{Null}(P)\Big\}.$$

The interior of $\sum_P$ is given by

$$\mathrm{int}(\textstyle\sum_P) = \{D \in \mathrm{Big}(X) \mid \mathrm{Neg}(D) = \mathrm{Null}(P) = \mathrm{Null}(P_D)\}.$$

The above result is proved in [4, Proposition 1.8]. It is also known that the sets $\sum_P$ cover $\mathrm{Big}(X)$; see [4, Proposition 1.6].

If H is an ample divisor, then the interior of the chamber $\sum_H$ is the ample cone and its closure is the nef cone. The Zariski chamber $\sum_H$ for any ample divisor H is called the *nef chamber*.

We recall the following useful observation.

Lemma 2.1. [3, Lemma 1.1] *The set of Zariski chambers on a smooth projective surface X that are different from the nef chamber is in bijective correspondence with the collection of sets of reduced divisors on X whose intersection matrix is negative definite.*

We define the following two sets.

$$\mathcal{I}(X) = \left\{ C \subset X \;\middle|\; \begin{array}{l} C \text{ is an irreducible and reduced} \\ \text{curve satisfying } C^2 < 0 \end{array} \right\}, \text{ and}$$

$$\mathcal{Z}(X) = \left\{ S \subseteq \mathcal{I}(X) \;\middle|\; \begin{array}{l} S \text{ is finite and the intersection} \\ \text{matrix of } S \text{ is negative definite} \end{array} \right\}.$$

The following remark is easy to prove and we use it repeatedly in our arguments: if $S \in \mathcal{Z}(X)$ and $S' \subset S$, then $S' \in \mathcal{Z}(X)$.

To every $S \in \mathcal{Z}(X)$, we associate a set of big divisors as follows:

$$Z_S = \{D \in \mathrm{Big}(X) \mid \mathrm{Neg}(D) = S\}.$$

In words, Z_S consists precisely of those big divisors whose negative part is supported on S.

Then Z_S are precisely the Zariski chambers. Indeed, let P be a big and nef divisor. We note that $\mathrm{Null}(P) \in \mathcal{Z}(X)$: since $\mathrm{Null}(P) = \mathrm{Null}(mP)$, for any positive integer m, we may suppose that $P = A + E$, where A is an ample divisor and E is an effective divisor. If $C \in \mathrm{Null}(P)$, then $N_E \cdot C < 0$. So C is a component of N_E. Since the irreducible components of N_E have a negative definite intersection matrix, so will the elements of $\mathrm{Null}(P)$. Now let $S := \mathrm{Null}(P)$. Then we get

$$\textstyle\sum_P = \{D \in \mathrm{Big}(X) \mid \mathrm{Neg}(D) = S\} = Z_S.$$

Conversely, given $S \in \mathcal{Z}(X)$, it can be shown that there exists a big and nef divisor P such that $\mathrm{Null}(P) = S$, so that $\sum_P = Z_S$.

By Lemma 2.1, it follows that the set of Zariski chambers is in bijective correspondence with the elements of $\mathcal{Z}(X) \cup \emptyset$. Note that $Z_\emptyset$ is precisely the nef chamber in the big cone of X. For $S, S' \in \mathcal{Z}(X)$, the Zariski chambers Z_S and $Z_{S'}$ are either disjoint or identical.

The following useful remark will be used in the proofs.

Remark 2.2. Let $C_1, \cdots, C_r \in \mathcal{I}(X)$ and let P be a nef divisor such that $P + c_1C_1 + \cdots + c_rC_r \in \mathrm{Big}(X)$ for some positive real numbers $c_1, \cdots, c_r$. Then $P + b_1C_1 + \cdots + b_rC_r \in \mathrm{Big}(X)$ for any positive real numbers $b_1, \cdots, b_r$. We give a brief argument for this fact below.

First, note that $P, C_1, \cdots, C_r$ belong to the closure of the big cone, since the closure K of the big cone is the closure of the cone of effective curves which contains the nef cone. Let $b_1, \cdots, b_r$ be any positive real numbers. If $P + b_1C_1 + \cdots + b_rC_r \notin \mathrm{Big}(X)$, then $P + b_1C_1 + \cdots + b_rC_r$ is contained in the boundary of K. Note that the interior of K is precisely $\mathrm{Big}(X)$ and it is disjoint with the boundary of K. If $P + b_1C_1 + \cdots + b_rC_r$ is contained in the boundary of K, then it spans a ray R in the boundary. This means that $P, C_1, \cdots, C_r \in R$ which in turn means that $P + c_1C_1 + \cdots + c_rC_r$ is contained in the boundary of K. This contradicts the hypothesis that $P + c_1C_1 + \cdots + c_rC_r$ is a big divisor.

For further details about the Zariski chambers, see [4].

2.4. **Weyl chambers.** Each curve $C \in \mathcal{I}(X)$ defines a hyperplane in the Néron-Severi space $N^1(X)_{\mathbb{R}}$ of X as follows:

$$C^{\perp} = \{D \in N^1(X)_{\mathbb{R}} \mid D \cdot C = 0\}.$$

These hyperplanes give a decomposition of the big cone $\mathrm{Big}(X)$. Namely, we consider the connected components of

$$\mathrm{Big}(X) \setminus \bigcup_{C \in \mathcal{I}(X)} C^{\perp}.$$

The connected components of the set $\mathrm{Big}(X) \setminus \bigcup_{C \in \mathcal{I}(X)} C^{\perp}$ are called the simple *Weyl chambers* of X.

Traditionally, the (simple) Weyl chambers are studied only if all the negative curves on X are (-2)-curves, i.e., smooth rational curves C satisfying $C^2 = -2$; see [2]. But in [10], authors studied Weyl chambers on arbitrary surfaces. In this note, we adopt the same approach. First we give a convenient alternate definition of Weyl chambers. This characterization was used in [2] for K3 surfaces.

Let $S \in \mathcal{Z}(X) \cup \{\emptyset\}$. Corresponding to S, we define the set W_S as follows:

$$W_S = \{D \in \mathrm{Big}(X) \mid D.C < 0 \ \forall C \in S \text{ and } D.C > 0 \ \forall C \in \mathcal{I}(X) \setminus S\}.$$

We show below that W_S are precisely the Weyl chambers of X. Note that $W_{\{\emptyset\}}$ is the ample cone in $\mathrm{Big}(X)$. From the definition, it is clear that each W_S is a convex set, hence connected.

We first prove the following lemma.

Lemma 2.3. *The following properties hold:*

(1) W_S is an open set for every $S \in \mathcal{Z}(X) \cup \{\emptyset\}$.
(2) For $S, S' \in \mathcal{Z}(X) \cup \{\emptyset\}$, $W_{S'} \cap W_S = \emptyset$ if and only if $S' \neq S$.

Proof. (1) To show W_S is open, it is enough to show that for any $D \in W_S$, and for any $E \in N^1(X)_{\mathbb{R}}$, $mD \pm E \in W_S$ for all sufficiently large integers m.

As the big cone is open, there exists an integer $m \gg 0$ such that $mD \pm E \in \mathrm{Big}(X)$.

For $C \in S$, we need to ensure $(mD \pm E) \cdot C < 0$. Since $D \cdot C < 0$ and the set of numbers $\{E \cdot C \mid C \in S\}$ is finite, we can choose a bigger m (if necessary) such that $(mD \pm E) \cdot C < 0$.

For $C \in \mathcal{I}(X) \setminus S$, we need to ensure that $(mD \pm E) \cdot C > 0$.

There exist only finitely many $C \in \big(\mathcal{I}(X) \setminus S\big) \cap \mathrm{Neg}(mD \pm E)$ such that $(mD \pm E) \cdot C < 0$. Then we can take large enough n such that $((mD \pm E) + nD) \cdot C > 0$ for all such C, as $D \cdot C > 0$.

This proves (1).

(2) Clearly, $W_{S'} \cap W_S = \emptyset$ implies $S' \neq S$.

For the converse, suppose that $S' \neq S$. We may assume, without loss of generality, that $S' \setminus S \neq \emptyset$. Let $C \in S' \setminus S$. Then $C \cdot D' < 0$, $\forall D \in W_{S'}$. Hence $D \notin W_S$. This proves (2). ◻

Now we show that W_S are precisely the Weyl chambers.

Lemma 2.4. *Connected components of Big$(X) \setminus \bigcup\limits_{C \in \mathcal{I}(X)} C^\perp$ are exactly W_S .*

Proof. Let $\text{Big}(X) \setminus \bigcup_{C \in \mathcal{I}(X)} C^\perp = \cup X_i$, where X_i are the connected components. Let us consider a big divisor D such that $D \in X_i$. We define the set $S_D = \{C \in \mathcal{I}(X) | D \cdot C < 0\}$. Note that S_D can be the empty set. Since $S_D \subset \text{Neg}(D)$, we have $S_D \in \mathcal{Z}(X)$.

We claim that $S_D = S_{D'}$ for any two $D, D' \in X_i$. Assume that the claim is not true. Then with out loss of generality, we can assume that there exists $C \in S_{D'}$, but $C \notin S_D$, i.e., $C \cdot D > 0$ but $C \cdot D' < 0$.

We define a map $f_C : N^1(X)_{\mathbb{R}} \to \mathbb{R}$ by $f_C(L) = C \cdot L$. It is easy to see that f_C is continuous. So $f_C(X_i)$ is a connected subset of $\mathbb{R}$. Note that $f_C(D) > 0$, and $f_C(D') < 0$. Thus there exists $D'' \in X_i$ such that $f_C(D'') = 0$. This contradicts the fact that X_i is in the complement of the set $\bigcup_{C \in \mathcal{I}(X)} C^\perp$. Hence, our assumption is wrong and the claim is proved. So $X_i = W_S$, where $S = S_D$, for any $D \in X_i$. ◻

As a consequence of the above two lemmas, we prove the following theorem. This result generalizes [2, Theorem 1.3] which proves the same statement for K3 surfaces.

Theorem 2.5. *Let X be a smooth projective variety. Then there is a bijection between the set of Zariski chambers in Big(X) and the set of Weyl chambers in Big(X).*

Proof. From Lemma 2.1, it follows that there is a bijective correspondence between the set of Zariski chambers and $\mathcal{Z}(X) \cup \{\emptyset\}$. From Lemma 2.4, it follows that there is a bijection between set of Weyl chambers and $\mathcal{Z}(X) \cup \{\emptyset\}$. ◻

We use the following result (see [2, Lemma A.1]) at several places in subsequent sections.

Lemma 2.6. *Let $S = (s_{i,j})$ be a negative definite matrix over $\mathbb{R}$ such that $s_{i,j} \geq 0$ for $i \neq j$. Then all the entries of the inverse matrix S^{-1} are less than or equal to 0.*

3. Main results

We now give necessary and sufficient conditions for a Weyl chamber to be contained in the interior of a Zariski chamber and vice versa. These results have been proved for K3 surfaces in [2]. We generalize them to an arbitrary surface. Our proofs are similar to the proofs given in [2].

Theorem 3.1. *Let X be a nonsingular projective surface and let $S \in \mathcal{Z}(X)$. We have $W_S \subseteq Z_S$ if and only if the following condition holds:*

If C' is a curve such that $C' \in \mathcal{I}(X) \setminus S$ and $S \cup \{C'\} \in \mathcal{Z}(X)$, then $C' \cdot C = 0$ $\forall C \in S$.

Proof. Suppose first that the given condition holds. We will show that $W_S \subseteq Z_S$. It suffices to show that $\mathrm{Neg}(D) = S$ for any $D \in W_S$.

Let $S = \{C_1, \dots, C_r\}$ and $D \in W_S$. By the definition of W_S, we have $D \cdot C_i < 0$ for $i = 1, 2, \dots, r$. Let $D = P_D + N_D$ be the Zariski decomposition of D. As P_D is the nef divisor and C_i's are irreducible curves, $P_D \cdot C_i \geq 0$. So $N_D \cdot C_i < 0$ $\forall i$. This implies each C_i is a component of N_D. Hence $S \subseteq \mathrm{Neg}(D)$.

We will now prove that $\mathrm{Neg}(D) \subseteq S$. Suppose not. Then $\mathrm{Neg}(D)$=$\cup_{i=1}^{r} C_i \cup_{j=1}^{s} C'_j$ with $C'_j \notin S$, $\forall j$ and $s > 0$. Let $N_D = \sum\limits_{i=1}^{r} a_i C_i + \sum\limits_{j=1}^{s} a'_j C'_j$.

Then by the given condition, we have $C_i \cdot C'_j = 0, \forall 1 \leq i \leq r, 1 \leq j \leq s$. As $D \in W_S$ and $C'_j \notin S$, we have $D \cdot C'_j > 0$. Let $b_j := D \cdot C'_j = N_D \cdot C'_j > 0$. This implies that $\sum\limits_{i=1}^{s} a'_i C'_i \cdot C'_j = b_j$ for $1 \leq j \leq s$.

Re-writing the above s equations in matrix form, we get $(a'_1, \cdots, a'_s)A = (b_1, \cdots, b_s)$, where A is the intersection matrix of $\{C'_1, \dots, C'_s\} \in \mathcal{Z}(X)$. This gives $(a'_1, \cdots, a'_s) = (b_1, \cdots, b_s)A^{-1}$. Note that A is a negative-definite matrix, and hence invertible. Moreover, by Lemma 2.6, all the entries of A^{-1} are non-positive. It follows that $a'_j < 0$ for all j. But this contradicts the fact that N_D is an effective divisor and C'_i are components of N_D. Thus $\mathrm{Neg}(D) = S$ and hence $W_S \subseteq Z_S$.

To prove the other direction of the theorem, suppose that $S = \{C_1, \dots, C_r\} \in \mathcal{Z}(X)$ does not satisfy the given condition. Let $C' \in \mathcal{Z}(X) \setminus S$ be such that $S \cup \{C'\} \in \mathcal{Z}(X)$ and $C' \cdot C > 0$ for some $C \in S$. We will show that $W_S \nsubseteq Z_S$, by exhibiting a divisor $D \in W_S$ which is not in Z_S.

The intersection matrix of the set $S \cup \{C'\}$ is negative definite. By Lemma 2.1, there exists a big divisor D'' such that $\mathrm{Neg}(D'') = S \cup \{C'\}$. In fact, there is a bijective correspondence between the Zariski chambers (different from the nef chamber) and the elements of $\mathcal{Z}(X)$. So we may choose a big divisor D'' in the interior of the Zariski chamber corresponding to $S \cup \{C'\}$. By [4, Proposition 1.8], we have $\mathrm{Null}(P_{D''}) = \mathrm{Neg}(D'') = S \cup \{C'\}$.

We will first construct a big divisor $D' = P_{D'} + c'C' + \sum\limits_{i=1}^{r} c_i C_i$, where $N_{D'} = c'C' + \sum\limits_{i=1}^{r} c_i C_i$ and $c', c_1, \dots, c_r$ are positive real numbers, such that $D' \cdot C' = N_{D'} \cdot C' < 0$ and $D' \cdot C_i = N_{D'} \cdot C_i < 0$, for all $i = 1, 2, \cdots, r$.

Let $b', b_1, \cdots, b_r$ be any negative real numbers. Consider the following system of $r + 1$ linear equations in $r + 1$ variables.

$$\begin{aligned} N_{D'} \cdot C' &= b' < 0, \\ N_{D'} \cdot C_i &= b_i < 0, i = 1, 2, \cdots, r. \end{aligned}$$

Since the intersection matrix of $N_{D'}$ is negative definite, there exists a unique solution $(c', c_1, \dots, c_r)$ for the above system. Moreover, by Lemma 2.6, $c', c_1, \cdots, c_r$ are positive real numbers. So we have a big divisor D' with the desired properties. Note that $P_{D'} = P_{D''}$.

Now we consider the divisor

$$D = P_{D'} + \frac{min\{c', c_i\}}{2|C'^2|}C' + \sum_{i=1}^{r} c_i C_i.$$

Then D is a big divisor and its Zariski decomposition is given by this above decomposition. We claim $D \in W_S$, but $D \notin Z_S$.

Since $\text{Neg}(D) \neq S$, clearly $D \notin Z_S$.

To prove that $D \in W_S$, we compute the intersection number $D \cdot C$ for all negative curves $C \in \mathcal{I}(X)$.

First, let $C \in S$. Then $D \cdot C = \frac{min\{c',c_i\}}{2|C'^2|}C' \cdot C + \sum_{i=1}^{r} c_i C_i \cdot C \leq N_{D'} \cdot C < 0$.

We have $D \cdot C' = \frac{min\{c',c_i\}}{2|C'^2|}C'^2 + \sum_{i=1}^{r} c_i C_i \cdot C' = \frac{-\min\{c',c_i\}}{2} + \sum_{i=1}^{r} c_i C_i \cdot C' > 0$, as $C' \cdot C_i > 0$ for some i.

Now let $C \notin S \cup \{C'\}$. Then $P_D \cdot C > 0$, since D'' is the interior of $Z_{S \cup \{C'\}}$ and $P_D = P_{D''}$. So $D \cdot C = P_D \cdot C + N_D \cdot C > 0$.

This proves that $D \in W_S$ and completes the proof the theorem. □

We will now give a criterion for the interior of a Zariski chamber to be contained in a Weyl chamber.

Theorem 3.2. *Let X be a smooth projective surface and let $S \in \mathcal{Z}(X)$. Let $\mathring{Z}_S$ be the interior of the Zariski chamber Z_S. Then we have $\mathring{Z}_S \subset W_S$ if and only if $C' \cdot C'' = 0$ for all curves $C', C'' \in S$.*

Proof. First, consider a set $S = \{C_1, \cdots, C_r\} \in \mathcal{Z}(X)$ satisfying the given condition. We will show that $\mathring{Z}_S \subset W_S$. Let $D \in \mathring{Z}_S$.

Let the Zariski decomposition of D be $D = P_D + N_D$, where $N_D = \sum_{i=1}^{r} a_i C_i$. Then $D \cdot C_i = N_D \cdot C_i = a_i C_i^2 < 0$, for all $i = 1, \cdots, r$. Let C be a negative curve on X which is not in S. Then $D \cdot C \geq 0$. If $D \cdot C = 0$, then $D \cdot C = P_D \cdot C + N_D \cdot C = 0$, which implies that $P_D \cdot C = N_D \cdot C = 0$. Hence, $C \in \text{Null}(P_D)$. This shows that $\text{Neg}(D) \neq \text{Null}(P_D)$. But since D is in the interior of the Zariski Chamber Z_S, we have Neg(D)=Null(P_D), by[4, Proposition 1.8]. Hence we must have $D \cdot C > 0$. So $\mathring{Z}_S \subset W_S$.

To prove the other direction of the theorem, consider $S \in \mathcal{Z}(X)$ such that there exist curves $C_i, C_j \in S$ satisfying $C_i \cdot C_j \neq 0$. We will show that there exists a big divisor $D \in \mathring{Z}_S$ such that $D \notin W_S$. Let $S = \{C_1, ..., C_r\}$ and suppose that $C_1 \cdot C_2 \neq 0$.

Fix an ample divisor H. For some unknown positive real numbers $a_1, \cdots, a_r, a_1^\star, \cdots, a_r^\star$, consider the following divisor:

$$D' = H + \sum_{i=1}^{r} a_i^\star C_i + \sum_{i=1}^{r} (a_i - a_i^\star) C_i,$$

where $P_{D'} = H + \sum_{i=1}^{r} a_i^\star C_i$ and $N_{D'} = \sum_{i=1}^{r}(a_i - a_i^\star)C_i$. Note that D' is a big divisor, being the sum of an ample divisor and an effective divisor.

Now we want to find values of $a_i, a_i^\star$, such that Neg(D') =Null$(P_{D'})$. To obtain this, we solve the following system of r linear equations in r variables:

$$0 = P_{D'} \cdot C_j = H \cdot C_j + \sum_{i=1}^{r} a_i^\star C_i.C_j.$$

This system has a unique solution $(a_1^\star, \cdots, a_r^\star)$ with $a_i^\star > 0$ for all $1 \le i \le r$. Choose real numbers a_i such that $0 < a_i^\star < a_i$ for all i. Finally, choose a real number k such that $0 < k < \frac{(a_2 - a_2^\star)}{|C_1^2|}$ and consider the divisor:

$$D = H + (a_1^\star + k)C_1 + \sum_{i=2}^{r} a_i C_i.$$

Then the Zariski decomposition of D is given by $D = P_D + N_D$, where $P_D = P_{D'} = H + \sum_{i=1}^{r} a_i^\star C_i$ and $N_D = kC_1 + \sum_{i=2}^{r}(a_i - a_i^\star)C_i$. As Neg$(D)$ = Null(P_D) = S, we have $D \in \mathring{Z}_S$. But $D \cdot C_1 = kC_1^2 + \sum_{i=2}^{r}(a_i - a_i^\star)C_i.C_1 \ge kC_1^2 + (a_2 - a_2^\star) > 0$, by the choice of k. So $D \notin W_S$.

This completes the proof. □

We recall the following result in [10] which determines when the interior of each Zariski chamber coincides with a Weyl chamber.

Theorem 3.3. [10, Theorem 3] *Let X be a smooth projective surface. The following conditions are equivalent:*

(a) the interior of each Zariski chamber on X coincides with a simple Weyl chamber,

(b) if two different irreducible negative curves $C_1 \neq C_2$ on X meet (i.e., $C_1 \cdot C_2 > 0$), then

$$C_1 \cdot C_2 \ge \sqrt{C_1^2 \cdot C_2^2}.$$

If the above equivalent conditions hold on a surface X, then [10] says that the Zariski chambers of X are *numerically determined.*

Theorems 3.1 and 3.2, determine when a specific Weyl chamber is contained in a Zariski chamber and when the interior of a specific Zariski chamber is contained in a Weyl chamber. Our results imply Theorem 3.3, as we show below.

We first give some equivalent formulations of condition (b) of Theorem 3.3.

Theorem 3.4. *Let X be a smooth projective surface. The following are equivalent.*

(1) If two irreducible negative curves $C_1 \neq C_2$ on X meet (i.e., $C_1 \cdot C_2 > 0$), then $C_1 \cdot C_2 \ge \sqrt{(C_1^2 \cdot C_2^2)}$.

(2) If C_1 and C_2 are any two negative curves such that $\{C_1, C_2\} \in \mathcal{Z}(X)$, then $C_1 \cdot C_2 = 0$.

(3) Let $S \in \mathcal{Z}(X)$. If $C' \in \mathcal{I}(X) \setminus S$, and $S \cup C' \in \mathcal{Z}(X)$, then $C' \cdot C = 0$ for all curves $C \in S$.

(4) Let $S \in \mathcal{Z}(X)$. Then $C_1 \cdot C_2 = 0$ for all curves $C_1, C_2 \in S$.

Proof. (1)$\Rightarrow$ (2): Assume that (2) does not hold. Let $S = \{C_1, C_2\} \in \mathcal{Z}(X)$ but $C_1 \cdot C_2 > 0$. As the intersection matrix of $S = \{C_1, C_2\}$ is negative definite, we have,

$$x^2C_1^2 + y^2C_2^2 + 2xyC_1 \cdot C_2 < 0$$

for any $(x, y) \neq (0, 0)$. For any $(x, y) \neq (0, 0)$, we have

$$C_1 \cdot C_2 < \frac{(x\sqrt{-C_1^2})^2 + (y\sqrt{-C_2^2})^2}{2xy\sqrt{C_1^2C_2^2}}\sqrt{C_1^2C_2^2}$$

Take $x = \sqrt{-C_2^2}$ and $y = \sqrt{-C_1^2}$. Then $C_1C_2 < \sqrt{C_1^2C_2^2}$. This violates (1).

(2)$\Rightarrow$ (1): Let C_1, C_2 be two different negative curves. If the intersection matrix of $S = \{C_1, C_2\}$ is negative definite, then $C_1 \cdot C_2 = 0$, by (2). So (1) clearly holds. If the intersection matrix of S is not negative definite, then there exists a tuple (x, y), where x, y are both nonzero real numbers having the same sign, such that

$$x^2C_1^2 + y^2C_2^2 + 2xyC_1 \cdot C_2 \geq 0$$

This implies,

$$C_1 \cdot C_2 \geq \frac{(x\sqrt{-C_1^2})^2 + (y\sqrt{-C_2^2})^2}{2xy\sqrt{C_1^2C_2^2}}\sqrt{C_1^2C_2^2}$$

If a, b are positive integers, then $(a - b)^2 \geq 0 \Rightarrow \frac{a^2+b^2}{2ab} \geq 1$. So the above inequality implies $C_1 \cdot C_2 \geq \sqrt{C_1^2 \cdot C_2^2}$.

(3)$\Rightarrow$ (2) and (4)$\Rightarrow$ (2): These implications are straightforward.

(2)$\Rightarrow$ (3): Let $S \in \mathcal{Z}(X)$. If $S \cup C' \in \mathcal{Z}(X)$ for some $C' \in \mathcal{I}(X) \setminus S$, and $C \in S$, then $\{C, C'\} \in \mathcal{Z}(X)$. So it follows that $C \cdot C' = 0$, as required.

(2)$\Rightarrow$ (4): Let $S \in \mathcal{Z}(X)$. If $C_1, C_2 \in S$, then $\{C_1, C_2\} \in \mathcal{Z}(X)$. So it follows that $C_1 \cdot C_2 = 0$, as required. □

Remark 3.5. We will now explain why our main results (Theorems 3.1 and 3.2) imply Theorem 3.3.

First, suppose that the condition (b) of Theorem 3.3 holds. This is same as the statement (1) of Theorem 3.4. So both statements (3) and (4) of Theorem 3.4 hold and this implies that the condition of Theorems 3.1 and 3.2 hold for all subsets $S \in \mathcal{Z}(X)$. So for every $S \in \mathcal{Z}(X)$, we have the equality $W_S = \mathring{Z}_S$.

Conversely, if the condition (a) of Theorem 3.3 holds, then we have $W_S = \mathring{Z}_S$ for every $S \in \mathcal{Z}(X)$. By Theorem 3.1, we conclude that the statement (3) of Theorem 3.4 is true and hence condition (b) of Theorem 3.3 follows.

We will give some examples later which show that Theorem 3.3 does not imply Theorem 3.1 or Theorem 3.2 (see Example 3.8, for instance).

Now we give a necessary and sufficient condition for a Weyl chamber and a Zariski chamber to have non-empty intersection.

Theorem 3.6. *Let X be a smooth projective surface and let $S, S_1 \in \mathcal{Z}(X)$. Then*

$$W_{S_1} \cap Z_S \neq \emptyset$$

if and only if $S_1 \subseteq S$ and any subset $S' \subseteq S \setminus S_1$ satisfies the following property: there exist $C' \in S'$ and $C \in S \setminus S'$ such that $C' \cdot C > 0$.

Proof. First assume that $W_{S_1} \cap Z_S \neq \emptyset$. Let $D \in W_{S_1} \cap Z_S$ and let its Zariski decomposition be $D = P_D + N_D$. Then for every curve $C' \in S_1$, we have $D \cdot C' < 0$, as $D \in W_{S_1}$. Moreover, $N_D \cdot C' < 0$, as P_D is nef. This implies that C' is an irreducible component of N_D. Since the set of irreducible components of N_D is precisely S (since $D \in Z_S$), we conclude $S_1 \subseteq S$.

Now suppose that there exists a subset $S' \subseteq S \setminus S_1$ which does not satisfy the given condition. In other words, for any $C' \in S'$, we have $C' \cdot C = 0$, for all $C \in S \setminus S'$. The Zariski decomposition of D can be written as

$$D = P_D + N_D = P_D + \sum_{C_i \in S'} a_i C_i + \sum_{C_i \in S \setminus S'} b_i C_i.$$

As $D \in W_{S_1}$, we have $D \cdot C_j > 0$ for all $C_j \in S' \subseteq S \setminus S_1$. Note that,

$$D \cdot C_j = \sum_{C_i \in S'} a_i C_i \cdot C_j > 0.$$

Since $S' \subset S \in \mathcal{Z}(X)$, the intersection matrix of S' is negative definite. Then it follows from Lemma 2.6 that $a_i < 0$, which is absurd since $C_j \in S'$ are irreducible component of the divisor $D \in Z_S$. Hence, our assumption on S' is wrong and this completes the proof of one direction of the theorem.

To prove the converse direction, let $S' \subseteq S$ be a subset satisfying the given condition. Our goal is to find a $D \in W_{S_1} \cap Z_S$.

First we show that $W_{S_1} \cap Z_{S_1} \neq \emptyset$. We will construct a divisor in this intersection by defining a valid Zariski decomposition. Fix an ample divisor H. For some positive real numbers a_i and a_i^* (to be determined), consider a divisor of the following form:

$$D_1 = H + \sum_{C_i \in S_1} a_i^* C_i + \sum_{C_i \in S_1} (a_i - a_i^*) C_i, \tag{3.1}$$

where $P_{D_1} := H + \sum_{C_i \in S_1} a_i^* C_i$ and $N_{D_1} := \sum_{C_i \in S_1} (a_i - a_i^*) C_i$. We will show that a_i^* and a_i can be chosen such that $D_1 \in W_{S_1} \cap Z_{S_1}$.

If the above decomposition is to be the Zariski decomposition of D_1, then we must have $P_{D_1} \cdot C_j = 0$, for all $C_j \in S_1$. Hence,

$$\sum_{C_i \in S_1} a_i^* C_i \cdot C_j = -H \cdot C_j < 0.$$

As the intersection matrix of S_1 is negative definite, we can solve uniquely for a_i^* in the above linear system of equations. Moreover, by Lemma 2.6, $a_i^* > 0$. In particular, we conclude that P_D is nef.

Fix a set $\{y_i < 0 \mid 1 \leq i \leq |S_1|\}$ of negative real numbers and consider the following linear system of $|S_1|$ equations in $|S_1|$ variables,

$$\sum_{C_i \in S_1} x_i C_i \cdot C_j = y_j, \text{ where } C_j \in S_1 \tag{3.2}$$

Again by the negative definiteness of the intersection matrix of S_1 and Lemma 2.6, there is a unique solution $x_i > 0$ for the above system. Set $a_i := a_i^* + x_i$. Now we conclude that $D_1 \in W_{S_1} \cap Z_{S_1}$, where D_1 is defined in (3.1). Indeed, D_1 is big, since it is the sum of an ample divisor and an effective divisor. Further, $D_1 = P_{D_1} + N_{D_1}$ is a Zariski decomposition by construction. Hence $\text{Neg}(D_1) = S_1$, so $D_1 \in Z_{S_1}$. Also, by (3.2), $D_1 \cdot C_j < 0$ for every $S_j \in S_1$. If $C \notin S_1$, then $D_1 \cdot C \geq H \cdot C > 0$. So $D_1 \in W_{S_1}$, as required.

Now, define

$$S_2 := \{C_2 \in S \setminus S_1 \mid C_2 \cdot C_1 > 0 \text{ for some } C_1 \in S_1\}.$$

By the condition given in the theorem, $S_2 \neq \emptyset$. By following a similar construction as above, we find a divisor $D_2 \in W_{S_1} \cap Z_{S_1 \cup S_2}$. We briefly explain the procedure, for clarity. Let

$$D_2 = H + \sum_{C_i \in S_1} \alpha_i C_i + \sum_{C_i \in S_2} \beta_i C_i + nN_{D_1} + \sum_{C_i \in S_2} C_i,$$

where $P_{D_2} = H + \sum_{C_i \in S_1} \alpha_i C_i + \sum_{C_i \in S_2} \beta_i C_i$ and $N_{D_2} = nN_{D_1} + \sum_{C_i \in S_2} C_i$. We claim that there exist positive real numbers α_i, β_i and a sufficiently large integer n such that $D_2 \in W_{S_1} \cap Z_{S_1 \cup S_2}$.

We can set-up linear equations as above to find positive real numbers α_i, β_i such that $D_2 = P_{D_2} + N_{D_2}$ is the Zariski decomposition of the big divisor D_2 for any positive integer n. It is also clear then $D_2 \in Z_{S_1 \cap S_2}$. Now we show that for a sufficiently large integer n, $D_2 \in W_{S_1}$.

Let $C \in S_1$. Then $D_2 \cdot C = nN_{D_1} \cdot C + \sum_{C_i \in S_2} C_i \cdot C$. Since $N_{D_1} \cdot C < 0$, we may choose $n \gg 0$ such that $D_2 \cdot C < 0$ for every curve $C \in S_1$. Now let $C \notin S_1$. If $C \notin S_2$, then clearly $D_2 \cdot C > 0$ (note that D_2 is a sum of the ample divisor H and an effective divisor supported on the curves in $S_1 \cap S_2$). If $C \in S_2$, then, by the definition of S_2, there exists $C' \in S_1$ such that $C \cdot C' > 0$. Now $D_2 \cdot C = nN_{D_1} \cdot C + \sum_{C_i \in S_2} C_i \cdot C$. Since all the terms in $N_{D_1} \cdot C$ are non-negative and at least one term is positive, we can choose $n \gg 0$ such that $D_2 \cdot C > 0$.

Proceeding this way, since S is a finite set, we find a divisor $D \in W_{S_1} \cap Z_S$, as claimed. This completes the proof of the theorem. □

3.1. Examples and remarks.

We now give some examples and make some remarks illustrating our results.

Example 3.7. Let $\pi : X \to \mathbb{P}^2$ be the blow up at four collinear points $P_1, P_2, P_3, P_4 \in \mathbb{P}^2$. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let $E_i = \pi^{-1}(p_i)$ be the exceptional divisors.

The set of irreducible negative curves of X is given by:

$$\mathcal{I}(X) = \{E_1, E_2, E_3, E_4, \tilde{L}_{1234}\},$$

where $\tilde{L}_{1234}$ is the strict transformation of the line joining P_1, P_2, P_3, and P_4. Note that $\tilde{L}_{1234} = H - E_1 - E_2 - E_3 - E_4$.

Let $S = \{E_1, E_2, \tilde{L}_{1234}\}$ and $S_1 = \{E_1, E_2\}$. It is easy to check that the intersection matrices of both S and S_1 are negative definite. So $S, S_1 \in \mathcal{Z}(X)$. Using Theorem 3.6, we can conclude that $W_{S_1} \cap Z_S \neq \emptyset$. We exhibit below an explicit divisor in the intersection.

Consider the divisor $D = 6H - E_3 - E_4 + E_1 + E_2$. Note that the Zariski decomposition of D is given by

$$D = (5H) + (\tilde{L}_{1234} + 2E_1 + 2E_2).$$

It follows from the construction that $D \in Z_S$ (note that $D \notin \mathring{Z}_S$). Note that, $D \cdot E_1 = -1$, $D \cdot E_2 = -1$, while $D \cdot E_3 = D \cdot E_4 = 1$ and $D \cdot \tilde{L}_{1234} = 6$. Hence $D \in W_{S_1}$ too.

Example 3.8. Let $X \to \mathbb{P}^2$ be a blow up at five points $P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2$ such that P_1, P_2, P_3 are collinear and no other triple is collinear. Let $C := \tilde{L}_{P_1P_2P_3}$ denote the strict transformation of the line containing P_1, P_2, P_3. Then $C^2 = -2$ and $C \cdot E_1 = 1$. So we see that the condition of Theorem 3.3 is not satisfied. In other words, X is not numerically determined. We exhibit two sets in $\mathcal{Z}(X)$ which behave differently with regard to the containment of Weyl and Zariski chambers.

It is easy to verify that the set $\mathcal{I}(X)$ of negative curves on X consists of the exceptional divisors, C and lines through all pairs P_i, P_j of points, where either P_i or P_j is not in $\{P_1, P_2, P_3\}$.

Let $S = \{E_1, \tilde{L}_{P_1P_2P_3}\}$ and $S' = \{E_4, E_5\}$. Note that $S, S' \in \mathcal{Z}(X)$.

It is clear that S' satisfies the condition of Theorem 3.2. It can be checked that S' also satisfies the condition of Theorem 3.1. If C' is a negative curve which meets either E_4 or E_5, then it turns out that $\{E_4, E_5, C'\} \notin \mathcal{Z}(X)$. For example, if $C' = H - E_4 - E_5$, then the intersection matrix of $\{E_4, E_5, C'\}$ has a positive eigenvaule. On the other hand, the intersection matrix of $\{E_4, E_5, H - E_1 - E_4\}$ is not even invertible. Thus we have $\mathring{Z}_{S'} = W_{S'}$.

On the other hand, the condition of Theorem 3.2 fails for S. So $\mathring{Z}_S \not\subset W_S$. And we can check that the condition of Theorem 3.1 holds for S. So $\mathring{Z}_S \supsetneq W_S$. In fact, by Theorem 3.6, $Z_S \cap W_{S''} = \emptyset$ for any $S'' \neq S$.

Example 3.9. Let $X \to \mathbb{P}^2$ be a blow up at ten points of intersection of five general lines $L_1, \cdots, L_5$ in $\mathbb{P}^2$. Suppose that the four points of intersection that lie on L_1 are $p_1, \ldots, p_4$. Then the strict transform $C_1 := H - E_1 - E_2 - E_3 - E_4$ of L_1 is a negative curve. Let $C_2 := E_1$. Then $S = \{C_1, C_2\} \in \mathcal{Z}(X)$. Since $C_1 \cdot C_2 = 1$ and $C_1^2 = -3$, the condition in Theorem 3.3 is not satisfied. So again the Zariski chambers are not numerically determined.

In fact, using Theorems 3.1 and 3.2, we see that neither of the inclusions $W_S \subset Z_S$ nor $\mathring{Z}_S \subset W_S$ hold. Indeed, since $C_1 \cdot C_2 \neq 0$, we know by Theorem 3.2 that $\mathring{Z}_S \not\subset W_S$. On other hand, if $C = E_2$, then it is easy to check that $S \cup \{C\} \in \mathcal{Z}(X)$. Since $C \cdot C_1 = 1 \neq 0$, we know by Theorem 3.1 that $W_S \not\subset Z_S$.

On the other hand, by Theorem 3.6, we know that $W_S \cap Z_S \neq \emptyset$.

Example 3.10. Let D be an irreducible and reduced plane cubic and let $X \to \mathbb{P}^2$ be a blow up of s very general points on D. It is well-known that the only the only negative curves on X are the strict transform C of D (when $s > 9$), and the (-1)-*curves*, i.e., smooth rational curves whose self-intersection is -1; see [8] or [6] (refer to the arXiv version of the latter paper for substantial changes made after publication). Since $C^2 = 9 - s$ and $C \cdot C' = 1$ for any other negative curve C', the Zariski chambers on X are not numerically determined for $s \geq 11$.

In fact, it is easy to check that $S = \{C, C'\} \in \mathcal{Z}(X)$ for any negative curve $C' \neq C$. Since $C \cdot C' = 1$, we have $\mathring{Z}_S \not\subset W_S$, by Theorem 3.2. On the other hand, $S \cup \{C''\} \notin \mathcal{Z}(X)$ for any negative curve $C'' \notin S$. So $W_S \subset Z_S$, by Theorem 3.1.

Remark 3.11. Let X be a Del Pezzo surface. Then X is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow up of $\mathbb{P}^2$ at eight or fewer general points. The number of Zariski chambers in each of these cases is known; see [3, Theorem].

There are no negative curves in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. If X is a blow up of $\mathbb{P}^2$ at eight or fewer general points, it is well-known that the only negative curves on X are (-1)-curves. Hence the condition given in [10, Theorem 3] (see Theorem 3.3) is satisfied. So the interior of each Zariski chamber coincides with a Weyl chamber.

Remark 3.12. Let $X \to \mathbb{P}^2$ be a blow up of $r \geq 0$ general points of $\mathbb{P}^2$. The (-1)-*curves Conjecture* (some times called the *Weak SHGH Conjecture*) predicts that the only negative curves on X are the (-1)-curves. This is known to be true when $r \leq 9$ (see Remark 3.11 above) but it is open when $r \geq 10$. If this conjecture is true, then the Zariski chambers on X are numerically determined.

Remark 3.13. Let X be a geometrically ruled surface over a nonsingular curve Y. The Zariski chamber and Weyl chamber decomposition of the big cone of X coincide. If $\mathcal{E}$ is a semistable bundle over Y, then we know that the big cone and the ample cone are the same in $X = \mathbb{P}_Y(\mathcal{E})$. So we have only one Zariski chamber and one Weyl chamber. If $\mathcal{E}$ is an unstable bundle over Y, then X has exactly one negative curve. Hence there are two Zariski chambers and two Weyl chambers. One of the Zariski chambers is the nef chamber and one of the Weyl chambers is the ample cone (corresponding to $S = \emptyset$). The other chamber corresponds to the unique negative curve on X.

Remark 3.14. The Weyl and Zariski chamber decomposition on K3 surfaces is studied in detail in [2]. Our results (Theorems 3.1 and 3.2) and their proofs are motivated by analogous results proved for K3 surfaces. In addition, [2] gives examples of K3 surfaces where the decompositions coincide and where the decomposition differ.

The case of Enriques surfaces is considered in [10]. The authors relate the coincidence of Weyl chambers and interiors of Zariski chambers to the properties of elliptic fibrations on an Enriques surface.

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